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Considering weights in real social networks: A review

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Network science offers powerful tools to model complex social systems. Most social network science research focuses on topological networks by simply considering the binary state of the links, i.e., their presence or absence. Nonetheless, complex social systems present heterogeneity in link interactions (link weight), and accounting for this heterogeneity, it is mandatory to design reliable social network models. Here, we revisit the topic of weighted social networks (WSNs). By summarizing the main notions, findings, and applications in the field of WSNs, we outline how WSN methodology may improve the modeling of several real problems in social sciences. We are convinced that WSNs may furnish ideas and insights to open interesting lines of new research in the social sciences.

KEYWORDS

social networks, weighted networks, weighted network analysis, network science, social physics

1 Introduction

Networks are prominent frameworks to model the functioning of complex social systems [1–4]. Social networks are composed of nodes (individual actors, people, or social organizations within the network) and the links (relationships or interactions) that connect them. Social networks have been used to describe patterns in friendship [5], science collaborations [6], criminal organizations [7, 8], and to model spreading dynamics involving, e.g., several real problems of interest, such as information diffusion [9, 10], social influence models [11–14], epidemic spreading [15, 16], and vaccination policies [17, 18].

Most of the social network science research has focused on the topological features of the networks by considering the binary state of the links only, i.e., presence or absence [19].

Nonetheless, complex social systems present heterogeneity in the interactions among nodes [4, 19, 20], and social networks should be specified not only by their topology but also by the importance of the links (link weights) that differentiate the interactions among nodes in terms of their strength, intensity, or capacity [2, 3, 21, 22]. For these reasons, considering the heterogeneity in the intensity of links/interactions is fundamental to understanding complex social systems [21].

This review focuses on social network research adopting a weighted network approach. We summarize the main notions, findings, and applications of weighted social networks (WSNs). Then, we outline how WSN methodology may improve the modeling of several complex social system problems and encourage new lines of research.

2 Basic notions

A binary network $G(N, L)$ consists of two sets, N and L, such that the elements of $N \equiv \{n_1, n_2, n_3, \ldots, n_N\}$ are the nodes (vertices, points), while the elements of $L \equiv \{l_1, l_2, l_3, \ldots, l_L\}$ are its links (edges, ties, or lines). A binary network with N vertices is represented by an $N \times N$ adjacency matrix A with elements a_{ij} equaling 1 if nodes i and j are connected and 0 otherwise [19]. Two nodes connected by a link are usually referred to as adjacent or neighboring or neighbors. A weighted network $G(N, L, W)$ consists of an additional third set of elements W, such that $W \equiv \{w_1, w_2, w_3, \ldots, w_L\}$ are the weights of the links. A weighted network is thus represented by an $N \times N$ weights matrix W with elements $w_{ij} \neq 0$ if nodes i and j are connected by a link of weight w_{ij}, and 0 otherwise [19]. In this review, we consider only weighted networks with positive link weights.

3 The "strength of weak ties" hypothesis

The "weak link hypothesis" is probably the most influential sociological theory of networks. In the famous "The Strength of Weak Ties" research [23], Granovetter modeled the individual interpersonal relationships (links or ties) by quantifying the strength of the friendship (link weight) as "strong," "weak," or "absent." Strong links represent friends, and weak links denote tenuous acquaintances. Granovetter stresses the importance of weak links involving secondary acquaintances outside the people community, which therefore represent preferential sources of new information and job opportunities [24].

The "weak link hypothesis" describes a specific social network structure in which strong links are located within dense communities (or groups) of similar individuals. In contrast, weaker links act as bridges between the different communities (Figure 1A). The weak acquaintance links play the important role of holding together groups with low levels of similarity, avoiding the segregation of different and disparate human communities, and thus preserving the cohesiveness of the social network. Consequently, social networks would be vulnerable to removing weak links since their removal would fragment the network into isolated communities [4, 23].

In recent years, the "weak link hypothesis" has been confirmed in different real-world WSNs. In mobile phone call networks, longer phone calls (strong links) generally occur within communities, whereas shorter-duration calls (weak links) take place from nodes/individuals in different communities [25]. In a teenager criminal network, links, defined as friends of friends, join distant communities of individuals and have a positive impact on criminal activities [26]. In cinematic collaboration networks, in which the nodes represent actors and link weight indicates the number of movies in which they appeared together, weak links are the main ones responsible for supporting network cohesiveness. Their removal triggered the quickest network disconnection [27].

4 Node distance in weighted social networks

The distance among network nodes is a fundamental metric in social network science, and it is based on the notion of path [20]. A "*path*" is a set of distinct and connected nodes. The path of minimum length between two nodes is usually called the shortest path (SP) [19]. The "*distance*" d_{uv} between two nodes u and v is the minimum length of a path joining them, if any; otherwise, $d_{uv} = \infty$ [28]. The distance d_{uv} is also known as the geodesic distance, shortest path distance, or shortest path length. In binary networks, where links are present or absent (or have the same weight), the distance d_{uv} is the minimum number of nodes (or links)

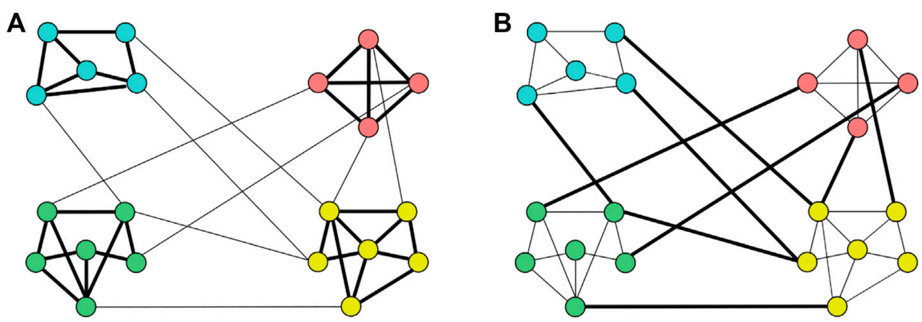

FIGURE 1
(A) Network structure underlying the "Granovetter weak link hypothesis" [23]. The thickness of the link indicates its weight, i.e., the intensity of the friendship between the nodes. Links of higher thickness are of higher weight (strong links), and links of lower thickness are of lower weight (weak links). Node color denotes different community memberships, and nodes of the same color are of the same community. The weak link hypothesis depicts a social network in which strong links denoting tight friendship generally occur within people communities, and weak acquaintance links connect distant individuals of different communities. Weak links are important to preserve the cohesiveness of the network. **(B)** The strong link hypothesis depicts a social network in which weak links generally occur within people communities, and strong friendship links connect individuals belonging to different communities. In this case, the strong links are the main players sustaining the cohesiveness of the social network.

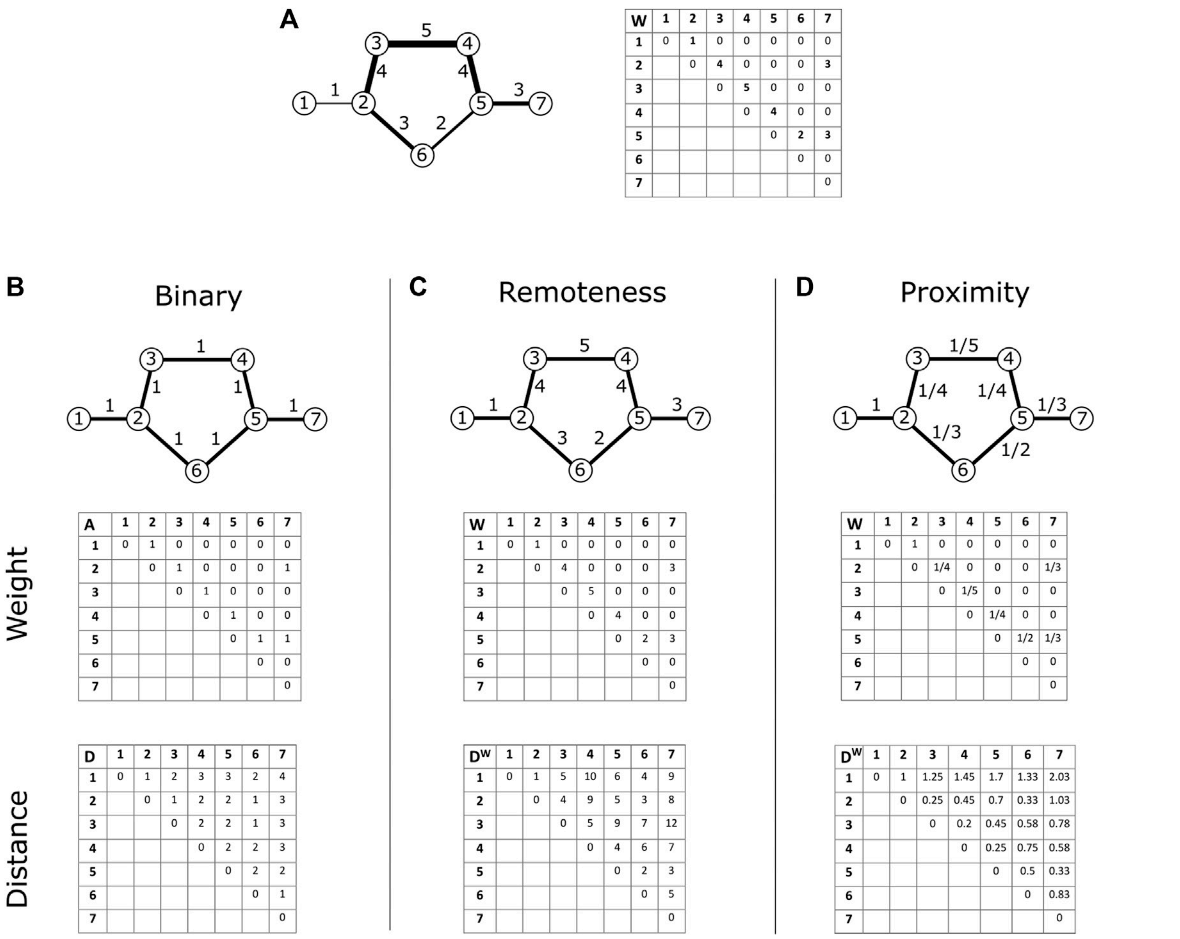

FIGURE 2
Binary and weighted node distance computational examples. (Top) Weighted network of seven nodes and seven links and its weight matrix W. (Left column) Binary network counterpart of the weighted network on top, with its adjacency matrix A and its distance matrix D of the binary node distances. (Middle column) The weighted network with link weight as remoteness, its weight matrix W and its distance matrix D^w of the weighted node distances d^w_{ij}. (Right column) The weighted network with link weight as proximity, its weight matrix W computed by reciprocating the link weights, and its distance matrix D^w of the weighted node distances d^w_{ij}. The element D_{ij} of the distance matrix indicates the distance d_{ij} (shortest path length) between node i and node j. Since the computation is for undirected networks, we depict only the upper triangular matrix of D.

to travel between the nodes [28]. In Figure 2 (left column), we give an example of binary node distance computation. From now on, d_{uv} indicates the shortest path length between nodes in binary networks.

In the case of weighted networks, to compute the shortest path, it is necessary to distinguish, in advance, the link weight meaning [3, 29]. The link weight may assume different meanings in WSNs, and a higher link weight value may indicate that nodes are either closer or farther apart. Let us take the example of a social network of employees. The response time to emails among nodes/employees in this working network indicates how far the connected nodes are in the social network, thus denoting less important working relationships. On the contrary, the time two nodes/employees spend calling each other, *calling time*, indicates that the nodes are closer in this social network. From now on, we define "proximity" the link weight, if its increase denotes a closer relationship between the nodes, and we define "remoteness" the link weight in the opposite scenario. *Proximity* has been defined as a "flow or capacity" in the literature, and remoteness is also named "cost, distance, or resistance" [19]. Examples of link weights as remoteness in social networks can be computed from the physical distance among nodes/individuals in friendship networks [30], the response time to emails among nodes/employees in email networks, the number of troublesome among nodes/individuals in friendship networks [31], or the antagonism number in working relationship networks [32]. Conversely, examples of link weights as proximity in social networks are the duration of a call between nodes/individuals in telephone networks [25, 33], the number of co-authored papers between scholars in scientific collaboration networks [6, 34], the number of emails among nodes/employees [35, 36], and the declared strength of friendship among nodes/individuals in social networks [5, 37–39]. In Table 1, we list important types of real-world WSNs, indicating whether the link weight meaning is proximity or remoteness.

In weighted networks, we call the shortest path between two nodes the "weighted shortest path" (WSP). The "*weighted distance*" d^w_{uv} between two nodes u and v in a weighted network is the length of the WSP, if any; otherwise, $d^w_{uv} = \infty$ [28]. From now on, d^w_{uv} indicates the shortest path length between nodes in weighted networks.

TABLE 1 Types of real-world weighted social networks. The "x" indicates if link weights are proximity or remoteness.

Network type	Description	Nodes	Links	Weight	Proximity	Remoteness	Reference
Advogato online social network	Trust relationship network among users on Advogato online community software developers.	Advogato webpages	Trust relationship	Trust strength	X		[40]
Actors	Actor coappearance network from the Internet Movie Database (IMDb), from 2009 and 2011. Nodes are actors, and two actors are linked if they appeared in a movie together.	Actors	Coappearances	Number of coappearances	X		[41]
Art	Artists' exhibition networks in the Museum of Modern Art	Artists	Co-exhibition	Co-exhibition number	X		[42]
	New York, from 1929 to 1968.						
Bitcoin	Network of who-trusts-whom relationships among users of the Bitcoin Alpha platform.	Traders	Trust relationship	Trust strength	X		[43]
Boards of Directors	Directors' affiliations network among Norwegian public limited companies (from 2002 to 2011)	Directors	Comemberships	Comembership number	X		[44]
Characters (book)	Network of coappearances of characters within 15 words in the Game of Thrones books series.	Characters	Coappearances in 15 words of the book	Coappearance number	X		[45]
Characters (movie)	Network of coappearances of characters in over 700 movies from the moviegalaxies.com website.	Characters	Coappearances in a scene	Number of coappearances in scenes	X		[46]
Cellphone users	Mobile phone network from 18 weeks of all mobile phone call record	Phone users	Calls	Total call duration (minutes)	X		[25]
Cellphone municipalities	Mobile phone calling network among municipalities in Colombia during a six-month period.	Cellphones	Calls	Total call duration (minutes)	X		[33]
Chess	Chess players' network.	Players	Matches	Number of matches	X		https://www.kaggle.com/c/chess/data
Crime	Criminal network	Criminals	Collaboration in criminal activities	Collaboration numbers	X		[47]
Drug	Networks representing communications among members of cocaine trafficking groups involved in Spain 2007–2009. Nodes are people, and an edge denotes intercepted communication between them.	Drug sellers	Collaborations	Number of collaborations	x		[48]
Email	Email network of peoples having received the same email in the 2016 Democratic National Committee email leak.	Persons	Reception of the same emails	Number of emails	X		[35]
Email	The Enron email network between employees of Enron between 1999 and 2003.	Employees	Email sending	Number of emails sent	X		[36]

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TABLE 1 (*Continued*) Types of real-world weighted social networks. The "x" indicates if link weights are proximity or remoteness.

Network type	Description	Nodes	Links	Weight	Proximity	Remoteness	Reference
Epidemics	Epidemic network of individuals infecting each other by sexual contact from "The National Survey" of the United Kingdom.	Persons	Infections	Infection probability	X		[15]
Friendship (offline)	Friendship network a United Kingdom faculty constructed with a questionnaire, where the individuals declared the strength of the friendship with others.	Individuals	Friendship	Declared friendship strength	X		[5]
Friendship (offline)	Friendships among freshmen at the University of Groningen, collected over 1997–1998. Link weight gives the friendship level from 1 (best friend) to 5 (troubled relation).	Individuals	Friendship	Troublesome level in the friendship		x	[31]
Friendship (online)	Online social networks, Gowalla and Brightkite.	Users	Friendships	Geographical spatial distance		x	[30]
Facebook Artist	Facebook artist online social networks represent the artist pages, and links are mutual likes among them. Data collected in November 2017.	Facebook artists' webpages	Mutual likes	Number of mutual likes	X		[49]
Facebook friendships	Friendship relationships and interactions (wall posts) for a subset of the Facebook social network in 2009, recorded over a 2-year period.	Facebook users	Friendships	Number of wall posts	X		[50]
Face-to-face	Face-to-face network built using wearable sensors to detect close-range interactions ("contacts") between individuals. Contact events were measured with a spatial resolution of about 1.5 m and a temporal resolution of 20 s.	Individuals	Close-range interactions	Interaction time	X		[51, 52]
Innovation spreading	Social networks that describe the idea spreading among individuals	Individuals	Interactions	Number of interactions number	X		[53]
Mafia	Mafia criminal network describing the number of times two individuals had a meeting (as reported by the police).	Mobsters	Meetings	Number of meetings	X		[7]
Online messages	Network of messages between the users of an online community of students from the University of California, Irvine.	Students	Message	Number of messages	X		[22]
Prostitution	Network of escorts and individuals who buy sex from them in Brazil, extracted from a Brazilian online community. Links represent a purchase of sexual intercourse.	Escorts/sex buyers	Sex	Number of sex activities	x		[54]

(Continued on following page)

TABLE 1 (*Continued*) Types of real-world weighted social networks. The "x" indicates if link weights are proximity or remoteness.

Network type	Description	Nodes	Links	Weight	Proximity	Remoteness	Reference
Rating Online	A network of ratings given between users at Libimseti.cz, a Czech online dating website. Links mean rating among users and corresponding link weight is the given rating, on a scale of 1–10.1	Individuals	Ratings	Rating value	X		[55]
Scientific collaboration	A co-authorship network among scientists working on network science, from 2006.	Scientists	Coauthorship	Number of papers	X		[56]
Scientific collaboration	Scientific collaborations of New Zealand institutions using Scopus bibliometric data (2010–2015)	Institutions	Coauthorship	Number of papers	X		[57]
Terrorists	Networks representing connections among the individuals associated with bombing or other terrorist events.	Terrorists	Friendship	Strength of friendship	X		[58]
Workers	Employees network from observational data of the Western Electric (Hawthorne Plant) factory	Employees	Working relationships	Number of antagonistic acts		x	[32]
Workers	Employees network from observational data of the Western Electric (Hawthorne Plant) factory	Employees	Working relationships	Number of helping acts	X		[32]

While computing d^w_{uv}, if the weights mean remoteness, the d^w_{uv} connecting two nodes is the minimum sum of the original link weights necessary to travel between them [29, 59, 60] (Figure 2, middle column). This procedure has the rationale to evaluate the more distant nodes connected by links with higher weight/remoteness.

In contrast, if the weights mean proximity, we first compute the reciprocal of the link weights, and then d^w_{uv} between two nodes is the minimum sum of the reciprocal of the link weights necessary to travel between them [3, 29, 60, 61]. This standard procedure is necessary to rightly evaluate higher link weight/proximity as a "faster and shorter route" between nodes, i.e., the higher the weight of a link, the faster the information flows between the linked nodes (and the closer the nodes). Conversely, the lower the link weight/proximity, the more distant the nodes [60]. This simple calculation has a straightforward interpretation in social systems. For example, in a telephone social network where link weights evaluate the calling times between nodes/individuals, the distance between them is just the inverse of the weight of their calling time. For example, if two individuals call each other twice as often as another couple of individuals, the distance between the first couple is half the distance between the second couple. Figure 2 (right column) depicts an example of weighted node distance computation for link weights as proximity.

Notably, while the interpretation of node distance is straightforward in binary, unweighted networks, whenever weights are reported, the meaning of the link weight must be considered to avoid wrong calculations and results. For example, Table 1 shows that since the link weights in most real-world WSNs present weights as proximity, the link weight is inversely proportional to the distance between network nodes. To correctly calculate the distances d^w_{uv} in social networks, it is essential to understand the structure of the relationships among nodes/individuals. On the one hand, considering link weights as proximity or remoteness may change the links forming the shortest paths between nodes (Figure 3). On the other hand, d^w_{uv} is the base to compute different node centralities such as closeness, betweenness, and delta centrality [19, 60, 62], and an incorrect d^w_{uv} assessment would lead to an incorrect node ranking. Lastly, d^w_{uv} is necessary to compute different network efficiency measurements [29, 60, 63, 64].

Mistakes in node distance computation in weighted networks may be more frequent than expected; it has been recently shown that in ecological network science, among 129 published research studies, 61% of these studies using shortest paths in weighted networks may contain errors in how WSPs are calculated [65].

5 Measures of node centrality

In network science, measures (or indicators) of node centrality assign rankings to nodes within a network corresponding to their position in the network structure [1, 66]. Node centrality measures were first developed in social network analysis to identify influential persons in social networks [1, 67, 68]. Ranking network nodes according to their network structural embedding helps address a

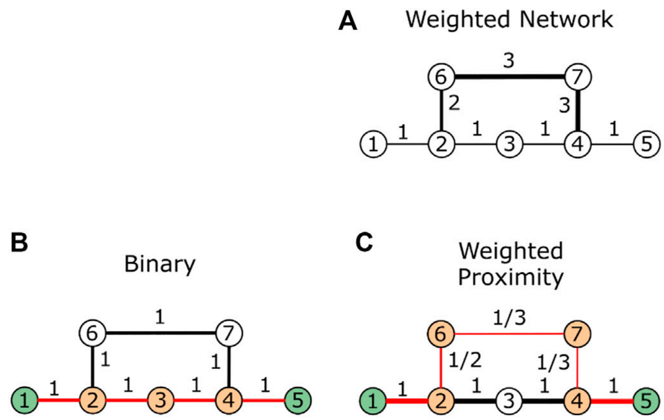

FIGURE 3
Finding the shortest path between nodes 1 and 5 in a weighted network. **(A)** The weighted network. **(B)** The binary shortest path. **(C)** Weighted "proximity" shortest path. **(D)** Weighted "remoteness" shortest path. The links forming the shortest path are in red, and nodes belonging to the shortest path are in orange. We can see how, by changing the methodology used to compute the distances among nodes in the social network, the shortest path between the nodes may change as well.

variety of problems in social networks, such as identifying the most influential persons in a friendship network [5], selecting the influential spreaders of news and information [69], and finding the most important nodes for vaccination to halt epidemic spreading, [17, 18, 70], etc [66]. Many node centrality measures conceived for binary networks were then adapted to rank nodes in WSNs.

The simplest measure of centrality is the node degree, i.e., the number of links to the node [1, 3].

The degree k_i of node i is given by:

$$k_i = \sum_{i=1}^{N} a_{ij}, \tag{1}$$

where N is the number of nodes in the network; a_{ij} equals 1 if there is a link connecting nodes i and j and equals 0 otherwise. The node strength, also named weighted degree, is the sum of the link weights to the node [3, 60].

The strength s_i of node i is:

$$s_i = \sum_{i=1}^{N} a_{ij} \cdot w_{ij}, \tag{2}$$

where $a_{ij} = 1$ if a link connects nodes i and j and 0 otherwise, and w_{ij} is the weight value of the link connecting nodes i and j. The degree and strength are simple and local measures, i.e., only the local structure around a node is required to calculate them. However, there are limitations: the degree and strength measures do not consider the global structure of the network. For example, nodes with few links might be located in a privileged position to reach others quickly to access resources, such as information or knowledge [3, 71]. The node closeness centrality can capture this feature and is defined as the reciprocal of the sum of the distances among the node and all other nodes in the network [1].

The closeness centrality of node i is:

$$C_i = 1 \Big/ \sum_{i \neq j \in N} d_{ij} \tag{3}$$

where d_{ij} indicates the binary distance from node i to node j. Therefore, in binary networks, nodes with more closeness are, on average, fewer steps away from the other nodes in the networks. In weighted networks, node closeness is defined by substituting the binary distance d_{ij} with the weighted distance d_{ij}^{w} among network nodes. Therefore, the weighted closeness centrality considers both the number of intermediary nodes and the link weights [3].

The betweenness is another common measure of node centrality, accounting for how many shortest paths among nodes lie on a node [1].

The betweenness centrality $g(i)$ of node i is:

$$g(i) = \sum_{s,t=1}^{N} \frac{\sigma_{st}(i)}{\sigma_{st}} \tag{4}$$

where σ_{st} is the total number of SP between nodes s and t, $\sigma_{st}(i)$ is the number of these SP passing through the node i, and N is the number of nodes. Betweenness is able to identify nodes that funnel the information flow in the network [72] and bridge-nodes connecting different communities [73, 74]. The weighted counterpart of the betweenness considers the WSP in the networks.

The weighted betweenness centrality $g^{w}(i)$ of node i is:

$$g^{w}(i) = \sum_{s,t=1}^{N} \frac{\sigma_{st}^{w}(i)}{\sigma_{st}^{w}} \tag{5}$$

where σ_{st}^{w} is the total number of the WSP between nodes s and t, $\sigma_{st}^{w}(i)$ is the number of these WSPs passing through node i, N is the number of nodes.

Passing from binary to node centrality measures accounting for the weight associated with the links might change the rank of node importance in social networks [3, 4, 19, 60]. In Figure 4, we show, on a simple network, how adopting the binary or the weighted version of the node centrality changes the node rank. Newman [6] showed that, by contrast, with the simple binary closeness measure, the list of scientists who are well-connected when ranking nodes with weighted closeness is no longer dominated by experimentalists, although the well-connected among them still score highly. The same has been proven true for other real-world social networks and

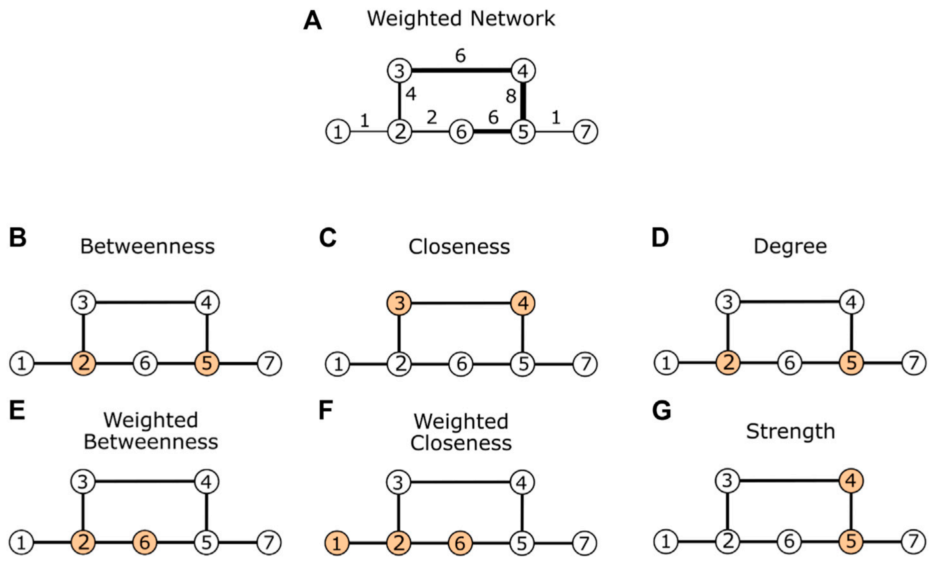

FIGURE 4
Node centrality ranking in binary and weighted networks. Orange nodes identify the highest centrality value nodes. **(A)** A weighted network with link weights as proximity. **(B)** First two nodes with a higher binary betweenness. **(C)** First two nodes with a higher binary closeness. **(D)** First two nodes with a higher degree. **(E)** First two nodes with higher weighted betweenness. **(F)** First three nodes with a higher weighted closeness. **(G)** First two nodes with higher strength. Passing from binary to weighted node centralities, the node rank changes as well.

different node rankings [60]. Therefore, for social network analyses, it is very important to adopt node centrality measures, considering the weight of the links.

Ranking important nodes is fundamental for finding influential spreaders in social networks [69, 75, 76], and most of the research investigating influential spreaders uses a binary network approach. In WSNs, Garas et al. [77] used a weighted SIR model to describe a general spreading process in networks of different natures and classified the nodes with a generalized method for calculating the weighted node coreness centrality. They show that the proposed weighted node coreness method places nodes with higher spreading potential closer to the network core, and it is more accurate in finding the best spreader nodes in WSNs [77]. With similar aims, Gao et al. [78] proposed a weighted version of the famous Hirsch index [79], usually called the H-index, to find influential spreader nodes. The authors defined the weighted h-index (h^w) and evaluated the effectiveness of the proposed measure with the SIR spreading model on three real-world social networks. The authors found that h^w may perform better than classic node centralities in finding influential spreaders in WSNs [78].

Finding influential spreader nodes is a pressing problem in case of a spreading epidemic [80, 81]. In this case, considering link weights and proper node centrality measures for weighted networks may be very important in selecting the best spreader node [4]. For example, let us take a social network where the link weights account for the face-to-face contact duration among people and consequently determine the probability that a susceptible person is infected after having been in contact with an infectious person. In this case, neglecting the link weights may hide the paths of higher infection probability, consequently selecting false influential spreader nodes and tracing unlikely infection trajectories (Figure 5).

In Table 2, we give a list of node centrality measures for WSNs with literature references.

6 Community structure in weighted social networks

The investigation of community structures in social networks is an important issue in many domains and disciplines since many real-world social networks present community structure [19, 61, 89]. Moreover, social networks offer a wide variety of possible community organizations: families, working and friendship circles, scholar collaborations, and social networking groups [90]. The classic weak link hypothesis is a paradigmatic example of social networks with community structure, in which densely connected communities of nodes/people are joined by sparsely weaker links (Figure 1).

The definition of community in binary networks is simple: a community is defined as a subset of nodes within the network such that the links among the nodes of the same community are denser than connections with the rest of the network [91]. In other words, communities are densely connected subgroups of network nodes with sparser connections among them. Synonyms of community are group, clique, module, class, and cluster.

In the following two paragraphs, we first describe node clustering, which can be viewed as the simple notion of the network community, and then the generalized community detection methodology.

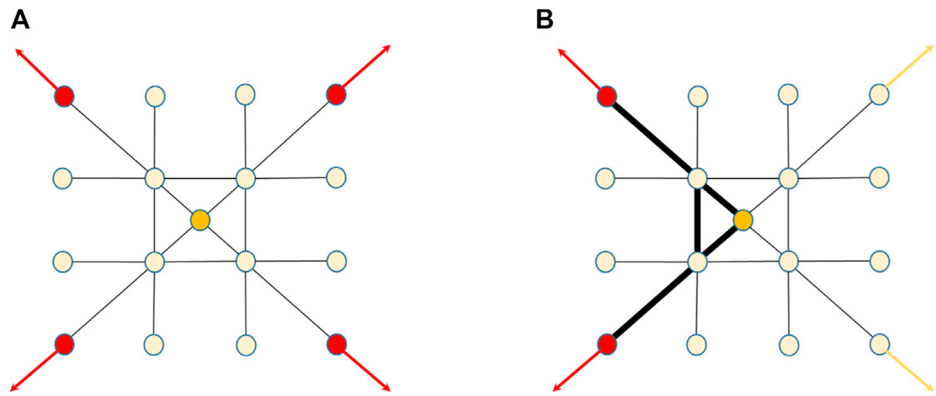

FIGURE 5
The paths of higher infection probability among nodes/individuals in a community. The orange node is the infected individual starting the epidemic; the red nodes are the individuals that may act as potential spreaders of infection outside the community, and the red arrows indicate the likely trajectories of the epidemic spreading outside the community. **(A)** A binary network where the four red nodes are all potential spreaders. **(B)** The weighted network counterpart of the binary network in **(A)**, where the most significant channels of infection emerge; in this weighted network, the potential spreaders to consider are the two red nodes only, and the likely trajectories of infection outside the community are only two (red arrows).

6.1 Clustering coefficient

A simple method to investigate the presence of communities is to evaluate the network clustering coefficient [20]. In binary networks, (binary) clustering is a measure that counts node triplets in the network. A triplet (or triangle) is a set of three nodes. A closed triplet is a full network of three nodes, i.e., a set of three nodes in which each node is connected by a link with the others. In other terms, a triplet is three nodes that are connected by either two (open triplet) or three (closed triplet) links.

The binary "local clustering coefficient" c_i of node i is defined as:

$$c_i = \frac{\Delta_i^{clo}}{\Delta_i^{tot}} \tag{6}$$

where Δ_i^{clo} is the number of closed triplets centered on node i, and Δ_i^{tot} is the total number of triplets (both open and closed) centered on node i [22].

Alternatively, the binary "local clustering coefficient" c_i may be written as follows:

$$c_i = \frac{1}{k_i(k_i-1)} \sum\nolimits_{j,h} a_{ij}a_{ih}a_{jh} \tag{7}$$

where k_i is the degree of node i, and a_{ij} is 1 if node i is connected with node j, and 0 otherwise.

By computing the triplets over the whole network, we can define the binary global clustering coefficient by generalizing Eq. 6:

$$C = \frac{\Delta^{clo}}{\Delta^{tot}} \tag{8}$$

where Δ^{clo} is the number of closed triplets, and Δ^{tot} is the total number of triplets (both open and closed) in the network [22].

The binary clustering coefficient evaluates the local group cohesiveness, and it is defined for any node in the network as the fraction of connected neighbors. In other words, the binary clustering coefficient indicates the intensity with which nodes tend to form tightly knit communities characterized by a relatively high density of links, i.e., a likelihood that tends to be higher than the average probability of links randomly drawn among nodes [92, 93].

There are many generalizations of the binary clustering coefficient for weighted networks [22, 87, 94, 95]. Barrat et al. [87] proposed the most commonly used generalization, which defined a weighted version of the local clustering coefficient defined in Eq. 7 as:

$$c_i^w = \frac{1}{s_i(k_i-1)} \sum\nolimits_{j,h} \frac{(w_{ij}+w_{ih})}{2} a_{ij}a_{ih}a_{jh} \tag{9}$$

where s_i is the strength of node i, k_i is the degree of node i, w_{ij} is the weight of the link connecting i and j, and a_{ij} is 1 if node i is connected with node j, and 0 otherwise. The global version of the weighted clustering coefficient is computed by averaging the local clustering in Eq. 9 over all nodes in the network [87].

Opsahl and Panzarasa [22] proposed another commonly used generalization of the global clustering coefficient in WSNs. Defining ω as the weight of a triplet, i.e., the average link weights of the triplet, we can generalize the global clustering coefficient to weighted networks as:

$$Cw = \frac{\sum_{|\Delta c|} \omega}{\sum_{|\Delta|} \omega} \tag{10}$$

where $\sum_{|\Delta c|} \omega$ is the total weight of the closed triplets, and $\sum_{|\Delta|} \omega$ is the total weight of all triplets [22].

The generalized weighted clustering coefficients proposed by Barrat et al. [87] and Opsahl and Panzarasa [22] are both measures of the local cohesiveness in weighted networks that take into account both the number of closed triplets in the neighborhood of a node and their total relative link weight with respect to the strength of the node. In Figure 6, we give examples of binary and weighted node clustering coefficient computation. As shown in Figure 6, the binary and weighted clustering coefficient values are different, and the

TABLE 2 Type of node centrality measures for weighted networks.

Node centrality	Definition	Formula	Reference
Strength	Sum of the weight of the links to the node	The strength s_i of the node i is $s_i = \sum_{i=1}^{N} a_{ij} \cdot w_{ij}$, where a_{ij} is 1 if there is a link joining nodes i and j and 0 otherwise, and w_{ij} is the weight of the link connecting nodes i and j.	[60, 64]
Betweenness	The weighted betweenness centrality of the node is the number of weighted shortest paths passing on it.	The weighted betweenness centrality $g^w(i)$ of the node i is $g^w(i) = \sum_{s,t=1}^{N} \frac{\sigma_{st}^w(i)}{\sigma_{st}^w}$, where σ_{st}^w is the total number of WSP between nodes s and t, $\sigma_{st}^w(i)$ is the number of these WSP passing through the node i, and N the number of nodes.	[82, 83]
Farness	The weighted farness is the sum length of the shortest paths between the i and all other nodes in the network	The weighted farness centrality of the node i is $F_i^w = \sum_{i \neq j \in N} d_{ij}^w$, where d_{ij}^w indicates the weighted distance from node i to node j.	[84]
Closeness	The weighted closeness is the reciprocal of the sum length of the shortest paths between the node and all other nodes in the network	The weighted closeness centrality of the node i is $C_i^w = 1/ \sum_{i \neq j \in N} d_{ij}^w$, where d_{ij}^w indicates the weighted distance from node i to node j.	[84]
Harmonic centrality	The harmonic centrality is the sum of the reciprocals of all distances to the node.	The harmonic centrality of node i is $\Theta_i^w = \sum_{i \neq j \in N} \frac{1}{d_{ij}^w}$, where d_{ij}^w indicates the weighted distance from node i to node j.	[85]
h^w Index	The weighted h index is the weighted counterpart of the classic h index.	The virtual weights set W_i of node i is $W_i = \{w_{ij,1}, \ldots, w_{ij,1}\}_{\forall j \epsilon \tau_i}$, where k_j is the degree of node j, w_{ij} is the weight of the link joining i and j, and τ_i is the neighbors set of i. In other words, for each neighbor j of node i, the weight w_{ij} of the link connecting i and j is repeated k_j times, where k_j is the degree of node j. Then, the weighted h index of node i is $h_i^w = \mathrm{H}(W_i)$, where $\mathrm{H}(\cdot)$ is an operator, which finds the maximum integer h such that there are at least h virtual weights of W_i whose value is no less than h.	[78, 86]
The virtual strength	The virtual strength of the node i is the sum of the weighted h index h_j^w of its j neighbors	The virtual strength of the node i is $s_i^h = \sum_{\forall j \epsilon \tau_i} h_j^w$, where h_j^w is the weighted h index of the node j, and τ_i is the neighbors set of i.	[78]
Transitivity	The weighted node transitivity considers the number and the strength of the closed triplets of the node.	The weighted node transitivity c_i^w of node i is $c_i^w = \frac{1}{s_i(k_i - 1)} \sum_{j,h} \frac{(w_{ij} + w_{ih})}{2} a_{ij} a_{ih} a_{jh}$, where s_i is the strength of i, k_i the degree of i, w_{ij} is the weight of the link connecting nodes i and j, and a_{ij} is 1 if there is a link between i and j.	[87]
Efficiency delta centrality	The efficiency delta centrality $\delta(i)$ of node i is the network efficiency decrease after the removal of i.	The weighted network efficiency is $Eff^w = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}^w}$, where N is the number of nodes and d_{ij}^w indicates the weighted distance from node i to node j. The efficiency delta centrality of node i is $\delta_i = \frac{Eff^w - Eff_i^w}{Eff^w}$, where Eff^w is the initial network efficiency and Eff_i^w the efficiency after the removal of node i.	[62]

(Continued on following page)

TABLE 2 (*Continued*) Type of node centrality measures for weighted networks.

Node centrality	Definition	Formula	Reference
Eigenvector	Eigenvector centralities correspond to the values of the first eigenvector of the adjacency matrix of the network	The weighted eigenvector centrality is	[68]
		$x_i^w = \frac{1}{\lambda}\sum_{j\in\tau_i} x_j^w = \frac{1}{\lambda}\sum_{j\in N} w_{ij}\cdot x_j^w,$	
		where τ_i is the set of neighbors of i, w_{ij} is the weight of the link connecting nodes i and j, N is the set of nodes, and λ is a constant.	
PageRank	The weighted PageRank evaluates the node importance, based on its links number, link weights, and the importance of the linked nodes.	The weighted Pagerank $WPR_i^{(t)}$ of node i is	[66]
		$WPR_i^{(t)} = \sum_{j=1}^{N} w_{ij}\cdot\frac{WPR_j^{(t-1)}}{s_j}$, where w_{ij} is the weight of the link connecting i and j, and s_j is the strength of node j.	
LeaderRank	The weighted LeaderRank evaluates the node importance, based on its link number, link weights, and the importance of the linked nodes.	The weighted LeaderRank rank WLR_i of node i is	[66]
		$WLR_i^{(t)} = \sum_{j=1}^{N+1} w_{ij}^{\alpha}\cdot\frac{WLR_j^{(t-1)}}{b_j},$	
		where w_{ij} is the weight of the link connecting i and j, α is a tunable parameter, and $b_j = \sum_i w_{ij}^{\alpha}$. When $\alpha = 1$, b_i is the strength s_i of the node i.	
h-Degree	The h-degree (hD_i^w) of node i is the largest natural number k such that i has at least k links each with weight at least equal to k.	The h-degree (hD_i^w) of node i is $hD_i^w = \{w_{i,j=1},\dots,w_{i,j=m}\}_{\forall j\in\tau_i},$	[88]
		where w_{ij} is the weight of the link connecting i and j, m is the number of neighbors of i, and τ_i is the neighbors set of i. The operator $\mathrm{H}(\cdot)$ is an operator returning the maximum integer h such that there are at least links whose weight value is no less than h.	
w-Lobby index	The w-lobby index (lD_i^w) of node i is defined as the largest integer k such that i has at least k neighbors with node strength at least k.	The w-lobby index (lD_i^w) of node i is $lD_i^w = \{s_{j=1},\dots,s_{j=m}\}_{\forall j\in\tau_i},$	[88]
		where s_j is the strength of node j, m is the neighbors number of i, and τ_i is the neighbors set of i. The operator $\mathrm{H}(\cdot)$ is an operator returning the maximum integer h such that there are at least h	
		nodes of strength s whose value is no less than h.	
Weighted coreness	The weighted coreness ranks nodes according to their centrality in the core of weighted networks.	A weighted degree of node i is defined: $k_i^W = (k_i^{\alpha}\cdot s_i^{\beta})^{\frac{1}{\alpha+\beta}}$	[66, 77]
		where k_i and s_i are the degree and the strength of i, respectively; α and β are tunable parameters. Then, to find the weighted coreness, the k-core decomposition process must be performed in weighted networks.	

weighted clustering provides additional information about the tightness of the node clustering.

The two measures described provide important results, and in all WSNs studied, the value of the weighted coefficient was greater than the value of the binary one [22]. These findings support Granovetter's [23] claim that in social networks, strong ties are more likely to be part of transitive triplets than weak ones [22].

The clustering coefficient measures the community structure of the networks only considering the triadic closure, i.e., evaluating the tendency of the nodes to cluster in communities of three nodes.

6.2 Community detections in weighted social networks

Social network communities are often composed of several nodes, and the node's propensity to cluster together in communities can be generalized to groups of any order.

The detection and characterization of community structure in social networks, meaning the appearance of densely connected groups of nodes with only sparser connections between groups, has increased in importance in the last decades [61, 96]. The node communities division can be performed by different methods of community detection, such as random walk searching [97], link betweenness [98], fast greedy modularity optimization [61], and spinning glass model [99] [90]. Proper indicators may then be used to evaluate the goodness of the communities division [90]. In the following, we describe the modularity Q [61], a community structure indicator that can properly work on binary and weighted social networks.

The modularity Q of a network measures how good the division of two node communities is or how separated the different node communities are from each other [61].

The modularity indicators Q for binary networks are defined as:

$$Q = \frac{1}{2L}\sum_{i,j}\left(a_{ij} - \frac{k_i k_j}{2L}\right)\delta\left(c_i c_j\right) \tag{11}$$

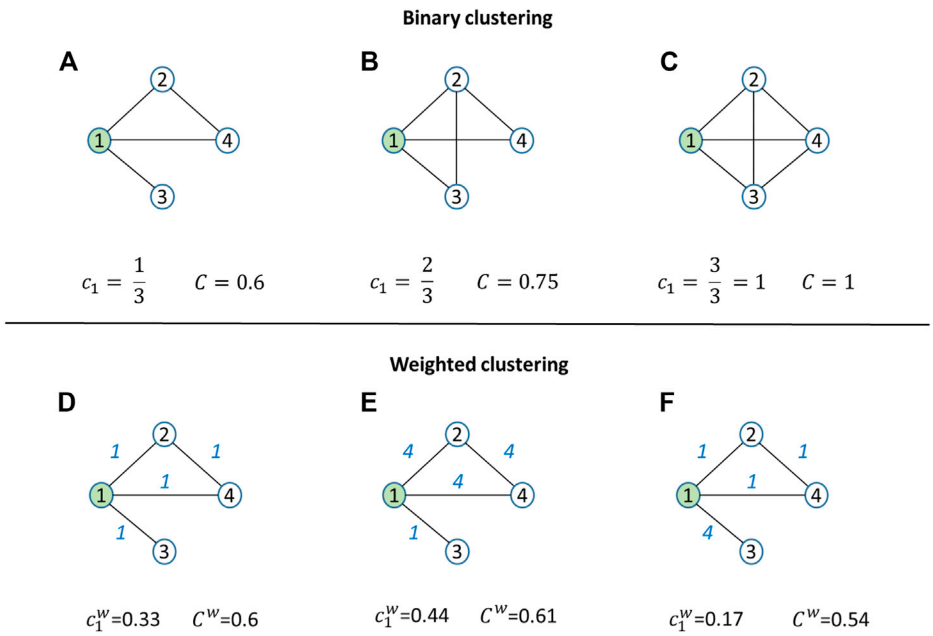

FIGURE 6
Clustering coefficients in binary and weighted networks. **(A-B-C)** The binary local clustering coefficient c_1 of node 1 is computed with Eq. 6 as a ratio of the closed triplets on node 1 over the total number of triplets centered on node 1 (three triplets). The global clustering coefficient is the ratio of closed triplets over the total number of triplets computed for the whole network (Eq. 8). In the network **(A)**, there is only one closed triplet over the three total triplets, the clustering of node 1 is $c_1 = \frac{1}{3}$, and the global clustering is $C = \frac{3}{5} = 0.6$. In the network **(B)**, there are two closed triplets over the three total triplets, thus $c_1 = \frac{2}{3}$, and the global clustering is $C = \frac{6}{8} = 0.75$. In the complete network **(C)**, all the triplets are closed with $c_1 = \frac{3}{3}$ and $C = 1$. **(D)** Weighted network in which the link weight is uniform and equals 1. In this network, the weighted clustering coefficient computed using Eq. 9 and its global version C^w return the same value for both the local and global binary clustering coefficients in panel **(A)**. **(E)** Weighted clustering coefficient of node 1 in a weighted network in which the closed triplet is made by strong links. **(F)** Weighted clustering coefficient of node 1 in a weighted network in which the closed triplet is made by weak links. The weighted local clustering coefficient c_1^w of node 1 is higher when the closed triplets on node 1 are made of strong links (panel **E**) than in the case where the closed triplets are built by weak links (panel **F**). Analogously, the global clustering coefficient in its global version C^w is higher when the closed triplets are composed of strong links (as in **E**).

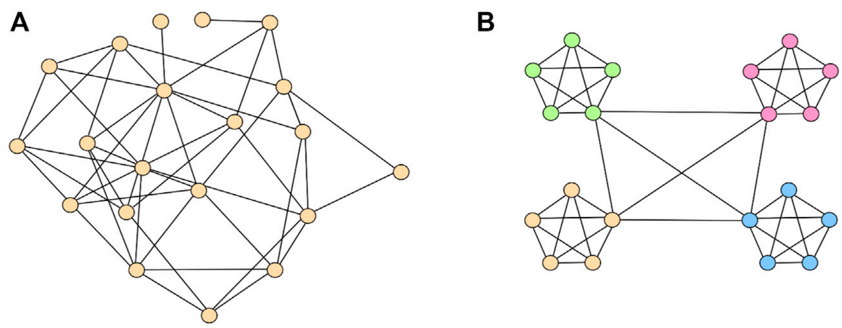

FIGURE 7
Community structure and the network modularity indicator Q [61]. **(A)** Random network with N = 20 nodes and L = 46 links. The random network does not present a community structure, and it returns a modularity indicator value of Q = 0.24. **(B)** A network of N = 20 nodes and L = 46 links with a strong community structure (node color distinguishes nodes belonging to the same community); this network is composed of four clearly separated communities, and Q = 0.62.

where L is the total number of links in the network; a_{ij} is the element i, j of the adjacency matrix, equal to 1 if i and j are connected, and 0 otherwise; k_i and k_j are the degrees of i and j, respectively; c_i and c_j are the modules (or community) of node i and j, respectively; and $\delta(x, y)$ is 1 if $x = y$ and 0 otherwise. The modularity Q represents the fraction of the links that fall within the given community minus the expected fraction if links are drowned at random. Positive Q indicates that the number of links within communities exceeds the

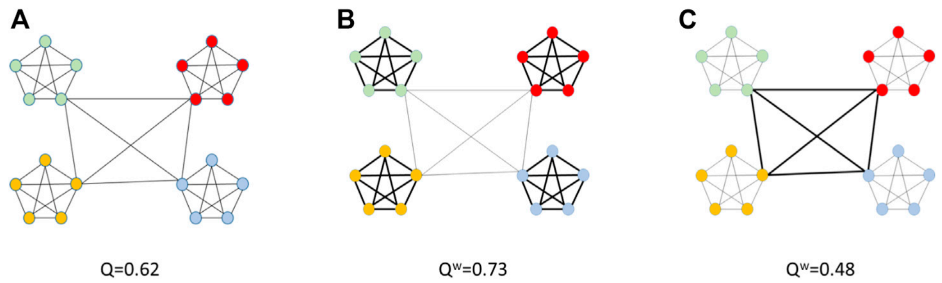

FIGURE 8
Community structure in weighted networks. **(A)** A binary network of N = 20 nodes and L = 46 links with a strong community structure (node color distinguishes nodes belonging to the same community); this network is composed of four clearly separated communities, and Q = 0.68. **(B)** The weighted network counterpart of the binary network in **(A)** in which strong links (higher link weights) are inside the communities with Q^w = 0.73. **(C)** The weighted network counterpart of the binary network in **(A)** in which strong links (higher link weights) are outside the communities, and for this network, Q^w = 0.48. The total link weights are the same for networks in **(B, C)**.

number expected on the basis of chance, the maximum possible value of Q is 1, non-zero values indicate deviations from randomness, and values around 0.3 or more usually indicate good divisions [19, 100]. In Figure 7, we depict two networks with the same number of nodes and links. The first network (Figure 7A) is a random network with no community structure, whereas the second network (Figure 7B) is a network with a strong community structure. The modularity indicator Q is able to quantify the community structure level, and it is higher for the network with a strong community structure in Figure 7B.

The modularity Q has been widely used to evaluate different aspects of the community structure in social networks [61, 90, 101–103].

The same rationale can be used to evaluate community structure in weighted networks. The modularity Q can evaluate weighted networks by replacing the node degree in Eq. 11 with the node strength and using the link weight values instead of the simple binary presence of the links among nodes [19].

The weighted modularity Q^w is thus defined:

$$Q^w = \frac{1}{2L^w} \sum_{i,j} \left(w_{ij} - \frac{s_i s_j}{2L^w} \right) \delta(c_i c_j) \qquad (12)$$

where L^w is the sum of the link weights; w_{ij} is the weight of the link connecting i and j; and s_i, s_j are the strengths of i and j, respectively.

The weighted modularity indicator considers both the binary presence of communities in the network and their total relative link weight. The weighted modularity Q^w represents the fraction of the link weights within the given community minus the expected fraction of the link weights when links are drowned at random. In other terms, the weighted modularity is higher when networks present both a higher number of intracommunity links and a higher weight of these links. Using the link weights in the weighted version of the community structure algorithm and the weighted version of the modularity Q^w may change the detected communities in the network with respect to the binary approach [19].

In Figure 8, we show that accounting link weights' heterogeneity may change the detected community structure of the network. Figure 8A depicts a network with a strong binary community structure, i.e., nodes belonging to four distinct communities. Then, we associate link weights to the network based on the following two criteria: delivering strong links inside (Figure 8B) and outside the network communities (Figure 8C). Finally, we calculate the modularity indicator Q for the three networks. The network with a binary community structure (Figure 8A) presents modularity Q = 0.62, whereas the network with strong links inside the communities (Figure 8B) presents higher modularity Q^w = 0.73. The modularity indicator Q shows how associating strong links within communities enhances the network's community structure. The network in Figure 8B with strong links inside the communities can be viewed as the network with both binary and weighted community structures, i.e., this network has a community structure even stronger than the binary network in Figure 8A. On the contrary, the network with strong links outside the communities (Figure 8C) presents the lowest value of the modularity Q^w = 0.48. This indicates that delivering strong links outside the binary communities decreases the community structure of the network. In other words, the network in Figure 8C presents a strong binary community structure but lacks a coupled weighted community structure.

7 Robustness of weighted social networks

Understanding the causes of the robustness of social systems is of concern to social scientists, who explore the stability of human societies in the face of disrupting forces such as epidemics, criminal activities, social segregation, famine, war, and changes in social and economic order [104]. Social networks are prominent frameworks for analyzing the robustness of social systems [4, 27, 41, 73, 85, 105–107].

"Network robustness" can be defined as the functionality (capacity) of the system to maintain its functions after removing nodes or links [104, 108]. Usually, the decrease in network functionality is evaluated by focusing on the connectivity degradation after node or link removal. The two most common measures of network connectivity are the largest connected component (LCC) and the network efficiency (Eff). LCC accounts for the maximum number of connected nodes in the network, and it is a pure topological measure that neglects the

TABLE 3 List of the network functioning (robustness) indicators.

Indicator	Symbol	Formula	Meaning	Refs
Total strength	S	The total strength is $S = \sum_{i=1}^{N} s_i$, where N is the number of nodes and s_i is the strength of the node i, i.e., the sum of the weights of the links connected to i.	The total strength is the sum of the strength of the network nodes.	[109]
Average node clustering	$\bar{C}$	The average binary node clustering coefficient is $\bar{C} = \frac{1}{N}\sum_{i=1}^{N} c_i$, where c_i is the binary node clustering coefficient of node i *and* N is the number of nodes in the network.	The average node clustering coefficient indicates the cohesiveness of the nodes' communities in the network.	[27]
Weighted efficiency	Eff^w	The weighted network efficiency is $Eff^w = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}^w}$, where N is the number of nodes and d_{ij}^w indicates the weighted distance from node i to node j.	Information spreading capacity through undirected weighted paths	[62]
Total flow	TF	The total flow is $TF = \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}$, where N is the number of nodes and w_{ij} is the weight of the link joining nodes i and j.	The total flow represents the actual or the potential flowing in the network, and it is the sum of link weights.	[64, 108]
Average weighted node distance	$\overline{d^w}$	The average weighted node distance in the network is $\overline{d^w} = \frac{1}{N(N-1)} \sum_{i \neq j \in G} d_{ij}^w$, where N is the number of nodes in the network G, and d_{ij}^w indicates the weighted distance from node i to node j.	The average weighted distance represents the average shortest path to traverse traveling among network nodes.	[29]
Weighted diameter	D^w	The weighted diameter is $D^w = \underbrace{\max}_{i,j \in N, i \neq j} (d_{ij}^w)$, where N is the number of nodes and d_{ij}^w indicates the weighted distance from node i to node j.	The diameter is the longest node distance in the network. It is the shortest distance between the two most distant nodes in the network.	[28, 118]

weighted structure of the networks [4, 109, 110]. The network efficiency evaluates how close the nodes are in the system and is a measure of the capacity to exchange information over the network [60, 63, 64]. Eff is defined as the average inverse distance between all nodes in the network [63]. The network efficiency is well defined for both binary and weighted networks.

The binary network efficiency Eff is:

$$Eff = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \tag{13}$$

where N is the number of nodes in the network G and d_{ij} is the binary node distance between nodes i and j [60, 63, 111]. Network efficiency can be viewed as the inverse of the harmonic mean of the node distances. Using the weighted version of the node distance d_{ij}^w, we can compute the weighted network efficiency Eff^w:

$$Eff^w = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}^w} \tag{14}$$

Other measures of network functioning have been proposed, such as the average node distance [112, 113], network diameter [114], algebraic connectivity [115], and others [116, 117]. In Table 3, we list network functioning measures for weighted networks.

Analyzing network robustness is also essential to determine the most important nodes and links in social networks, i.e., the nodes and links whose removal causes the greatest amount of damage in the social system, which reveal the links/nodes acting as key players for network functioning [110].

In the following, we review the main results and applications in the field of weighted social network robustness by focusing separately on link and node removal.

7.1 Link removal

Link removal (LR), also known as bond percolation [75], link attack [108, 119], or link pruning [120], studies how the robustness of networks decreases by removing links [108].

The "weak link hypothesis" [23, 24] can be viewed as one of the first applications of LR analysis in social networks. The "weak link hypothesis" describes a specific social network structure in which strong links are located within dense communities (or groups) of similar individuals. In contrast, weaker links act as bridges between different communities. As a consequence, social networks would be vulnerable to the removal of weak links since the removal of the weak link (acquaintances), which serve a cohesive function in social systems, may rapidly disconnect the LCC of the network in isolated components [23, 108]. As explained in the previous paragraph, the "weak link hypothesis" was confirmed in several real-world social networks [26, 27, 121]; these social systems would be vulnerable to the removal of weak links.

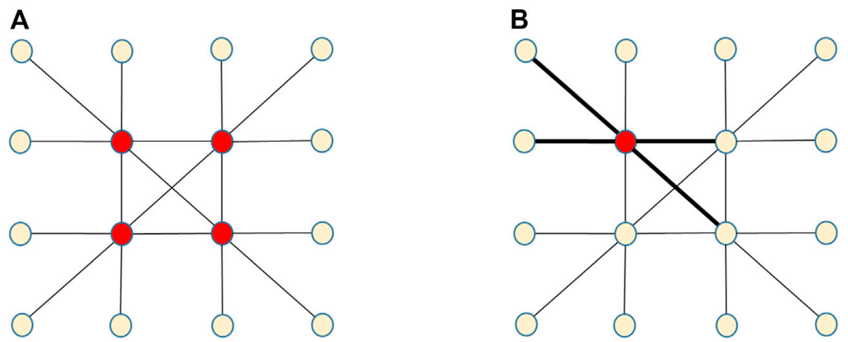

FIGURE 9
Node vaccination priority in social networks. **(A)** A binary network where the nodes to vaccinate (red nodes) are selected considering the node degree; in this network, the four red nodes show the same vaccination priority. **(B)** The weighted network counterpart of the binary network in (**A**); in this network, the nodes to be vaccinated are ranked considering their strength. The node in red has the highest strength and the highest priority for vaccination. In the case of face-to-face networks in which link weights indicate the contact time between individuals, vaccinating the node sharing strong links would eliminate the most significant channels of infection.

However, network science investigations have uncovered a seeming "paradox of weak ties," suggesting that strong links (i.e., links of higher weight) may be more valuable than weak links to serve the cohesiveness and robustness of the system [4]. For example, science co-authorship networks are formed by nodes (scientists) and links weighted by the number of co-authored papers, presenting dense local neighborhoods mainly consisting of weaker links, while strong links connect senior scientists leading different research groups [27, 107]. In contrast to what occurred in other social networks, the *LCC* of the scientific networks shrinks faster when the strongest links are removed first [27, 107]. On the other hand, recent large-scale correlational investigations of the weak link hypothesis, have uncovered that strong links are more valuable than weak links in generating job transmissions [24, 122, 123].

The aforementioned studies investigating the robustness of the social network to LR evaluated the robustness using the *LCC*, thus focusing on the binary-topological structure of the system only. These studies are based on the principle that once a fraction of links are removed from the network, the remaining links are equally capable of supporting the system's functioning, regardless of their weight. Bellingeri et al. [108] showed that, using the weighted network efficiency Eff^w to evaluate the functioning of the WSN after LR, the removal of strong and higher weighted betweenness centrality links in friendship and science co-authorship WSNs triggered a much faster Eff^w decrease than the weak link removal. This result shed light on the problem of the weak link hypothesis, showing that weak links, which support the connectivity of the system (measured by the *LCC*), would not be able to support the social network functioning when deprived of their strong links, such as supporting the information delivery efficiency (Eff^w) in real-world social systems.

Furthermore, Bellingeri et al. [64] found that removing a very small fraction of strong links of higher weight from science co-authorship and friendship social networks triggers an abrupt collapse of Eff^w, while the *LCC*, which only evaluates binary-topological connectedness, remains almost unaffected. These findings suggested that the robustness of social networks might be overestimated when focusing only on their binary-topological connectedness.

LR may properly model various real problems in social systems, such as describing the effect of the interruption of friendship, work, or science collaboration relationships [4, 27]. Moreover, LR can be used to model social distancing and non-pharmaceutical interventions (NPIs) to curb epidemic spreading in complex social networks [124, 125]. In the simplest binary-topological model, LR may describe how reducing social interactions can fragment the social network, thus confining the epidemics to a small part of the network. On the other hand, coupling link removal and dynamical epidemiological models, i.e., SIR or SEIRS models [126], has the advantage of investigating important aspects of the NPIs, such as modeling how the NPIs affect the temporal dynamics of the infection [125, 127, 128]. Nonetheless, most research studies focusing on NPIs and social network epidemics fail to consider the weight of the links in the analyses. Including the weight heterogeneity in models describing the epidemic dynamics is fundamental in real-world social networks, such as face-to-face networks, which are highly promising frameworks to model epidemic spreading in real-world social systems [51]. Face-to-face networks describe the physical interactions among individuals, and the weight of the link accounts for the interaction time. Since longer interactions imply higher epidemic transmission probabilities, to properly model the epidemic dynamics in real-world WSNs, it is fundamental to develop models considering the link weight heterogeneity. For example, a correct description of the epidemic dynamics in face-to-face networks should assign higher infection probabilities to links with higher weights representing long interaction times.

7.2 Node removal

Node removal (NR), also known as node attack [74, 129] and site percolation [130], studies how the removal of nodes in the network affects its robustness [4]. The study of real-world network robustness after node removals has drawn great attention in recent years [73, 110, 131].

In social network analyses, NR may describe a variety of real problems. For example, in friendship networks, NR may be useful to individuate social hubs, i.e., more important individuals for the connectedness of the friendship networks [132]. In science co-authorship networks, it may be a tool to identify more important scholars for the connectedness of the scientific collaboration networks [133]. In criminal networks, NR may describe how arresting criminals affects the structure of interpersonal relationships in crime with the aim of developing policies to halt criminal activities [134]. Finally, in social contact networks within which a disease can spread, NR may simulate the effect of node vaccination [18] and quarantine [128] on the spread of the disease.

The majority of these studies focused on NR with a binary approach, failing to evaluate the difference in link weights. However, adopting a binary approach to NR may produce inaccurate modeling in real-world social networks. On the one hand, predicting the robustness of social networks when subjected to NR may be misleading. For example, removing a few nodes from WSNs may abruptly collapse the weighted network efficiency (Eff^w), while the LCC stays roughly constant [64]. In this case, the widely used LCC may overestimate the robustness of real-world WSNs.

On the other hand, ignoring the weighted structure of the network may trigger an erroneous ranking of the node's importance [60, 82, 109]. As explained in Figure 4, using weighted node centralities instead of the simple binary node centralities may change node rank. Therefore, in real-social networks presenting decoupling with binary and weighted node centralities, the inaccuracy in node ranking may be very high. For example, in real-world WSNs with a low correlation between the degree and the strength of the nodes, ranking nodes according to their degree would give more importance to nodes connected with several weak links than to nodes connected with a few stronger ones. This may induce many inaccurate models of real-world social systems. For example, in epidemic networks, such as face-to-face networks, neglecting the weighted structure may hide the important nodes to be vaccinated (Figure 9).

8 Conclusion

Network analysis is a promising tool for modeling and studying several problems in social systems. In many social networks, links have a naturally associated weight. We have shown that a more complete and proper view of WSNs is provided by considering the heterogeneity of the interactions defining the links between these social systems. On the one hand, this review aims to summarize the results of recent studies in WSNs. On the other hand, this review offers an overview of many open and pressing problems in social network science by outlining how adopting a weighted network perspective may improve the reality of social system descriptions. It is worth noting that artificial intelligence (AI) and data mining methodologies offer new and powerful tools to collect social systems information. For this reason, data are increasingly available, and they will permit the construction of new and well-resolved WSNs. Our review may prompt social science researchers to exercise social system analysis from a weighted network perspective.

Author contributions

MB and DC conceived the manuscript. All the authors wrote the manuscript.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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