Dynamic fishing with endogenous habitat damage

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Abstract

The nature of fishing activities is such that marine habitats can be deteriorated when employing destructive fishing gear. This makes even more complex the determination of sustainable fishing policies and has led some authors to propose dynamic models which take into account this habitat degradation. In this work, we analyze in detail one of these models, an extension of the single-species Gordon-Schaefer model to two state interrelated variables: stock of fish and habitat. The model assumes that stock and carrying capacity are positively linked, and that the fishing activity has a direct and negative impact on the carrying capacity. We extend and characterize Clark's most rapid approach optimal solution to this case.

Keywords: Bio-economic models, Gordon-Schaefer model, Marine habitats, Fishery management, Singular Control

1 Introduction

This paper focuses on the problem of habitat degradation, that is one of the most important causes of the over-exploitation of marine resources: see, e.g., Barbier (2000), Fluharty (2000), Kaiser and de Groot (2000), Botsford et al. (1997). See also Auster and Langton (1999), Barbier and Strand (1998), Barbier et al. (2002) for practical applications. This literature highlights the interdependent relationship between commercial fishing and fish habitats. In one way habitats are deteriorated by fishing activities. This reduction of habitats in turn leads to a reduction in the growth rate of the fish stock.

A number of papers have tried to model some aspects of the two-way relationship between fish stock and the habitat stock. Among those, and at the origin of the present work, the paper Long, Zaccour, and Tidball (2020) takes into account an interrelated fishery-habitat model with myopic agents that act non-cooperativelly and shows that there exists a steady-state solution to the planner's problem that can be supported by different tax schemes. The model assumes that the damage of fishing on the environment is proportional to harvest (effort times stock of fish). Habitat evolves in time following a logistic growth function and affects both the stock of fishing via the carrying capacity and the natural growth. Profit of fishermen are revenue (price times harvesting) minus a cost that is quadratic in the effort. In this model the planner's problem and the definition of the tax schemes are only treated at the steady state owing to the lack of tractable optimal social planner solution. Being concerned about the lack of information on the optimal *dynamic* path, Professor Long suggested, in one of the last discussions, to develop a simpler model in order to obtain more analytical results, while keeping the main ingredients about the relation between stock of fish and habitat.

1.1 Purpose and Contribution

Following this objective, we study the dynamic controlled model specified by:

$$\dot{X} = rX(1 - \frac{X}{K}) - qEX, \qquad \dot{K} = \rho K(1 - \frac{K}{\overline{\overline{K}}}) - \beta EK, \tag{1}$$

where X is the stock, K is the carrying capacity and E is the fishing effort. This is a simplification of the model in Long, Zaccour, and Tidball (2020) by assuming that the natural growth rate is now constant, and a modification since the damage of fishing on the environment is now proportional to effort, not to harvest. In addition, we will remove the quadratic part from the fisherman's revenue to end up with a linear function of effort.

Thus simplified, the model can also be seen as a direct extension of Clark's model (Clark (1990)), to which it reduces by a proper choice of parameters and initial values. It is well known that the solution to Clark's model is a "most rapid approach" path leading to an optimal extraction maintaining the resource at an optimal level. By selecting a model linear in the control variable E, we expect the solution to be similar in nature, with possibly an explicit expression.

We start (Section 2) with an extension of the standard singular control model to two variables: stock and carrying capacity, with general growth functions. We derive first-order optimality equations, identify a singular curve in the state space (Proposition 1), and compute the optimal singular extraction

(Proposition 2). Proposition 4 describes how non-singular trajectories connect to the singular one. In Proposition 3, we consider the particular case where the fishing effort does not impact the carrying capacity, which nevertheless evolves over time (e.g. when $\beta = 0$ in (1)). In that case, the equation of the singular curve is obtained explicitly.

We then specialize the growth functions to the logistic form of model (1) (Section 3). We describe the set of possible steady states (Lemma 1), and provide results on their existence and on their properties (Lemma 2). Explicit formulas for non-singular trajectories are derived in Section 3.2 and the special case $\beta = 0$ is analyzed in Section 3.3.

Finally, we present numerical illustrations in which the pieces of analysis are glued together to provide a global solution to the optimization problem (Section 4). We conclude with a summary and perspectives in Section 5.

1.2 Literature

We review now related models and results of the literature and their relationship with the present paper,

The theory of optimal management of fisheries classically refers to the "Gordon-Schaefer model". In this model, the stock of fish X is endowed with a natural dynamics $\dot{X} = F(X)$ which is modified by the fishing effort E to become $\dot{X} = F(X) - H(X, E)$. In this representation, H is the harvest function, depending on stock and effort and taking into account the "catchability" of fish. The economic part of the model is represented by a profit function $P(X, E)$ and its optimization, either over time, or at the steady-state of the system.

The general purpose for extending the standard model with an explicit consideration of habitat, is to propose the implementation of a policy apt at achieving a given economic and/or ecological objective. Some contributions are concerned with the use of different fishing technologies as in Nichols et al. (2018b). Others consider the deployment of a tax system to combat the tragedy of commons, as in Long et al. (2020). Models involving a spatial component aim in particular at deciding if managing marine reserves is economically beneficial/optimal or not Moeller and Neubert (2013); Nichols et al. (2018a); Kelly et al. (2019), or study the implementation of a quota system as in Holland and Schnier (2006). In contrast, the primary purpose of the present paper is to explore how far one can go in the explicit solution of a model involving stock and habitat as variables. We leave a detailed parametric analysis and potential applications to further studies.

Taking habitat into account in this standard model has been done principally in two ways.[1] The first one is to consider habitat as a parameter that affects the intrinsic growth function F (the bio-physical part), the harvest function, or the profit function. Foley et al. (2012) review this part of the bio-economic literature and discuss how habitat can be embedded in the model by affecting growth rate, carrying capacity or catchability parameters. More

[1] A third way, appearing recently in Poulton et al. (2023), links directly habitat quality and fishing effort.

recent developments in this family of models include the paper by Moeller and Neubert (2013) which takes into account spatial features in the growth rate of the resource. A general conclusion of this branch of the literature is that the effect of fishing on habitat should not be neglected, and this calls for an explicit modeling of this dynamic effect.

Accordingly, a second family of papers endows the habitat, represented by the carrying capacity of the stock, with its own dynamics. This dynamics is negatively affected by the fishing effort. Udumyan (2012) reviews and compares some of these models, which vary in the assumptions on the natural dynamics of stock and habitat, as well as on the effect of fishing on habitat. The model (1) we will analyze is referred to as "model H_2" in this reference. It is characterized by two assumptions concerning the dynamics of the habitat, represented by the carrying capacity, which we discuss now.

One assumption is that the natural dynamics of the habitat is governed by a logistic growth function. This appears to be a natural, first-approach assumption if this habitat is itself a living stock submitted to resource constraints. In addition to Udumyan (2012), the assumption is present in Nichols et al. (2018a,b) (in discrete-time models) and Kelly et al. (2019).

The second assumption is that the reduction rate in carrying capacity due to the habitat damage caused by fishing is assumed to be proportional to fishing effort, and not to harvest as in e.g. Armstrong et al. (2017); Kahui et al. (2016); Vondolia et al. (2020); Long et al. (2020). This assumption is also made in Nichols et al. (2018a,b); Kelly et al. (2019); Poulton et al. (2023). It is justified by the fact that trawling causes the damage to some habitats (see e.g. Sainsbury et al. (1993), cited by Turner et al. (1999)), increasingly with the surface of fishing gear, but independently of the actual quantity of fish caught.

The paper Kelly, Neubert, and Lenhart (2019) makes these two assumptions in a continuous-time model as we do, but also considers the spatial dimension of the problem and its profit function involves a quadratic function of effort. However, it reduces to (1) when the parameters of the spatial component and the profit function are set appropriately.

Although model (1) is already present in the literature, the analysis we make of it is different from previous ones. Indeed, in Udumyan (2012) the analysis mostly concentrates on the steady-state analysis of the Bioeconomic Equilibrium, the Maximum Sustainable Yield and the Maximum Economic Yield. The dynamic optimization problem in infinite horizon is not solved. Likewise, the spatial model of Kelly et al. (2019) leads to optimality conditions which are solved numerically.

By extending Clark's singular model to two state variables, we aim at identifying necessary conditions, and possibly closed-form solutions which can enjoy bioeconomical interpretations, and also serve as benchmark for numerical solutions of more general models.

2 The general model

In this section, we present the general model we study in this paper, and the general results that can be stated. In the following sections, we shall specify functional forms for the dynamics and derive more precise results.

2.1 Dynamics and control

We consider the case of one representative agent managing a fishery. This agent takes into account the effect of fishing on the resource and on the habitat when maximizing his infinite-horizon discounted utility. An increase in fishing effort triggers two different effects in the dynamics. Firstly, an increase in the current period's aggregate fishing effort will reduce the future fish stock and thus lower the next period marginal product of effort. Secondly, this increase in fishing effort will have a negative effect on the habitat, which in turn leads to a lower stock of fish in the future (through the adverse effect of reduced habitat on spawning). Following this idea the dynamic model is:

$$\dot{X} = F(X, K) - EX \tag{2}$$

$$\dot{K} = G(K) - \beta EK \tag{3}$$

where F, G are the growth functions of X (stock of fish) and K (carrying capacity) respectively, β is a positive constant that measures the destructive effects of fishing on the habitat, and the function $E(\cdot)$ is a control variable (E is the fishing effort, EX is the harvesting).

The optimization problem is

$$\max_{E(\cdot)} \int_0^\infty e^{-\delta t}(p - c(X))\ EX\ \mathrm{d}t \tag{4}$$

where $E(\cdot)$ belongs to the class of measurable functions such that $0 \leq E(t) \leq \overline{E}$ for all $t \geq 0$. The specification of the profit per unit of harvest is $p - c(X)$. The parameter p is a fixed price and $c(X)$ is a unit cost decreasing in X. For compactness of notation we shall write $P(X) = (p - c(X))X$: then $P(X)$ is the net marginal profit per unit of effort. Finally, δ is the discount rate.

The model thus specified contains as particular case the case where $\beta = 0$. This is the case where fishing does not damage the habitat. It occurs for instance when fishermen use a respectful technology (as in Nichols et al. (2018b)) or inside a reserve (as in Nichols et al. (2018a)). In that situation, the carrying capacity $K(t)$ evolves independently of the control $E(t)$. For instance, it may be recovering from a degraded state after some change of technology or some moratorium on fishing. It turns out that more analytical results can be obtained in this case. Of course, the ultimate objective of the paper is to understand the case where damage on the habitat does occur. Fortunately, the analytical solution of $\beta = 0$ helps reveal the general structure of the solution when $\beta > 0$, since these solutions are qualitatively similar. It may also be useful for testing the numerical simulations.

The model of Clark (1990) is itself a particular case of the case $\beta = 0$, when $K(t) = \overline{K}$ is constant: either because $G = 0$ (no dynamics at all) or because $G(K(0)) = 0$ (for instance in the logistic growth model (1) if $K(0) = \overline{K}$). Nevertheless the case $\beta = 0$ cannot be reduced to Clark's analysis since when the carrying capacity evolves, this affects the growth rate of fish. If the initial carrying capacity is low (less than its steady state $\overline{K}$) this must induce the fisherman to harvest less than in the case with constant $\overline{K}$. The contrary occurs when the initial carrying capacity is high.

2.2 First-order conditions

We derive here the first-order conditions with general growth functions. We shall study specifically the case where the growth is logistic in Section 3.

The Hamiltonian of the problem is:

$$\mathcal{H} \;=\; P(X)E + \lambda_X\left(F(X,K) - EX\right) + \lambda_K\left(G(K) - \beta EK\right) + \gamma_E E + \mu_E(\overline{E} - E)\;,$$

where λ_X and λ_K are the adjoint variables of X and K respectively, and γ_E, μ_E the Lagrange multipliers for the constraints on effort E. From standard theorems (see *e.g.* Seierstad and Sydsæter (1987), Chapter 6), the adjoint variables can be assumed to be continuous functions of time. Then the necessary first-order conditions are:

$$\frac{\partial \mathcal{H}}{\partial E} = 0 = P(X) - X\lambda_X - \beta K\lambda_K + \gamma_E - \mu_E \tag{5}$$

$$\dot{\lambda}_X = \delta\lambda_X - \frac{\partial \mathcal{H}}{\partial X} = \delta\lambda_X - EP'(X) - \lambda_X\left(F_X(X,K) - E\right) \tag{6}$$

$$\dot{\lambda}_K = \delta\lambda_K - \frac{\partial \mathcal{H}}{\partial K} = \delta\lambda_K - \lambda_X F_K(X,K) - \lambda_K\left(G'(K) - \beta E\right)\;, \tag{7}$$

at every time instant where $E(\cdot)$ is continuous. Here and in the following, F_X and F_K denote the partial derivatives of function F with respect to variables X and K. Complementarity conditions include $\gamma_E \geq 0$ and $\mu_E \geq 0$, and if $\gamma_E > 0$ then $E = 0$ and if $\mu_E > 0$, then $E = \overline{E}$. Considering (5), we are led to introduce:

$$\gamma_E - \mu_E \;=\; \chi \;:=\; X\lambda_X + \beta K\lambda_K - P(X)\;. \tag{8}$$

From this, a rule can be sketched for the optimal control $E(\cdot)$, as follows.

Proposition 1 (Optimal control) *On any optimal trajectory, we have necessarily*

$$E(t) \;=\; \begin{cases} 0 & \textit{if } \chi(t) > 0 \\ \overline{E} & \textit{if } \chi(t) < 0 \\ \textit{some } E \in [0,\overline{E}] & \textit{such that } \chi(t) = 0. \end{cases}$$

This structural result is *independent from the growth functions* and the specific functional form taken by $P(X)$. As a consequence, optimal trajectories will necessarily consist of segments where either of the three cases occur. In the following Section 2.3 we focus first on the third case, the "singular" one. In Section 2.4, we consider the pieces of optimal trajectories satisfying the other conditions.

2.3 The singular curve $\chi \equiv 0$

Proposition 1 opens the possibility that optimal trajectories may be such that $\chi(t) \equiv 0$ at least for some interval of time. We call such trajectories *singular trajectories*, and we call the corresponding curve in the (K, X) plane the *singular curve*.

In this paragraph, we investigate the properties of this singular curve $\chi = 0$, which is central to the solution of the optimization problem. For such a control problem, it is expected that the optimal control and the corresponding adjoint variables can be expressed as a feedback of the state. The following results exhibit explicit formulas for this. In Proposition 2, the result is general. In Proposition 3, the additional assumption that $\beta = 0$ is made, and more precise results are obtained.

The property at the origin of these results is that the functions $\chi(t)$ and $\dot{\chi}(t)$ do not depend formally on the control, as shown in Equation (37) of Appendix A, where both propositions are proved.

Proposition 2 *If an optimal trajectory is constrained to stay on the curve defined by $\chi = 0$, then: there exist two functions $\varphi(K, X)$ and $\psi(K, X)$, explicitly constructed, such that the optimal control is $E(t) = \varphi(K(t), X(t))/\psi(K(t), X(t))$.*

Although Proposition 2 provides explicitly the value of the singular feedback control, it provides neither the equation of the singular curve, nor clear interpretations. It is nevertheless useful for testing numerical algorithms for this general case and it introduces elements needed for describing how optimal trajectories connect to the singular curve in Proposition 4. When $\beta = 0$ the following Proposition 3 does provide explicitly the singular curves and interpretations.

Proposition 3 *Assume $\beta = 0$. If an optimal trajectory is constrained to stay on the curve defined by $\chi = 0$, then:*

a) the following relations hold

$$\lambda_X = \frac{P}{X} = p - c(X) \tag{9}$$

$$\delta = F_X + F\,\frac{P'X - P}{PX}. \tag{10}$$

b) Let

$$\theta(X,K) = \delta P - FP' + P\left(\frac{F}{X} - F_X\right).$$

The optimal control on this piece of trajectory is given by:

$$E = \frac{F}{X} + \frac{G}{X}\frac{\theta_K}{\theta_X}. \tag{11}$$

Proposition 3 can be interpreted referring to Clark's Most Rapid Approach (Clark, 1990, Section 2.6). Equation (10) is the equation of the singular curve in the (K,X) plane. Clark's singular solution is solution of Eq. (2.16) *op. cit.*: $\delta = F' - Fc'/(p-c)$. It is easily checked that for each K, equation (10) solved for X gives exactly this singular solution, so that it is optimal to maintain X at this solution as $K(t)$ evolves. To achieve this, the optimal harvest must be $EX = F + G\theta_K/\theta_X$ according to (11). We find the optimal harvest $EX = F$ of Clark when K is constant (or equivalently $G = 0$).

2.4 Connecting to the singular curve

In view of Proposition 1, an optimal trajectory will necessarily consist of continuous connections (for the state variable as well as for the adjoint variables) of pieces of trajectories where one of three conditions prevails: $E = 0$, $E = \overline{E}$ or E maintains the state on the singular curve. The question is to determine at which states (K,X) these changes occur. The next proposition will give us a piece of information about these changes. It will be used in the construction of the optimal solution in the numerical simulations (see section 4).

We consider in this section the case where an optimal piece of trajectory, different from the singular curve, connects to it at some point (K_0,X_0). Then we have the following property. It refers to functions φ and ψ introduced in Proposition 2.

Proposition 4 *Assume an optimal trajectory connects to the singular curve at point* (K_0,X_0) *at time* t_0*. Then:*

a) $\chi(t_0) = \dot\chi(t_0) = 0$*;*
b) if the optimal control is $E = 0$ *then necessarily* $\varphi(K_0,X_0) \geq 0$*;*
c) if the optimal control is $E = \overline{E}$ *then necessarily* $\varphi(K_0,X_0) \leq \overline{E}\psi(K_0,X_0)$*.*

The proof of *a)* is another consequence of the fact that $\dot\chi$ does not depend on the control E. The other statements follow from Equation (40) in Appendix A.1, which writes as: $\ddot\chi = \delta\dot\chi + \varphi(K,X) - E\psi(K,X)$, and from the continuity of all variables at t_0.

3 Specific functional forms

In this section, the analysis is pursued with logistic growth functions:

$$F(X,K) = rX(1 - X/K) \qquad\qquad G(K) = \rho K(1 - K/\overline{K})$$

where $\overline{K}$, r and ρ are positive constants. Accordingly, the dynamics of the system (2)–(3) now write as in (1):

$$\dot{X} = rX(1 - \frac{X}{K}) - EX \tag{12}$$

$$\dot{K} = \rho K(1 - \frac{K}{\overline{K}}) - \beta EK. \tag{13}$$

As in Clark's example, we consider $c(X) = c/X$: our profit function reads

$$(pX - c)E. \tag{14}$$

Remark 1 The parameter β measures the degradation of the carrying capacity K due to the harvesting effort, relatively to the degradation of the stock X. Depending on the value of β and the effort E, there can be situations where the stock, initially smaller than the current carrying capacity, can become larger than it. Indeed, consider the case where $X = K$. The harvesting effort drives the system (K, X) in the direction $(\rho K(1-K/\overline{K}) - \beta E, -E)$. If β is large enough, the first component can be made smaller than $-E$, so that the dynamics points outside the region $\{X \leq K\}$. Actually, from the dynamics of the ratio X/K, it can be seen that the impact of some effort E on the ratio X/K is negative if $\beta < 1$ (any effort makes the growth rate of ratio smaller), whereas this impact is *positive* if $\beta > 1$.
Stocks and capacities X, K, $\overline{K}$ are in units of mass: kg. Rates r, ρ, δ and E are in s^{-1}. β is dimensionless. $P(\cdot)$ is expressed in €, adjoint variables λ_X and λ_K in €kg^{-1}, Lagrange multipliers γ_E and μ_E in €, and the Hamiltonian $\mathcal{H}$ in €s^{-1}.

This specification does not allow in itself to determine more precisely the singular curve than in the general result of Proposition 2. The specification of general formulas of Section 2 is presented in details in Appendix B.1. In particular, the explicit form of functions φ and ψ of Proposition 2 can be deduced from Equation (49). However, some explicit formulas do become possible: for steady states and for trajectories. Also, in the particular case of $\beta = 0$, the singular curve has an explicit expression, which becomes linear when $c = 0$. We develop these formulas in this section.

3.1 Stationary states

Stationary states are such that the values of the state and adjoint variables does not depend on time. Such states are the only possible limits of trajectories (including adjoint variables in the trajectory). Accordingly, we denote such states as K^∞, X^∞, λ_K^∞ and λ_X^∞.

Assuming stationarity, the control E is some constant E^∞ and we have the four algebraic equations deduced from (12), (13), (6) and (7):

$$0 = rX(1 - \frac{X}{K}) - E^\infty X \tag{15}$$

$$0 = \rho K(1 - \frac{K}{\overline{K}}) - \beta E^\infty K \tag{16}$$

$$0 = \delta\lambda_X - pE^\infty - \lambda_X \left(r - \frac{2rX}{K} - E^\infty \right) \tag{17}$$

$$0 = \delta\lambda_K - r\lambda_X \frac{X^2}{K^2} - \lambda_K \left(\rho - \frac{2\rho K}{\overline{K}} - \beta E^\infty \right) \ . \tag{18}$$

The set of stationary states is then described in the following result, the proof of which is presented in Appendix B.2.

Lemma 1 *The set of stationary states satisfies*

$$K^\infty = \overline{K} \left(1 - \frac{\beta E^\infty}{\rho} \right) \tag{19}$$

and: either $X^\infty = 0$, $\lambda_K^\infty = 0$ *and*

$$\lambda_X^\infty = \frac{pE^\infty}{\delta - r + E^\infty}, \tag{20}$$

or:

$$X^\infty = K^\infty \left(1 - \frac{E^\infty}{r} \right) \ = \overline{K} \left(1 - \frac{\beta E^\infty}{\rho} \right) \left(1 - \frac{E^\infty}{r} \right) \tag{21}$$

$$\lambda_X^\infty = \frac{pE^\infty}{\delta + r - E^\infty} \tag{22}$$

$$\lambda_K^\infty = \lambda_X^\infty \left(1 - \frac{E^\infty}{r} \right)^2 \frac{r}{\delta + \rho - \beta E^\infty} \ , \tag{23}$$

for some effort E^∞. *It is necessary that* $E^\infty \leq \rho/\beta$ *(and* $E^\infty \leq r$ *in the case* $X^\infty \neq 0$*) and* $0 \leq E^\infty \leq \overline{E}$ *for such a stationary solution to exist.*

Remark 2 From Equations (19), (21) and (22), we can conclude the following influence of steady-state effort on steady-state stock and habitat. If, all other parameters being constant, E^∞ is decreased, then both K^∞ and X^∞/K^∞ increase while λ_X^∞ decreases. Reducing effort is, of course, beneficial to the habitat, but also to the *occupancy* of the habitat.

Remark 3 The choice in Lemma 1 is to express quantities as a function of E^∞ alone, but other identities are possible. For instance, eliminating E^∞ between (15) and (17), then between (16) and (18), we obtain the following ones:

$$\lambda_X^\infty = \frac{pr(1-X^\infty/K^\infty)}{\delta + rX^\infty/K^\infty} \qquad \lambda_K^\infty = \lambda_X^\infty \frac{(X^\infty/K^\infty)^2}{\delta+\rho K/\overline{K}},$$

connected with Lemma 1 by the equation $E^\infty = r(1 - X^\infty/K^\infty)$.

Lemma 1 identifies possible stationary states of the dynamical system, without considerations for optimality. If an *optimal* trajectory converges to a steady state (or stays there), then Proposition 1 applies to it and the control E^∞ must be consistent with the sign of the function χ, defined in (8). Assume that the system is stationary and $X^\infty \neq 0$: then equations (19)–(23) apply. Replacing in the equation $0 = X^\infty\lambda_X^\infty + \beta K^\infty \lambda_K^\infty - P(X^\infty)$, we obtain that E^∞ is a root of a polynomial of degree 4 in E, $Pol(E)$.

$$\begin{aligned} Pol(E) &= -c(E\beta - \delta - \rho)r(E-\delta-r)\rho \\ &\quad + p\overline{K}(E-r)(\beta E-\rho)(3E^2\beta - (\beta\delta + 2\beta r + 2\delta + 2\rho)E + (\delta+r)(\delta+\rho)). \end{aligned} \tag{24}$$

Some properties of these roots are described in the next result, the proof of which is presented in Appendix C.

Lemma 2 *Assume $\rho > 0$ and $\beta > 0$. Optimal stationary solutions have the following properties:*

a) If $\overline{K}p - c > 0$, $c > 0$, then:

- *there exists a unique solution E^∞ to $Pol(E) = 0$, in the interval $(0, \min(r, \rho/\beta))$;*
- *when considered as a function of c, $E^\infty(c)$ is decreasing: $\partial E^\infty/\partial c < 0$.*

b) When $c = 0$ denote with $E_0^\infty \equiv E^\infty|_{c=0}$. Let

$$\begin{aligned} &E_1 \\ &= \frac{\beta\delta + 2\beta r + 2\delta + 2\rho - \sqrt{(\delta+2r)^2\beta^2 - 4(\delta+\rho)(2\delta+r)\beta + 4(\delta+\rho)^2}}{6\beta}. \end{aligned} \tag{25}$$

Then:

- *If $\delta < r \leq \rho/\beta$, then $E_0^\infty = E_1$;*
- *If $\delta < r$ and $r > \rho/\beta$ then $E_0^\infty = \min(\rho/\beta, E_1)$.*

The results about the variations of E^∞ with respect to c can be coupled with Remark 2 to conclude that if c increases then the stationary ratios X^∞/K^∞ and $K^\infty/\overline{K}$ increase as well.

3.2 Equations of trajectories

From Proposition 1, we know that some parts of optimal trajectories will have a constant control, $E = 0$ or $E = \overline{E}$. It turns out that the system of differential equations (12)–(13) can be solved in closed form when E is constant. These solutions are useful when developing numerical examples.

Assuming that $(K(0), X(0)) = (K_0, X_0)$, the functions obtained are:

$$K(t) = \frac{K_0\,\overline{K}(\rho - \beta E)\,\mathrm{e}^{(\rho-\beta E)t}}{(\rho - \beta\,E)\overline{K} + \rho K_0\,(\mathrm{e}^{(\rho-\beta E)t} - 1)} \tag{26}$$

$$\begin{aligned} X(t) = \Bigg[& \frac{e^{(E-r)t}}{X(0)} \\ & + r\left(\frac{1}{K_0} - \frac{\rho}{\rho - \beta E}\frac{1}{\overline{K}}\right)\frac{e^{(\beta E-\rho)t} - e^{(E-r)t}}{(\beta-1)E + r - \rho} + \frac{r\rho}{(\rho-\beta E)\overline{K}}\frac{1 - e^{(E-r)t}}{r - E}\Bigg]^{-1}. \end{aligned} \tag{27}$$

In addition, in the specific case where $E = 0$, it is also possible to obtain an explicit formula for the adjoint variable λ_X. Indeed, it can be shown that the function $\Theta_X = X\lambda_X$ satisfies the relation $\dot{\Theta}_X/\Theta_X = \delta + r - \dot{X}/X$, which leads to the expression:

$$\lambda_X(t) = \lambda_X(0)\frac{X_0^2}{X(t)^2}\;e^{(\delta+r)t}\;. \tag{28}$$

The derivation of these formulas and more details are provided in Appendix B.3.

3.3 The singular curve in the case $\beta = 0$

When $\beta = 0$, the singular curve takes a polynomial form. Indeed, from (10), and the specification $P(X) = pX - c$,

$$\begin{aligned} & 0 = (pX - c)\left(\delta + r(1 - \frac{X}{K}) - r(1 - \frac{2X}{K})\right) + prX(1 - \frac{X}{K}) \\ \iff \quad & 0 = \frac{2pr}{K}X^2 + ((\delta - r)p - \frac{cr}{K})X - \delta c. \end{aligned} \tag{29}$$

This equation gives the following solutions, as a function of K or of X:

$$X^*(K) = \frac{cr - pK(\delta - r) + \sqrt{(p(\delta - r)K - cr)^2 + 8prc\delta K}}{4pr} \tag{30}$$

$$K^*(X) = \frac{rX(2pX - c)}{\delta c - (\delta - r)pX}. \tag{31}$$

Observe that the point (K^∞, X^∞), given by (19) and (21), where E^∞ is solution of (24), is located on this curve, which is expected by construction. As observed previously, this point is a steady state only if the condition

$0 \le E^\infty \le \overline{E}$ is satisfied. Also, observe that $K^*(c/p) = c/p$, which means that the point $(c/p, c/p)$ is located on this curve as well. In the particular case where $c = 0$, the singular curve is a straight line: from (29) we obtain: $K^*(X) = 2rX/(r - \delta)$. This curve is meaningful if $\delta < r$, an economic condition for a sustainable exploitation of the resource already appearing in the description of steady states in Lemma 2.

While these explicit results provide valuable information for constructing optimal controls, they do not provide all the details. We will complement this theoretical analysis with numerical examples in the next section.

4 Numerical illustrations

In this section, we present two numerical examples based on the analysis of Section 3, that is, with the logistic specification for the dynamics and the linear specification for the profit function $P(\cdot)$.

In a first example, developed in Section 4.1 we set β to 0. This allows us to use the strong result of Proposition 3 and its application in Section 3.3 for determining the equation of the singular curve. The knowledge of this curve is sufficient to determine the pieces of optimal trajectories that use a constant control. This way, we obtain a global view of optimal trajectories and optimal control.

In a second example we set β to 1/3. Then, the equation of the singular curve is not known, but we check with a numerical approximation that the global configuration of optimal trajectories is similar.

4.1 Example 1 with $\beta = 0$

Consider the following parameters: in addition to $\beta = 0$, let:

$$\overline{K} = 1, \quad r = 8, \quad \delta = 0.05, \quad \rho = 4, \quad p = 1, \quad c = \frac{205}{808} \simeq 0.2537. \tag{32}$$

The value of $\overline{E}$ will be specified later. The equation of the singular curve (29) is:

$$258560X^2 - 128472XK - 205K - 32800X = 0\ .$$

The (potential) stationary optimal point (with non-zero X) is located on this curve, and from (19), (21) and (24) it follows that:

$$E^\infty = 3, \quad K^\infty = 1, \quad X^\infty = \frac{5}{8}.$$

As noted in Lemma 1, these values actually correspond to a steady-state optimal solution if $E^\infty \le \overline{E}$. The optimal feedback control on the singular curve, given by (11), has the form:

$$E^*(K, X) = 8 - \frac{8X}{K} + \frac{4K\,(1-K)\,(128472X + 205)}{X\,(128472K - 517120X + 32800)}. \tag{33}$$

It turns out that this value is positive, for (K, X) on the singular curve, only for $K \geq K_s \simeq 0.399316$ (corresponding to $X \geq X_s \simeq 0.326237$). The point (K_s, X_s) located on the singular curve is called the "zero-control singular point" (see the forthcoming Conjecture 5).

For a better understanding of the optimal trajectories, it is useful to look more closely to the dynamics of the system when $E = 0$. As computed in (27), the trajectory of $X(t)$ is given by:

$$X(t) \;=\; \left(\frac{e^{-rt}}{X(0)} \;+\; r\left(\frac{1}{K(0)} - \frac{1}{\overline{K}}\right)\frac{e^{-\rho t} - e^{-rt}}{r-\rho} + \frac{1-e^{-rt}}{\overline{K}}\right)^{-1}.$$

Depending on the values of $K(0) = K_0$ and $X(0) = X_0$, the term within parentheses may vanish for some value $t_\ell < 0$. In that case, $\lim_{t\to t_\ell^+} X(t) = +\infty$. The necessary and sufficient condition for this is, from Lemma 3 (in Appendix B.3):

$$X_0 \;>\; \zeta(K_0) := \frac{K_0\overline{K}}{2\overline{K} - K_0}. \tag{34}$$

We call the curve of equation $X = \zeta(K)$ the "divergence curve", to highlight the fact that trajectories with $E = 0$ located above it tend to infinity (when considered backwards in time) for some finite value of $t < 0$ with some finite value of K. Trajectories located below it are bounded. They tend to $X = 0$ when $K \to 0^+$, that is, when $t \to -\infty$. The divergence curve is itself a trajectory with $E = 0$.

The location of (K_0, X_0) with respect to the divergence curve is important when considering the function $\chi(t) = \lambda_X(t)X(t) - P(X(t))$. Indeed, according to (28), along a trajectory with $E = 0$,

$$\chi(t) \;=\; \lambda_X(0)\frac{X(0)^2}{X(t)}e^{(\delta+r)t} - pX(t) + c\,.$$

Therefore, if t_0 is such that when $t \to t_0^+$, $X(t) \to +\infty$, then $\chi(t) \to -\infty$. So, if we consider a trajectory with $E = 0$ which joins the singular curve at time $t = 0$ at some location (K_0, X_0) located above the divergence curve, we have the following properties: a) $\chi(t) > 0$ in a neighborhood of $t = 0^-$ (which can be checked using Proposition 4 b)); b) there exists some t_0 such that $\lim_{t\to t_0^+} \chi(t) = -\infty$. This implies the existence of $t_1 \in (t_0, 0)$ with $\chi(t_1) = 0$ and with $\chi(t) > 0$ for $t \in (t_1, 0)$. The corresponding trajectory is then consistent with Proposition 1. The locus of the points $(K(t_1), X(t_1))$ forms a curve which is called the "switching curve".

On the other hand, if (K_0, X_0) is located below the divergence curve, the function $\chi(t)$ remains positive for all $t < 0$, as can be observed (but not proved) from numerical evidence. Trajectories with $E = 0$ are therefore consistent with Proposition 1 until they join with the singular curve, whatever their starting point.

Consider now trajectories with $E = \overline{E}$. When they are supposed to join, at time $t = 0$, some position (K_0, X_0) where $\chi(0) = 0$, then numerical evidence is that $\chi(t) < 0$ for all $t < 0$. These trajectories are then consistent with Proposition 1 until they joint the point (K_0, X_0). Two situations may occur. In the first one, the location (K_0, X_0) is on the singular curve (which is indeed characterized by $\chi = 0$). A trajectory which follows the singular curve from then on is consistent with Proposition 1. In the second one, the location (K_0, X_0) is some point defined as $(K(t_1), X(t_1))$ above: in that case $\chi(0) = 0$ also, but an optimal trajectory cannot be such that $\chi(t) = 0$ for all $t > 0$. Then it is optimal to switch the control to $E = 0$ and continue with the trajectory constructed above. This motivates the name "switching curve" chosen for such locations.

Summing up, we conjecture the following result, numerically validated in the present example:

Conjecture 5 *Assume $\overline{E} \geq E^\infty$. Then, there exists a "switching curve" $X = \sigma(K)$, which connects to the singular curve at the "zero-control singular point", thus forming a "separating curve", and which is such that the optimal feedback control is:*

$$E^*(K, X) = \begin{cases} 0 & \textit{if } (K, X) \textit{ is below (or on) the separating curve} \\ \overline{E} & \textit{if } (K, X) \textit{ is above the separating curve} \\ \textit{the control } (33) & \textit{if } (K, X) \textit{ is on the singular curve.} \end{cases}$$

The situation is represented in Figure 1, obtained with the value $\overline{E} = 5$. The figure shows the singular curve (violet), limited by the "zero-control singular point" in its left, and the steady state point on the line $K = \overline{K}$. The figure further shows the divergence curve (blue), which joins the singular curve at the "divergence point". Its part to the right of the divergence point is not relevant and is not displayed. A third curve is the switching curve: it was numerically computed using differential equations, continuity of the adjoint variables and finding the time instant at which $\chi(t) = 0$. Optimal trajectories are represented in green if the optimal control is $E = 0$, or in red if $E = \overline{E}$.

Figure 1 also displays the line $X = c/p$. All points below this line represent economically unprofitable stocks, and indeed, the optimal control is always to do $E = 0$ for such stocks.

In a second experiment, we set $\overline{E} = 5/2$, so that now $\overline{E} < E^\infty$. It is not possible anymore for an optimal trajectory to follow the singular curve forever. The situation is displayed in Figure 2. The global diagram is on the left, a zoom appears on the right. The elements of these diagrams are as follows.

The singular curve is, of course, the same as in Figure 1. However, there is now a point at which $E_s(t) = \overline{E}$. It is called "end point" and has approximate coordinates $(K_e, X_e) \simeq (0.882533, 0.566600)$. But it turns out that it is *not* optimal for the optimal trajectory to remain on the singular curve until the end point. There is actually another point with approximate coordinates $(K_\ell, X_\ell) \simeq (0.800911, 0.526016)$ at which it is optimal to leave the singular curve and switch the control to $\overline{E}$. For this reason, we call this point the "leave

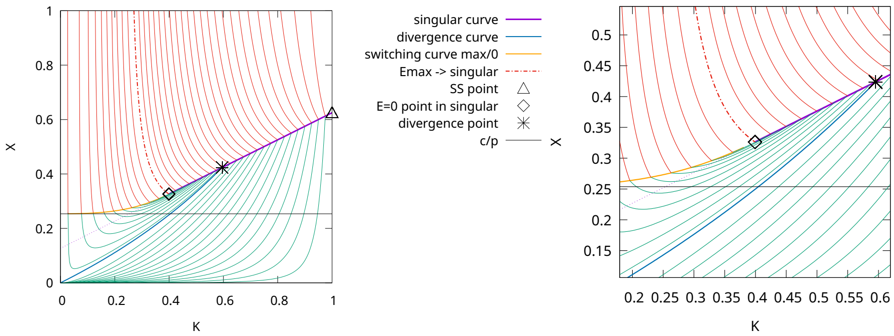

Fig. 1 Optimal trajectories in the example with $\beta = 0$, with $\overline{E} > E^\infty$

point". All optimal trajectories ultimately use the control $\overline{E}$. For this reason, they all converge to the steady state of the dynamical system with $E = \overline{E}$, labelled as "actual SS point" in the diagram.

As in Figure 1, there is a "divergence curve" which attaches to the singular curve at a "divergence point". The switching curve max/0 is also identical to Figure 1, despite the fact that $\overline{E}$ is different. Indeed, this curve is computed using only trajectories with $E = 0$, and the value of $\overline{E}$ is nowhere involved.

The new element is a switching curve which appears for large values of K: since joining and following the singular curve is not possible anymore, optimal trajectories switch directly from $E = 0$ to $E = \overline{E}$. This new switching curve is labelled as "switching 0/max" in Figure 2. It connects with the singular curve and joins the line $K = \overline{K}$ at some stock level $X_s \leq X^\infty$.

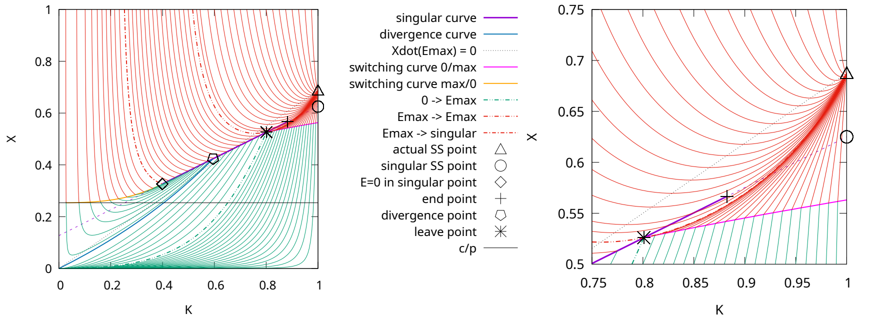

Fig. 2 Optimal trajectories in the example with $\beta = 0$, with $\overline{E} < E^\infty$

The general behavior of the optimal trajectories can now be summarized, taking as reference the separating curve ς composed of: a) the switching curve max/0, until the starting point; b) the singular curve between the starting point and the leave point; c) the switching curve 0/max:

- For initial states located under the curve ς, use control $E = 0$ until:

a) either the trajectory encounters the singular curve between the starting and the leave points: then use the singular control until the leave point, then the control $\overline{E}$;
b) or the trajectory encounters the switching 0/max curve: then switch directly to $\overline{E}$;

The groups of trajectories corresponding the the different situations are separated in Figure 2 by the particular curve labelled as "0/Emax".

- For initial states located over the curve ς, use control $E = \overline{E}$ until:

a) either the trajectory encounters the switching max/0 curve: then switch to $E = 0$ and continue as indicated above;
b) or the trajectory encounters the singular curve between the starting and the leave points: then follow it as above;
c) else continue applying $E = \overline{E}$ forever.

The groups of trajectories corresponding to the different situations are separated in Figure 2 by the particular curves labelled as "Emax→singular" and "Emax→Emax".

Connection to Clark's optimal singular solution.

As already noted, the model of Clark arises when $\beta = 0$ and $K = \overline{K}$. The solution constructed in these numerical examples indeed corresponds to Clark's solution. In the first situation where $\overline{E}$ is large enough, the optimal trajectory in Figure 1 starting with $K_0 = \overline{K}$ and $X_0 < X^\infty$ obviously consists in applying $E = 0$ until $X(t) = X^\infty$, then staying there. Similarly if $X_0 > X^\infty$, applying $E = \overline{E}$. In the situation of Figure 2, the optimal control for small X_0 is to apply $E = 0$ until some value X_s at which the optimal control switches to $E = \overline{E}$. This situation is described in Clark (1990), Section 4.1.

In both cases, the practical conclusion is that the optimal control taking into account the state and the dynamics of the carrying capacity K, is substantially different from the uninformed control which would consider that $K = \overline{K}$ and apply the usual most rapid approach.

4.2 Example 2 with $\beta \neq 0$

In this second experiment, we take the same parameters as in Example 1 (see (32)), except that we set β to 1/3. For these values, the values of steady-state variables are:

$$E^\infty \simeq 2.1494, \quad K^\infty \simeq 0.8209, \quad X^\infty \simeq 0.6003.$$

We first run the experiment with $\overline{E} = 3.5$ so that $\overline{E} > E^\infty$. Applying the Value Iteration method to a discretized version of the optimization problem,[2]

[2]The Value Iteration algorithm was applied to a 256×256 discretization of the state space with time increment $h = 10^{-4}$ and iterations until the L1 distance between two successive functions was less than $\varepsilon = 10^{-6}$. Running time was of the order of minutes on a 4-core Intel i7-6820HQ CPU@2.70GHz laptop.

we compute numerically the value function and the optimal control. The latter is displayed in Figure 3, left. We then set $\overline{E} = 1.8$ so that $\overline{E} < E^{\infty}$ and the optimal trajectories cannot finish at the steady state. The result is displayed in Figure 3, right.

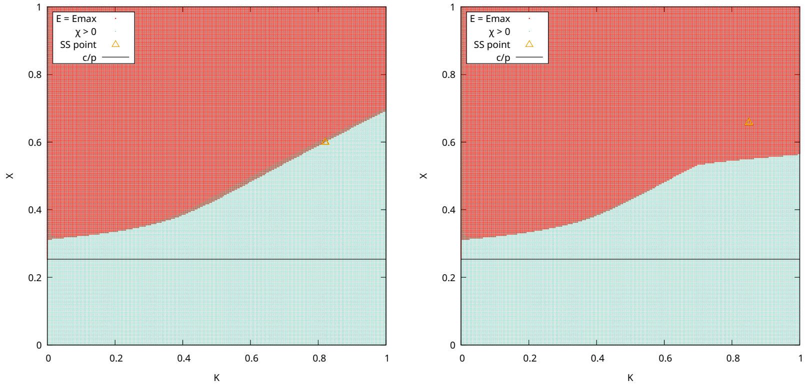

Fig. 3 Optimal control for Example 2: $\overline{E} = 3.5$ (left), $\overline{E} = 1.8$ (right)

More precisely, the figure displays, in red, the zone of points (K, X) for which the value iteration algorithm concluded that $E = \overline{E}$ is optimal. For all other points, $E = 0$ was the best control found. The points represented in green are those where the *approximation* of χ was found to be positive. This approximation was built using the (approximate) value function V obtained at grid points, approximating $\lambda_X = \partial V/\partial X$ and $\lambda_K = \partial V/\partial K$ with the symmetric difference quotient. Other elements of the figure are the steady state and the line where $X = c/p$.

A first observation is that the red and green regions slightly overlap in a zone that likely includes the singular curve. The overlap is not extended, which validates the accuracy of numerical methods. Indeed, around the singular curve we have $\chi = \dot{\chi} = 0$ on optimal trajectories, so that deciding numerically the sign of χ is challenging.

A second observation is that the shape of the separating curve (switching curve and singular curve, see Conjecture 5) is globally the same as in Example 1. It can be checked numerically that, when all parameters but β are the same (including $\overline{E}$), this curve is located above the one for $\beta = 0$. Practically speaking, this means that for a given value of the carrying capacity, the fishing threshold for the stock X is larger when the impact of fishing on the habitat is larger.

5 Discussion and conclusion

In this work we have made an extension of the singular control model (Clark's model, most rapid approach) when habitat is taken into account. In general we have obtained some analytical optimal solutions which, completed with simulations, give the optimal solution of the problem. As a special case, Clark's solution follows when some parameters are zero ($\rho = \beta = 0$). The case $\beta = 0$, allows us to compute the singular curve analytically and the case $\beta > 0$, is completed numerically via Value Iteration that converges quickly. We have found regions in the plane (K, X) where the optimal control consists in *four* different phases: bang ($E = \overline{E}$)-bang ($E = 0$)-bang (E in the singular curve)-bang ($E = \overline{E}$) trajectories. This is the case when carrying capacity is small enough as compared with the stock of fish. This kind of change in the harvesting that takes into account the state of the habitat must be taken into account when proposing public policies.

The immediate technical followup to this work should be to investigate further the properties of the singular and the switching curves: characterize analytically their equations in general, and characterize the location of "leave points".

Also, several extensions of the model can be envisioned. For instance, it would be interesting to analyze the case where β is a control and to investigate how trajectories differ from the case with β constant. In a dynamic game theoretic context, the present model can be seen as describing a social optimum from the point of view of a regulator. Following the ideas of Long et al. (2020), a decentralized situation with myopic (of almost myopic[3]) agents can be considered and economic instruments can be proposed to achieve or approach this social optimum.

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Declarations

Ethical Approval

This research does not involve human or animal studies.

Competing interests

The authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

[3] We call 'almost myopic' an agent that maximises his instantaneous payoff plus a scrap value taking into account the environment; in our framework environment can be the stock of fish or the habitat or both.

Authors' contributions

Both authors contributed equally to the research and to the writing of the manuscript.

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Availability of data and materials

The data was produced by code written by the authors. The code and the data is available upon request.

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A Proofs of Propositions 2 and 3

It may be convenient in calculations to manipulate the following functions:

$$\Theta_X(t) = X(t)\lambda_X(t) \qquad \Theta_K(t) = K(t)\lambda_K(t) \ ,$$

with which χ can be written as:

$$\chi(t) \ = \ \Theta_X(t) + \beta\Theta_K(t) - P(X(t)).$$

Using (6) and (7), the first-order derivatives are given by:

$$\begin{aligned}
\frac{\mathrm{d}\Theta_X}{\mathrm{d}t} &= \dot{X}\lambda_X + X\dot{\lambda}_X \\
&= \delta\Theta_X + (F(X,K) - XF_X(X,K))\lambda_X - EXP'(X) \qquad (35)\\
\frac{\mathrm{d}\Theta_K}{\mathrm{d}t} &= \dot{K}\lambda_K + K\dot{\lambda}_K \\
&= \delta\Theta_K + (G(K) - KG'(K))\lambda_K - KF_K(X,K)\lambda_X. \qquad (36)
\end{aligned}$$

Using (36) and (35) for the derivative of χ, one obtains:

$$\begin{aligned}
\dot{\chi}(t) &= \dot{\Theta}_X + \beta\dot{\Theta}_K - \dot{X}P'(X) \\
&= \delta(\Theta_X + \beta\Theta_K) + (F - XF_X)\lambda_X - EXP'(X) \\
&+ \beta(G - KG')\lambda_K - \beta KF_K\lambda_X \\
&- (F - EX)P'(X) \\
\dot{\chi} &= \delta(\chi + P(X)) + \lambda_X\,(F - XF_X - \beta KF_K) + \beta\lambda_K\,(G - KG') - FP'(X). \qquad (37)
\end{aligned}$$

This derivative does not depend, formally, on the control function $E(t)$. Of course, the trajectories of X, K, λ_X and λ_K do depend on the control.

A.1 Proof of Proposition 2

If the trajectory is constrained to stay on the curve defined by $\chi = 0$, then we must have, for all t, $\chi(t) = \dot{\chi}(t) = 0$. With (37), this leads to the system of equations:

$$\begin{aligned}
X\lambda_X + \beta K\lambda_K &= P \\
\lambda_X\,(F - XF_X - \beta KF_K) + \beta\lambda_K\,(G - KG') &= FP' - \delta P,
\end{aligned}$$

which is linear in the variables λ_X and $\beta\lambda_K$. Solving this linear system, we obtain

$$\lambda_X = \frac{1}{\Delta}\left((G - KG')P + K(\delta P(X) - FP')\right) \qquad (38)$$

$$\begin{aligned}\beta\lambda_K &= -\frac{1}{\Delta}\left(X(\delta P - FP') + (F - XF_X - \beta KF_K)P)\right) \\ \Delta &= X(G - KG') - K(F - XF_X - \beta KF_K).\end{aligned} \tag{39}$$

For statement b), we start from (37) but introduce the new functions $F^{(1)}(X,K) = F(X,K) - XF_X(X,K) - \beta KF_K(X,K)$, $G^{(1)}(K) = G(K) - KG'(K)$ and $P^{(1)}(X,K) = F(X,K)P'(X)$. Accordingly:

$$\begin{aligned}&\dot\chi = \delta(\chi + P) + \lambda_X F^{(1)} + \beta\lambda_K G^{(1)} - P^{(1)} \\ &\ddot\chi = \delta(\dot\chi + \dot X P') + \dot\lambda_X F^{(1)} + \lambda_X(\dot X F_X^{(1)} + \dot K F_K^{(1)}) + \beta\dot\lambda_K G^{(1)} + \beta\lambda_K \dot K G^{(1)\prime} \\ &\quad - (\dot X P_X^{(1)} + \dot K P_K^{(1)}).\end{aligned}$$

Then, applying (2), (3), (6) and (7), separating the terms with E from the rest, we have:

$$\begin{aligned}\ddot\chi = \delta\dot\chi &+ \Big\{\delta FP' + \lambda_X(\delta - F_X)F^{(1)} + \lambda_X(FF_X^{(1)} + GF_K^{(1)}) \\ &+ \beta\lambda_K(\delta - G')G^{(1)} - \beta\lambda_X F_K G^{(1)} + \beta\lambda_K G G^{(1)\prime} - (FP_X^{(1)} + GP_K^{(1)})\Big\} \\ &- E\Big[\delta XP' + (P' - \lambda_X)F^{(1)} + \lambda_X(XF_X^{(1)} + \beta KF_K^{(1)}) - \beta^2\lambda_K G^{(1)} + \beta^2 K\lambda_K G^{(1)\prime} \\ &- (XP_X^{(1)} + \beta KP_K^{(1)})\Big].\end{aligned} \tag{40}$$

If the trajectory is constrained to stay on the singular curve, then $\ddot\chi(t) = 0$. We can also eliminate λ_X and λ_K using (38) and (39). The function $\varphi(K,X)$ is defined as the term within braces, and the function $\psi(K,X)$ is defined as the term within square brackets, after this substitution. Then, solving $\ddot\chi = 0$, taking into account $\dot\chi = 0$, implies $0 = \varphi(K,X) - E\psi(K,X)$. This proves Proposition 2.

A.2 Proof of Proposition 3

When β is set to 0 in (37), the expression of $\dot\chi$ does not depend on λ_K anymore. Also, $\chi = 0$ is equivalent to (9). Then eliminating λ_X from (37) gives (10). This proves statement a) of Proposition 3.

For the proof of statement b), we first expose preliminary considerations. Assume we want the state to stay on a particular curve, given in general by the equation $f(X,K) = 0$. Differentiating with respect to t, we have the identities:

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t} f(X(t), K(t)) \\ &= \dot X\frac{\partial f}{\partial X} + \dot K\frac{\partial f}{\partial K} = (F(X,K) - EX)\, f_X + (G(K) - \beta EK)\, f_K\end{aligned} \tag{41}$$

$$
\begin{aligned}
E\left(Xf_X + \beta K f_K\right) &= F(X,K)f_X + G(K)f_K \\
E &= \frac{F(X,K)f_X + G(K)f_K}{Xf_X + \beta K f_K}.
\end{aligned}
\tag{42}
$$

The expression (42) gives the control as a feedback of the state. If one plugs (42) in the dynamics of the state, one gets the diffential system:

$$
\begin{aligned}
\dot{X} &= f_K \, \frac{\beta K F(X,K) - X G(K)}{Xf_X + \beta K f_K} \\
\dot{K} &= f_X \, \frac{X G(X) - \beta K F(X,K)}{Xf_X + \beta K f_K}
\end{aligned}
$$

and it is checked that this dynamics satisfies (41).

When $\beta = 0$, these expressions are simplified as:

$$
\begin{aligned}
E(X,K) &= \frac{F(X,K)}{X} + \frac{G(K)}{X} \, \frac{f_K}{f_X} \\
\dot{X} &= -G(K)\frac{f_K}{f_X} \qquad \dot{K} = G(K),
\end{aligned}
\tag{43}
$$

which proves statement b) of Proposition 3. In (43), the term $F(X,K)/X$ is the extraction that would make the stock X constant. It is corrected with a term, proportional to $G(K)$, taking into account the growth of the carrying capacity K.

B Logistic growth functions

In this section, we list the formulas obtained by specifying the general results of Section 2 to the logistic growth functions.

B.1 Dynamics and the singular control

Assume the control has the objective of keeping the function χ equal to 0. Then we have the following identities, in which we use the notation introduced in Section A.1.

$$
\begin{aligned}
G^{(1)} &:= G(K) - KG'(K) = \frac{\rho K^2}{\overline{K}} \\
F^{(1)} &:= F(X,K) - XF_X(X,K) - \beta K F_K(X,K) = r(1-\beta)\frac{X^2}{K}.
\end{aligned}
\tag{44}
$$

B.1.1 Adjoint variables

When using the logistic form of the growth functions, we obtain for the expressions (6) and (7):

$$\dot{\lambda}_X = \delta\lambda_X - pE - \lambda_X\left(r - \frac{2rX}{K} - E\right) \tag{45}$$

$$\dot{\lambda}_K = \delta\lambda_K - r\lambda_X\frac{X^2}{K^2} - \lambda_K\left(\rho - \frac{2\rho K}{\overline{K}} - \beta E\right) \tag{46}$$

B.1.2 The function $\chi(t)$

The formulas for the first derivatives of $\chi(t)$ are obtained from the generic formulas (37) and (40). Using the identities (44), we obtain:

$$\chi = X\lambda_X + \beta K\lambda_K - pX + c \ = \ \Theta_X + \beta\Theta_K - pX + c \tag{47}$$

$$\dot{\chi} = \delta\left(\chi + pX - c\right) + \Theta_X(1-\beta)\frac{rX}{K} + \beta\Theta_K\frac{\rho K}{\overline{K}} - prX(1-\frac{X}{K}) \tag{48}$$

$$\begin{aligned}
\ddot{\chi} &= \delta\left[\dot{\chi} + prX(1-\frac{X}{K})\right] \\
&\quad + r(1-\beta)\frac{X}{K}\Theta_X\left[\delta + r - \rho(1-\frac{K}{\overline{K}})\right] + \rho\beta\frac{K}{\overline{K}}\left[\Theta_K\left(\delta+\rho\right) - r\Theta_X\frac{X}{K}\right] \\
&\quad - rXp\left(r(1-\frac{X}{K})(1-\frac{2X}{K}) + \rho\frac{X}{K}(1-\frac{K}{\overline{K}})\right) \\
&\quad - E\left[r(1-\beta)^2\frac{X}{K}\Theta_X + \rho\beta^2\frac{K}{\overline{K}}\Theta_K + \delta Xp + rXp\left((3-2\beta)\frac{X}{K} - 1\right)\right].
\end{aligned} \tag{49}$$

B.1.3 Staying on the curve $\chi = 0$

If the trajectory follows the curve $\chi(t) \equiv 0$, then $\chi = \dot{\chi} = 0$. Using (38) and (39) (or directly solving for Θ_X and Θ_K in (47) and (48)), we obtain the following values:

$$\Theta_X = \frac{1}{\Delta}\left((pX-c)(\delta + \rho\frac{K}{\overline{K}}) - rXp(1-\frac{X}{K})\right) \tag{50}$$

$$\beta\Theta_K = \frac{1}{\Delta}\left(-(pX-c)(\delta + (1-\beta)r\frac{X}{K}) + rXp(1-\frac{X}{K})\right) \tag{51}$$

$$\Delta = \rho\frac{K}{\overline{K}} - (1-\beta)r\frac{X}{K}. \tag{52}$$

B.2 Proof of Lemma 1

The proof consists in enumerating the solutions of the system of four equations (15)–(18) with four unknowns. We exclude $K = 0$ as a solution of (16) since in that case the dynamics (12) are not well defined. Then the solution of (16) is

$$K^{\infty} = \overline{K}\left(1 - \frac{\beta E}{\rho}\right). \tag{19}$$

Next, $X = 0$ is clearly a solution of (15). Replacing in (17) and (18), we find:

$$\begin{aligned} pE^{\infty} &= \lambda_X \left(\delta - r + E^{\infty}\right) \\ 0 &= \lambda_K \left(\delta - \rho + E^{\infty}\right), \end{aligned}$$

leading to $\lambda_K = 0$ and the value of λ_X in (20). Assuming now that $K \neq 0$ and $X \neq 0$, we have successively:

$$\begin{aligned} X^{\infty} = K^{\infty}\left(1 - \frac{E^{\infty}}{r}\right) &= \overline{K}\left(1 - \frac{\beta E^{\infty}}{\rho}\right)\left(1 - \frac{E^{\infty}}{r}\right) \qquad (21) \\ pE^{\infty} &= \lambda_X^{\infty}\left(\delta - r + 2r\left[1 - \frac{E^{\infty}}{r}\right] + E^{\infty}\right) \\ r\lambda_X^{\infty}\left(1 - \frac{E^{\infty}}{r}\right)^2 &= \lambda_K^{\infty}\left(\delta - \rho + 2\rho\left[1 - \frac{\beta E}{\rho}\right] + \beta E^{\infty}\right) . \end{aligned}$$

The two last equations rewrite as (22) and (23). This concludes the proof of Lemma 1.

B.3 Trajectories for *E* constant

We investigate in this section the solutions to the differential systems when E is constant. To that end, it is convenient to perform the change of functions $Z = K^{-1}$ and $Y = X^{-1}$. This leads to the differential system:

$$\dot{Y} = (E - r)Y + rZ \tag{53}$$

$$\dot{Z} = (\beta E - \rho)Z + \frac{\rho}{\overline{K}} . \tag{54}$$

Trajectories of the state.

The solution of (13) is, classically, (26). This is obtained for instance using the function $Z = K^{-1}$ and the *linear* differential equation (54), whose solution is:

$$Z(t) = \frac{1}{K(t)} = \frac{1}{K_0}\, e^{(\beta E - \rho)t} + \frac{\rho}{\overline{K}}\, \frac{e^{(\beta E - \rho)t} - 1}{\beta E - \rho} . \tag{55}$$

The solution of (12) can be deduced as follows. Using the function $Y = X^{-1}$ and (53) we have:

$$\dot{Y} = (E - r)Y + re^{(\beta E - \rho)t}\left(\frac{1}{K_0} - \frac{\rho}{\rho - \beta E}\frac{1}{\overline{K}}\right) + \frac{r\rho}{(\rho - \beta E)\overline{K}} . \tag{56}$$

Consequently, the solution will be:

$$\begin{aligned}
Y(t) &= Y(0)e^{(E-r)t} \\
&\quad + \int_0^t \left[re^{(\beta E-\rho)u} \left(\frac{1}{K_0} - \frac{\rho}{\rho-\beta E}\frac{1}{\overline{\overline{K}}} \right) + \frac{r\rho}{(\rho-\beta E)\overline{\overline{K}}} \right] e^{(E-r)(t-u)} \mathrm{d}u \\
&= Y(0)e^{(E-r)t} \\
&\quad + r\left(\frac{1}{K_0} - \frac{\rho}{\rho-\beta E}\frac{1}{\overline{\overline{K}}} \right) \frac{e^{(\beta E-\rho)t} - e^{(E-r)t}}{(\beta-1)E+r-\rho} + \frac{r\rho}{(\rho-\beta E)\overline{\overline{K}}}\frac{1-e^{(E-r)t}}{r-E} \, .
\end{aligned}$$

This equation directly leads to (27).

Alternative representations.

Using the asymptotic quantities when $t \to \infty$, derived in Section 3.1, the solution for E constant may be rewritten as:

$$Z(t) \;=\; \frac{1}{K(t)} = \left(\frac{1}{K_0} - \frac{1}{K^\infty} \right) e^{(\beta E-\rho)t} + \frac{1}{K^\infty} \tag{57}$$

$$Y(t) \;=\; \frac{1}{X(t)} = \left(\frac{1}{X_0} - \frac{1}{X^\infty} \right) e^{(E-r)t} \;+\; r\left(\frac{1}{K_0} - \frac{1}{K^\infty} \right) \frac{e^{(\beta E-\rho)t} - e^{(E-r)t}}{(\beta-1)E+r-\rho} + \frac{1}{X^\infty} \, . \tag{58}$$

Equations in the (K, X) ***plane.***

Provided $K_0 \neq K^\infty$, the variable t can be eliminated from (57):

$$t \;=\; \frac{1}{\rho-\beta E} \log\left(\frac{\frac{1}{K_0} - \frac{1}{K^\infty}}{\frac{1}{K} - \frac{1}{K^\infty}} \right) \;=\; \frac{1}{\rho-\beta E} \log\left(\frac{K^\infty - K_0}{K^\infty - K}\,\frac{K}{K_0} \right) \, .$$

This is the amount of time needed to go from state K_0 to state K. Then, replacing in (58), we have:

$$\begin{aligned}
\frac{1}{X} - \frac{1}{X^\infty} &= e^{(E-r)t}\left(\frac{1}{X_0} - \frac{1}{X^\infty} - \left(\frac{1}{K_0} - \frac{1}{K^\infty} \right) \frac{r}{(\beta-1)E+r-\rho} \right) \\
&\quad + \; e^{(\beta E-\rho)t}\left(\frac{1}{K_0} - \frac{1}{K^\infty} \right) \frac{r}{(\beta-1)E+r-\rho} \\
&= \left(\frac{K^\infty - K_0}{K^\infty - K}\,\frac{K}{K_0} \right)^{\frac{E-r}{\rho-\beta E}} \left(\frac{1}{X_0} - \frac{1}{X^\infty} - \left(\frac{1}{K_0} - \frac{1}{K^\infty} \right) \frac{r}{(\beta-1)E+r-\rho} \right) \\
&\quad + \; \left(\frac{1}{K} - \frac{1}{K^\infty} \right) \frac{r}{(\beta-1)E+r-\rho} \, .
\end{aligned} \tag{59}$$

This is the equation of the trajectory in the (K, X) plane, parameterized by any point (K_0, X_0) on the curve.

If $K_0 = K^\infty$, the trajectory is included in the line $\{K = K^\infty\}$. If needed, the time variable may be eliminated from (58):

$$t = \frac{1}{r-E} \log\left(\frac{\frac{1}{X_0} - \frac{1}{X^\infty}}{\frac{1}{X} - \frac{1}{X^\infty}}\right) = \frac{1}{r-E} \log\left(\frac{X^\infty - X_0}{X^\infty - X} \frac{X}{X_0}\right).$$

This is, in particular, the case in the standard model with K constant.

We now investigate the behaviors of trajectories in the (K, X) plane. From (59), and since with $E = 0$ we have $X^\infty = K^\infty = \overline{K}$, these are the curves of equation

$$\frac{1}{X} = \frac{1}{\overline{K}} + \left(\frac{\overline{K} - K_0}{\overline{K} - K} \frac{K}{K_0}\right)^{-r/\rho} \left(\frac{1}{X_0} - \frac{1}{\overline{K}} - \left(\frac{1}{K_0} - \frac{1}{\overline{K}}\right)\frac{r}{r-\rho}\right) + \left(\frac{1}{K} - \frac{1}{\overline{K}}\right)\frac{r}{r-\rho}$$

parametrized by any point (K_0, X_0) on the curve.[4] These curves have the following property.

Lemma 3 *Let $f(K)$ be one of these curves, parametrized by (K_0, X_0). There exists some $K_1 \in (0, \overline{K})$ such that $\lim_{K \to K_1^+} f(K) = +\infty$ if:*

a) $\rho > r$, or
b) when and only when

$$X_0 > \frac{K_0 \overline{K}(r-\rho)}{r\overline{K} - \rho K_0} \tag{60}$$

if $\rho < r$.

The proof consists in analyzing the behavior of the curves when $K \to 0$. If $\rho > r$, we find that $f(K) \sim K(r-\rho)/r < 0$: then necessarily $f(\cdot)$ changes sign between $K = 0$ and $K = \overline{K}$, which can occur only by diverging to $\mp\infty$. If $\rho < r$, we find that $f(K) \sim C(K_0, X_0)K^{r/\rho}$ for some value $C(K_0, X_0)$. If this value is negative, which is equivalent to condition (60), this is the same situation as when $\rho > r$ and there is a point K_1 where $f(\cdot)$ diverges. If this value is 0, the curve f has actually the equation

$$f_c(K) = \frac{K\overline{K}(r-\rho)}{r\overline{K} - \rho K_0}$$

and it does not diverge. If $C(K_0, X_0) > 0$, the trajectory lies below the curve $f_c(K)$ (solutions to the differential system cannot cross each other) so that it cannot diverge.

[4] When $r = \rho$, a different formula applies.

Adjoint variable trajectories with $E = 0$.

In the specific case where $E = 0$, it is possible to obtain a formula for the adjoint variable λ_X. Start with the differential equation for $\Theta_X = X\lambda_X$

$$\begin{aligned}\dot{\Theta}_X = \dot{X}\lambda_X + X\dot{\lambda}_X \;&=\; \Theta_X\left(\delta + r\frac{X}{K}\right)\\ \frac{\dot{\Theta}_X}{\Theta_X} &= \delta + r - \frac{\dot{X}}{X}\\ \frac{\Theta_X(t)}{\Theta_X(0)} &= e^{(\delta+r)t}\;\frac{X(0)}{X(t)}\\ \Theta_X(t) &= \Theta_X(0)X(0)\frac{e^{(\delta+r)t}}{X(t)}\end{aligned}$$

and this last equation yields (28).

C Proof of Lemma 2

Proof a) Consider the polynomial in (24), that is,

$$\begin{aligned}Pol(E) &= -c(E\beta - \delta - \rho)r(E - \delta - r)\rho\\ &\quad + p\overline{K}(E - r)(\beta E - \rho)\left(3E^2\beta - (\beta\delta + 2\beta r + 2\delta + 2\rho)E + (\delta + r)(\delta + \rho)\right).\end{aligned}$$

Recall that we are looking for *admissible* roots of $Pol(E)$, that is, such that $0 \le E \le \min(r, \rho/\beta)$. This polynomial has the following properties.

$$Pol(0) = r\rho(\delta + \rho)(\delta + r)(\overline{K}p - c),$$

$$Pol(r) = rc\delta\rho(\beta r - \delta - \rho), \quad Pol\left(\frac{\rho}{\beta}\right) = -\frac{\rho c\delta r(\beta\delta + \beta r - \rho)}{\beta}.$$

The existence part is proved by showing that $Pol(\cdot)$ changes sign over the interval. The uniqueness follows from convexity.

Under the condition $\overline{K}p > c > 0$, $Pol(0) > 0$. Consider first the case $r \le \rho/\beta$. This implies $Pol(r) < 0$, hence $Pol(E) = 0$ for at least one $E \in (0, r)$. Through tedious yet mechanical computations, using the assumption that $\overline{K}p - c > 0$, one shows that $Pol''(E) > 0$ if $E \in [0, r]$. Then in is not possible to have three roots of $Pol(\cdot)$ in the interval. When $r > \rho/\beta$, then $Pol(\rho/\beta) < 0$, and the reasoning is similar. From this analysis we conclude, not only that $Pol(\cdot)$ has a single root, but also that $Pol'(E) < 0$ in the region where E is admissible.

Moreover from (24), and for admissible E,

$$\frac{\partial Pol}{\partial c}(E) \;=\; -\,r\rho(E\beta - \delta - \rho)(E - \delta - r) \;<\; 0.$$

Then using total differentiation for the root $E^\infty(c)$, we have:

$$\frac{\partial E^\infty}{\partial c}(c) = -\frac{1}{Pol'(E^\infty(c))}\;\frac{\partial Pol}{\partial c}(E^\infty(c)) < 0. \tag{61}$$

b) Now we consider $c = 0$. In this case, with $Pol_0(E) \equiv Pol(E)|_{c=0}$,

$$Pol_0(E) = p\overline{K}(E - r)(\beta E - \rho)\left(3E^2\beta - (\beta\delta + 2\beta r + 2\delta + 2\rho)E + (\delta + \rho)(\delta + r)\right).$$

Let $Q(E)$ denote the last polynomial of degree 2 in E. The discriminant of $Q(E)$ is:

$$\Delta_\beta := (\delta + 2r)^2\beta^2 - 4(\delta + \rho)(2\delta + r)\beta + 4(\delta + \rho)^2.$$

This quadratic polynomial of β has in turn as discriminant: $48(\delta+\rho)^2(\delta-r)(\delta+r)$. Then, if $\delta < r$, this discriminant is negative and $\Delta_\beta > 0$ for all values of β. As a consequence, $Q(E)$ has two real and positive roots $E_1 < E_2$, where E_1 is given explicitly by (25). We proceed with locating E_1 with respect to the other roots of $Pol_0(E)$: r and ρ/β. We have:

$$Q(0) > 0,$$
$$Q\left(\frac{\delta+r}{2}\right) = \frac{\beta(\delta-r)(\delta+r)}{4}.$$

If $\delta < r$, this implies that $0 < E_1 < (\delta + r)/2$. If furthermore $r \leq \rho/\beta$, then $(\delta + r)/2 < \min(r, \rho/\beta)$ and therefore $E_1 = E_0^\infty$.

If $\delta < r$ but $r > \rho/\beta$, we just have $E_0^\infty = \min(E_1, \rho/\beta)$. □