An adaptive observer for time-varying nonlinear systems - application to a crop irrigation model

Gildas Dadjo, Denis Efimov, Jérôme Harmand, Alain Rapaport, Rosane Ushirobira

An adaptive observer for time-varying nonlinear systems - application to a crop irrigation model

M.G. Dadjo, D. Efimov, J. Harmand, A. Rapaport, R. Ushirobira

***Abstract*— We propose an adaptive observer for a class of nonlinear time-varying systems, for which the regressor depends not only on the known input-output signals but also on all the unmeasured states. There are state disturbances, and measurements are corrupted by noise. The Lyapunov function method is used to prove that the coupled system-observer dynamics admits a state-independent input-to-output stability property in estimation errors from unknown inputs. Numerical simulations illustrate the presented approach on a three-dimensional crop irrigation model.**

I. Introduction

Observer theory has been a powerful tool for several decades to estimate the state and the parameters of an uncertain dynamical system in real time, and many researchers have worked in this field; see, for instance, [11]. Nevertheless, there is no general method to design observers for nonlinear systems in the current literature, and usually, certain canonical model representations are considered. This is also related to the lack of a standard selection approach for Lyapunov functions, which are needed in the stability analysis.

We consider in this note the problem of simultaneously reconstructing the unmeasured state variables and the constant unknown parameters in the presence of disturbances and measurement perturbations. For this purpose, adaptive observers have been proposed in the literature in the linear and the nonlinear contexts, starting from earlier works [6], [15], [2], [16], where two main ways for adaptive observer design were recognized: when the unknown parameters appear in the derivative of the output, or when they appear for the first time in higher order derivatives, and additional filters are needed to compensate the high relative degree obstruction. Significant results have been formulated at the beginning of this century, e.g., [3], [22], where canonical forms of systems admitting nonlinear adaptive observers were defined, and the canonical structure of the auxiliary filters was given. It is worth highlighting that an important restriction for a majority of the results is the requirement of dependence of regressor (the gain function that multiplies the vector of unknown parameters) on measured input and output signals only, which was resolved in [10] for the cases of linear and nonlinear parameterization. Another common drawback is the lack of analysis in the presence of state disturbances and measurement noises, where only special cases were studied [18], [8].

This work has been motivated by a concrete problem of crop irrigation. Usually, soil humidity and biomass can be measured or estimated [1], but the estimation of nitrogen content is often described as challenging, while it is crucial for growth prediction [19]. These variables' dynamics are nonlinear, time-varying, and rely on several parameters that need to be estimated for each soil-crop system [4], [7]. Moreover, the measurements are noisy, and the state dynamics contains uncertain external inputs. So, we aim to propose a new approach for the adaptive estimation of this model, which due to its special characteristics, does not fall within the scope of previous works on observers.

In this paper, we propose an adaptive observer for a class of non-autonomous nonlinear dynamics with unknown inputs, for which the regressor depends not only on the known input-output signals but also on the unmeasured state, and the measurements are noisy. The analysis is based on utilization of the theory of input-to-state stability (ISS) and other related properties. A Lyapunov function candidate is defined to prove that the system is state independent input-to-output stable from unknown inputs to the estimation errors, relying on a condition of persistently excitation of the control variables.

The paper is organized as follows. In Section II the used results from ISS theory are reviewed. In Section III, we give assumptions, present our observer and prove our main result. In Section IV, we introduce the irrigation problem and show how apply there our main result. Finally, Section V presents numerical simulations and compare the performances of the adaptive observer with a high-gain observer.

Notations

- $\mathbb{R}_+$, $\mathbb{R}_+^*$ denote the sets of non-negative real numbers and positive real numbers, respectively.
- $\mathbb{R}^p$ and $\mathbb{R}^{n\times m}$ denote the real vector space of dimension p, and the set of matrices with real coefficients of dimension $n\times m$, respectively.
- I_n denotes the identity matrix of dimension $n\times n$.

M.G. Dadjo, D. Efimov, and R. Ushirobira are with Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France; M.G. Dadjo and A. Rapaport are with MISTEA, Univ. Montpellier, INRAE, Institut Agro, 34060 Montpellier, France; J. Harmand is with LBE, INRAE, Univ. Montpellier, 11100 Narbonne, France; E-mails: mahugnon.dadjo@inrae.fr, denis.efimov@inria.fr, jerome.harmand@inrae.fr, alain.rapaport@inrae.fr, rosane.ushirobira@inria.fr

- $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced norm for matrices.
- For a Lebesgue measurable function $u\colon \mathbb{R}_+ \to \mathbb{R}^m$, define the norm $\|u\|_{[t_1,t_2)} = \operatorname{ess\,sup}_{t\in[t_1,t_2)}\|u(t)\|$ for $t_1,\ t_2 \in \mathbb{R}_+$. We denote by $\mathcal{L}_\infty^m$ the set of functions u with $\|u\|_\infty := \|u\|_{[0,\infty)} < +\infty$.
- A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r,\cdot)$ is a decreasing function going to zero for any fixed $r \in \mathbb{R}_+$.
- $A^\top$ (resp. $\xi^\top$) denotes the transpose of the matrix A (resp. the vector ξ).
- For a matrix M of $\mathbb{R}^{n\times n}$, $M \preceq 0$ (resp. $M \prec 0$) means $\xi^\top M\xi \le 0$ (resp. $\xi^\top M\xi < 0$) for all $\xi \in \mathbb{R}^n$.

II. Preliminaries

Following [20] consider a class of nonlinear systems:

$$\begin{aligned} \dot{x}(t) &= f(x(t), d(t)),\ t \ge 0, \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the external perturbation, with $d \in \mathcal{L}_\infty^m$, and $y(t) \in \mathbb{R}^p$ is the output vector. Moreover, $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz continuous function, $f(0,0) = 0$, and $h\colon \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function. For an initial state $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$, we denote the corresponding solution of the system (1) by $x(t, x_0, d)$ for the values of $t \ge 0$ the solution exists, so the corresponding output is $y(t, x_0, d) = h\,(x(t, x_0, d))$.

The system (1) is called *forward complete* if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$, the solution $x(t, x_0, d)$ is uniquely defined for all $t \ge 0$.

Definition 1: A forward complete system (1) is said to be:

1) *practical input-to-output stable* (pIOS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $c \in \mathbb{R}_+$ such that

$$\|y(t, x_0, d)\| \le \beta\,(\|x_0\|, t) + \gamma(\|d\|_\infty) + c,\ \ \forall t \ge 0$$

for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$. The system is called *input-to-output stable* (IOS) if $c = 0$. In the special case when $y = x$, the IOS property is called *ISS*.

2) *state-independent input-to-output stable* (SIIOS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|y(t, x_0, d)\| \le \beta(\|h(x_0)\|, t) + \gamma(\|d\|_\infty),\ \forall t \ge 0$$

for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$.

As we can deduce from this definition, SIIOS is a direct extension of the ISS property to the systems demonstrating convergence only with respect to a part of the variables.

Definition 2: A forward complete system (1) is said to be *uniformly bounded-input-bounded-state stable* (UBIBS) if there exists $\sigma \in \mathcal{K}$ such that

$$\|x(t, x_0, d)\| \le \max\{\sigma(\|x_0\|), \sigma(\|d\|_\infty)\},\ \forall t \ge 0$$

for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$.

Definition 3: For the system (1), a smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is:

1) an IOS-*Lyapunov function* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, and $\alpha_3 \in \mathcal{KL}$ such that

$$\begin{gathered} \alpha_1(\|h(x)\|) \le V(x) \le \alpha_2(\|x\|), \\ V(x) \ge \chi(\|d\|) \Rightarrow \nabla V(x) f(x,d) \le -\alpha_3(V(x), \|x\|) \end{gathered} \tag{2}$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

2) a SIIOS-*Lyapunov function* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and χ, $\alpha_3 \in \mathcal{K}$ such that

$$\begin{gathered} \alpha_1(\|h(x)\|) \le V(x) \le \alpha_2(\|h(x)\|), \\ V(x) \ge \chi(\|d\|) \Rightarrow \nabla V(x) f(x,d) \le -\alpha_3(V(x)) \end{gathered}$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

Theorem 1 ([21]): A UBIBS (forward complete) system (1) is IOS (SIIOS) if and only if it admits an IOS (SIIOS)-Lyapunov function.

Despite that all definitions and results above are given for an autonomous system (1), the same formulations hold for time-varying counterparts, and the sufficient part of Theorem 1 is valid.

III. Main result

Consider a dynamical system of the form:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + \varphi(y(t), u(t)) \\ &\quad + G(y(t), u(t), x(t))\theta + d(t), \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output measurement vector, $\theta \in \mathbb{R}^m$ is the vector of unknown parameters, $u(t) \in \mathbb{R}^d$ is the known input vector, $v(t) \in \mathbb{R}^p$ is a measurement noise, $d(t) \in \mathbb{R}^n$ is a disturbance, the functions $G : \mathbb{R}^{n+p+d} \to \mathbb{R}^{n\times m}$ and $\varphi : \mathbb{R}^{p+d} \to \mathbb{R}^n$ are assumed to be continuous and guaranteeing existence of the solutions in forward time.

We require the following hypotheses.

Assumption 1: Let $d \in \mathcal{L}_\infty^n$, $v \in \mathcal{L}_\infty^p$, and $\|\theta\| \le \theta_{\max}$ with a known bound $\theta_{\max} > 0$.

Hence, the unknown inputs are bounded without a known upper limit, but the set of admissible values for the vector of unknown parameters is given. The latter is not a hard restriction since the parameters usually must respect physical constraints.

Assumption 2: There exists $\lambda > 0$ such that $\|G(y,u,x) - G(y,u,\hat{x})\| \le \lambda\|x - \hat{x}\|$ for all $y \in \mathbb{R}^p$, $u \in \mathbb{R}^d$ and $x,\ \hat{x} \in \mathbb{R}^n$.

Assumption 3: There is $G_{\max} > 0$ such that $\|G(y,u,x)\| \le G_{\max}$ for all $y \in \mathbb{R}^p$, $u \in \mathbb{R}^d$ and $x \in \mathbb{R}^n$.

These restrictions imply that G is uniformly Lipschitz continuous in the third argument and globally bounded.

Following [22], we propose an observer in the form:

$$\begin{aligned}
\dot{\hat{x}}(t) &= A(t)\hat{x}(t) + \varphi(y(t),u(t)) + G(y(t),u(t),\hat{x}(t))\hat{\theta}(t) \\
&\quad + L(t)(y(t) - C\hat{x}(t)) + \Omega(t)\dot{\hat{\theta}}(t), \\
\dot{\Omega}(t) &= (A(t) - L(t)C)\Omega(t) + G(y(t),u(t),\hat{x}(t)), \\
\dot{\hat{\theta}}(t) &= \gamma\Omega^\top(t)C^\top(y(t) - C\hat{x}(t)),
\end{aligned} \tag{4}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate vector for $x(t)$, $L(t) \in \mathbb{R}^{n\times p}$ is the observer gain providing the desired stability property for the matrix $A(t) - L(t)C$, $\Omega(t) \in \mathbb{R}^{n\times m}$ is an intermediate filter state variable, and $\gamma > 0$ is the adaptation gain. We detail other conditions below (further, if the dependence on time of a variable is evident, it may be omitted for brevity of presentation).

Next, by introducing an auxiliary estimation error and a parameter estimation error

$$\begin{aligned}
\delta &= x - \hat{x} - \Omega(\theta - \hat{\theta}), \\
\tilde{\theta} &= \theta - \hat{\theta},
\end{aligned}$$

after straightforward computations, we obtain the following dynamics:

$$\begin{aligned}
\dot{\delta} &= (A - LC)\delta + (G - \hat{G})\theta - Lv + d, \\
\dot{\tilde{\theta}} &= -\gamma\Omega^\top C^\top(C\delta + C\Omega\tilde{\theta}),
\end{aligned}$$

where, with a slight abuse of notation, we use the shorthand notations $G = G(y,u,x)$ and $\hat{G} = G(y,u,\hat{x})$.

We now set $e := x - \hat{x}$ as the state estimation error. We aim to prove that e and $\tilde{\theta}$ converge to 0 in the absence of the disturbances d and v and demonstrate a SIIOS property for bounded perturbations.

Theorem 2: Let assumptions 1, 2 and 3 hold. Assume there are symmetric matrix functions $P_1 : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $P_2 : \mathbb{R}_+ \to \mathbb{R}^{m\times m}$ satisfying the following conditions for all $t \geq 0$:

(i) $0 \prec \underline{a}I_n \prec P_1(t) \prec \overline{a}I_n$, $0 \prec \underline{a}I_m \prec P_2(t) \prec \overline{a}I_m$, for some $\underline{a}$, $\overline{a} \in \mathbb{R}_+$;

(ii) $\dot{P}_1(t) + P_1(t)(A(t) - L(t)C) + (A(t) - L(t)C)^\top P_1(t) \prec -Q_1$, for some $Q_1^\top = Q_1 \succeq 0$;

(iii) for some $Q_2^\top = Q_2 \succeq 0$,

$$\dot{P}_2(t) - \gamma P_2(t)\Omega^\top(t)C^\top C\Omega(t) - \gamma\Omega^\top(t)C^\top C\Omega(t)P_2(t) \preceq -Q_2;$$

(iv) there exist β_1, β_2, γ_1, $\gamma_2 > 0$ such that

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{21}^\top & P_1 & -P_1 L & P_1 \\ \Gamma_{21} & \Gamma_{22} & 0 & \Gamma_{24} & 0 \\ P_1 & 0 & -\gamma_1 I_n & 0 & 0 \\ -L^\top P_1 & \Gamma_{24}^\top & 0 & -\gamma_2 I_p & 0 \\ P_1 & 0 & 0 & 0 & -I_n \end{pmatrix} \prec 0$$

with $\Gamma_{11} = -Q_1 + (\beta_1 + 2\lambda^2\theta_{\max}^2)I_n$, $\Gamma_{22} = -Q_2 + (\beta_2 + 2\lambda^2\theta_{\max}^2\|\Omega\|^2)I_m$, $\Gamma_{21} = \Gamma_{24}C$, and $\Gamma_{24} = -\gamma P_2\Omega^\top C^\top$.

Then the system (3)-(4) is SIIOS in errors $x - \hat{x}$ and $\theta - \hat{\theta}$ from the inputs d and v, and the variable Ω is bounded.

In this theorem, it is assumed that the gain function $L(t)$ is chosen in a way to ensure the stability of the time-varying matrix $A(t) - L(t)C$, which is equivalent to the existence of a positive definite matrix $P_1(t)$. It is also assumed that the control u and the perturbations d, v are persistently exciting, providing the stability of a symmetric time-varying matrix $\Omega^\top(t)C^\top C\Omega(t)$. Then the existence of a positive definite matrix $P_2(t)$ follows. The remaining restrictions are needed to prove robust stability and convergence in the closed-loop system and are used in the proof below.

Proof: We will consider a SIIOS-Lyapunov function candidate defined by:

$$V = \delta^\top P_1\delta + \tilde{\theta}^\top P_2\tilde{\theta}$$

where the matrices P_1 and P_2 are introduced in the formulation of the theorem. We have:

$$\begin{aligned}
\dot{V} =& \delta^\top\left(\dot{P}_1 + P_1(A - LC) + (A - LC)^\top P_1\right)\delta + \delta^\top P_1 d \\
& + \tilde{\theta}^\top\left(\dot{P}_2 - \gamma P_2\Omega^\top C^\top C\Omega - \gamma\Omega^\top C^\top C\Omega P_2\right)\tilde{\theta} + \\
& \theta^\top\left(G - \hat{G}\right)^\top P_1\delta + \delta^\top P_1\left(G - \hat{G}\right)\theta + d^\top\delta P_1 - \\
& v^\top L^\top P_1\delta - \delta^\top P_1 Lv - \left(\gamma\Omega^\top C^\top C\delta\right)^\top P_2\tilde{\theta} - \\
& \gamma\tilde{\theta}^\top P_2\Omega^\top C^\top C\delta - \gamma\tilde{\theta}^\top P_2\Omega^\top C^\top v - \left(\gamma\Omega^\top C^\top v\right)^\top P_2\tilde{\theta} \\
& \leq -\delta^\top Q_1\delta - \tilde{\theta}^\top Q_2\tilde{\theta} + \left(\left(G - \hat{G}\right)\theta\right)^\top P_1\delta + \\
& \delta^\top P_1\left(G - \hat{G}\right)\theta + d^\top\delta P_1 + \delta^\top P_1 d - \\
& v^\top L^\top P_1\delta - \delta^\top P_1 Lv - \left(\gamma\Omega^\top C^\top C\delta\right)^\top P_2\tilde{\theta} - \\
& \gamma\tilde{\theta}^\top P_2\Omega^\top C^\top C\delta - \gamma\tilde{\theta}^\top P_2\Omega^\top C^\top v - \left(\gamma\Omega^\top C^\top v\right)^\top P_2\tilde{\theta} \\
& \leq X^\top MX - \beta_1\|\delta\|^2 - \beta_2\|\tilde{\theta}\|^2 + \gamma_1\|d\|^2 + \gamma_2\|v\|^2 + \\
& \left(\left(G - \hat{G}\right)\theta\right)^\top\left(G - \hat{G}\right)\theta \\
& \leq X^\top MX - \beta_1\|\delta\|^2 - \beta_2\|\tilde{\theta}\|^2 + \gamma_1\|d\|^2 + \gamma_2\|v\|^2 + \\
& 2\lambda^2\|\theta_{\max}\|^2\left(\|\delta\|^2 + \|\Omega\|^2\|\tilde{\theta}\|^2\right) \\
& \leq X^\top\Gamma X - \beta_1\|\delta\|^2 - \beta_2\|\tilde{\theta}\|^2 + \gamma_1\|d\|^2 + \gamma_2\|v\|^2
\end{aligned}$$

where $X = \begin{pmatrix} \delta^\top & \tilde{\theta}^\top & d^\top & v^\top & \left(\left(G - \hat{G}\right)\theta\right)^\top \end{pmatrix}^\top$,

$$M = \begin{pmatrix} -Q_1 + \beta_1 I_n & -\gamma C^\top C\Omega P_2 & P_1 & -P_1 L & P_1 \\ -\gamma P_2\Omega^\top C^\top C & -Q_2 + \beta_2 I_m & 0 & -\gamma P_2\Omega^\top C^\top & 0 \\ P_1 & 0 & -\gamma_1 I_n & 0 & 0 \\ -L^\top P_1 & -\gamma_2 C\Omega P_2 & 0 & -\gamma_2 I_p & 0 \\ P_1 & 0 & 0 & 0 & -I_n \end{pmatrix}$$

and we used Assumption 2 to bound $G - \hat{G}$:

$$\begin{aligned}
\left(\left(G - \hat{G}\right)\theta\right)^\top\left(G - \hat{G}\right)\theta &\leq \theta_{\max}^2\|G - \hat{G}\|^2 \\
&\leq \lambda^2\theta_{\max}^2\|e\|^2 \leq \lambda^2\theta_{\max}^2\|\delta + \Omega\tilde{\theta}\|^2.
\end{aligned}$$

Since $\Gamma \prec 0$ by the conditions of the theorem, then V is a SIIOS-Lyapunov function by Theorem 1.

Using Assumption 3 and Lyapunov function $W = \Omega^\top P_1 \Omega$, by repeating the same calculations for W, it is possible to show the boundedness of Ω. ■

Note that the matrix Γ used in the formulation of this theorem is time-dependent. So, to get a linear matrix inequality, additional constraints can be imposed:

Corollary 1: Let all conditions of Theorem 2 be satisfied, and consider the matrix

$$\Delta = \begin{pmatrix} \Delta_{11} & 0 & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ 0 & \Delta_{22} & 0 & 0 & 0 \\ \Gamma_{31} & 0 & \Gamma_{33} & 0 & 0 \\ \Gamma_{41} & 0 & 0 & \Delta_{44} & 0 \\ \Gamma_{51} & 0 & 0 & 0 & \Gamma_{55} \end{pmatrix}$$

where $\Delta_{11} = -Q_1 + \left(\beta_1 + \frac{\|\Gamma_{21}\|^2}{\epsilon} + 2\lambda^2\theta_{\max}^2\right) I_n$, $\Delta_{22} = -Q_2 + (\beta_2 + \epsilon + \epsilon' + 2\lambda^2\theta_{\max}^2\Omega_{\max}^2) I_m$, $\Delta_{44} = \left(\frac{\|\Gamma_{42}\|^2}{\epsilon'} - \gamma_2\right) I_p$ and $\|\Omega\| \le \Omega_{\max} < \infty$ (such a bound $\Omega_{\max}$ exists in the conditions of Theorem 2), with some ϵ, $\epsilon' > 0$. If $\Delta \preceq 0$, then (4) is SIIOS in the errors $x - \hat{x}$ and $\theta - \hat{\theta}$ for the inputs d and v, and the variable Ω stays bounded.

Proof: It is straightforward to verify that:

$$2\left(\delta^\top \Gamma_{12}\right)\tilde{\theta} \le \frac{\|\Gamma_{12}\|^2}{\epsilon}\|\delta\|^2 + \epsilon\|\tilde{\theta}\|^2,$$

$$2\left(v^\top \Gamma_{42}\right)\tilde{\theta} \le \frac{\|\Gamma_{42}\|^2}{\epsilon'}\|v\|^2 + \epsilon'\|\tilde{\theta}\|^2.$$

It is easy to check that the following inequality holds: $X^\top \Gamma X \le X^\top \Delta X$. ■

IV. Application to a Crop Irrigation Model

A fundamental concern in agriculture is related with the state of the soil ensuring the good conditions for crop growth (such as tomatoes, lettuce, etc.), especially under climate change and drought events. Having visibility on the water and nutrient needs of crops is crucial for reliable predictions and stimulation of the production. Humidity is a crucial state variable in most of the crop models, and therefore humidity sensors are widely used to evaluate the behavior of this quantity. However other crucial information, necessary to obtain an accurate prediction, is contained in the parameters as a well as in other state variables such as the nitrogen concentration [14], [9]. Most of the time, sensors of chemical composition are not available on the field, or not accurate, or too costly, etc. This is where the software sensors enter into the picture as an expected cheap alternative.

We consider here a simplified crop irrigation model, inspired from [17], [5], where we explicitly include the dynamics of nitrogen, in addition to humidity one.

$$\begin{aligned} \dot{S} &= k_1\left(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)\right) \\ \dot{B} &= \varphi(t)K_S(S) f\left(\frac{N}{S}\right) \\ \dot{N} &= -k_3\varphi(t)K_S(S) f\left(\frac{N}{S}\right) + k_4 C_N^{\text{in}} u(t) \end{aligned}$$

where S denotes the soil humidity level (between 0 and 1) and $B, N \in \mathbb{R}_+$ are biomass and nitrogen content per unit of soil surface. As often met in crop modelling, terms $\varphi(t)K_S(S)$, $(1-\varphi(t))K_R(S)$ represent crop transpiration and soil evaporation. Functions K_S, K_R are usually piecewise linear non-decreasing from $[0,1]$ to $[0,1]$. The time function φ is the crop radiation interception efficiency, which is usually C^1 and increasing, while f is a piecewise C^1 non-decreasing function, which regulates the growth depending on the nitrogen concentration. The input variable u is the irrigation flow rate, and C_N^{in} the nitrogen concentration of the irrigation water. Typical, instances of the functions K_S, K_R, f are:

$$K_S(S) = \begin{cases} 0 & \text{if } S \in [0, S_w] \\ \dfrac{S - S_w}{S^* - S_w} & \text{if } S \in (S_w, S^*], \\ 1 & \text{if } S > S^* \end{cases} \qquad 0 < S_w < S^*,$$

$$K_R(S) = \begin{cases} 0 & \text{if } S \in [0, S_h] \\ \dfrac{S - S_h}{1 - S_h} & \text{if } S > S_h \end{cases} \qquad S_h > 0,$$

$$f\left(\frac{N}{S}\right) = \begin{cases} \dfrac{N}{\eta_c S} & \text{if } \dfrac{N}{S} \in [0, \eta_c) \\ 1 & \text{if } \dfrac{N}{S} \in [\eta_c, 1] \end{cases} \qquad \eta_c > 0,$$

where $S^\star$ is the water stress threshold, S_w is the humidity threshold above which the plant wilts, S_h is the hydroscopic point, and η_c is the nitrgoen saturation coefficient. The online measurements are the humidity and the biomass

$$y_1 = S + v_1(t), \quad y_2 = B + v_2(t),$$

where v_1, v_2 are unknown measurement noises. We shall consider the beginning of the season ($t = 0$) for which the initial humidity level $S(0)$ can be assumed to be high, i.e., larger than the threshold $S^\star$ and the unknown initial nitrogen $N(0)$ can be small, i.e., such that $N(0)/S(0) < \eta_c$. We then look for two estimation problems on a time window for which S stays above $S^\star$ and N/S below η_c. In this sub-domain of the state space, the dynamics can be written, using the expressions of K_S, K_R, f given above, as follows:

$$\dot{S} = k_1\left(-\varphi(t) - (1-\varphi(t))K_R(S) + k_2 u(t)\right) \tag{5}$$

$$\dot{B} = \varphi(t)\frac{N}{\eta_c S} \tag{6}$$

$$\dot{N} = -k_3\varphi(t)\frac{N}{\eta_c S} + k_4 C_N^{\text{in}} u(t) \tag{7}$$

One can notice that the system has a cascade structure, i.e., the dynamics of the first variable S is independent of B and N. We consider two estimation problems:

A. reconstruct parameters k_1, k_2, S_h related to soil characteristics and the humidity dynamics (5) with the single measurement y_1, assuming that the functions φ and u are known;

B. reconstruct the nitrogen variable N and parameter k_3 (related to soil-crop system) with both measurements y_1, y_2, assuming that parameters η_c, $k_4 C_N^{in}$ and functions φ and u are known.

A. Estimation of k_1, k_2 and S_h

Equation (5) can be written as

$$\dot{S} = k_1\left(-1 - \frac{1}{1-S_h}(1-\varphi(t))(S-1) + k_2 u(t)\right) = \eta(t)^\top \theta$$

where we posit

$$\eta(t) = \begin{bmatrix} -1 \\ -(1-\varphi(t))(S(t)-1) \\ u(t) \end{bmatrix}, \ \theta = \begin{bmatrix} k_1 \\ \frac{k_1}{1-S_h} \\ k_1 k_2 \end{bmatrix}.$$

Then, one can consider the sub-system

$$\dot{S} = \eta(t)^\top \theta \tag{8}$$

$$y_1 = S + v_1 \tag{9}$$

where the parameter vector θ is unknown, i.e., here we consider that k_1, k_2 and S_h are unknown. It is then possible to reconstruct these parameters with a classical adaptive observer with the single measurement y_1:

Proposition 1: Consider the system

$$\dot{\hat{\xi}} = \omega(t)^\top \hat{\theta} + L\left(y - \hat{\xi}\right)$$
$$\dot{\hat{\theta}} = \Gamma \omega(t)\left(y - \hat{\xi}\right)$$

where $\omega(t) = [-1 \ -(1-\varphi(t))(y_1(t)-1)\ u(t)]^\top$ and $L > 0$, $\Gamma > 0$. If ω is persistently exciting, then $\hat{\theta}$ is an asymptotic estimator of θ for system (8) when $v_1 = 0$.

Proof: Denote $e := S - \hat{\xi}$ and $\tilde{\theta} = \theta - \hat{\theta}$. One has

$$\dot{e} = \omega(t)^\top \tilde{\theta} - Le$$
$$\dot{\tilde{\theta}} = -\Gamma \omega(t) e$$

and the conclusion follows by using Theorem 3 (see Appendix), when ω has a persistent excitation. ■

Robustness to noise can be established similarly to Theorem 2, see [8].

B. Estimation of N, k_1 and k_3

We write dynamics of the system(5)-(6)-(7) as follows

$$\dot{S} = k_1\left(-\varphi(t) - (1-\varphi(t))K_R(S) + k_2 u(t)\right),$$
$$\dot{B} = \frac{\varphi(t)}{\eta_c S} N$$
$$\dot{N} = -\kappa \frac{\varphi(t)}{\eta_c S} N + (\kappa - k_3)\frac{\varphi N}{\eta_c S} + k_4 C_N^{in} u(t)$$

where κ is any positive number. In the noise-free case it takes the form of (3) with

$$x = \begin{bmatrix} S \\ B \\ N \end{bmatrix}, \ A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\varphi(t)}{\eta_c y_1(t)} \\ 0 & 0 & -\kappa \dfrac{\varphi(t)}{\eta_c y_1(t)} \end{bmatrix},$$

$$\theta = \begin{bmatrix} k_1 \\ \\ \kappa - k_3 \end{bmatrix}, \ G(t,x,y,u) = \begin{bmatrix} G_1(t,x,u) & 0 \\ 0 & 0 \\ 0 & \dfrac{\varphi(t) N}{\eta_c y_1} \end{bmatrix}$$

$$\phi(u) = \begin{bmatrix} 0 \\ 0 \\ k_4 C_N^{in} u \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with $G_1(t,x,u) = -\varphi(t) - (1-\varphi(t))K_R(S) + k_2 u$. In the presence of the noise v the discrepancy can be hidden in the disturbance term d. We then consider an observer of the form:

$$\dot{\hat{x}} = A(t)\hat{x} + G(t,\hat{x},y,u(t))\hat{\theta} + L(t)\left(y - C\hat{x}\right) + \Omega\dot{\hat{\theta}} + \phi(u(t))$$
$$\dot{\Omega} = (A(t) - L(t)C)\Omega + G(t,\hat{x},y,u(t))$$
$$\dot{\hat{\theta}} = \gamma \Omega^\top C^\top \left(y - C\hat{x}\right)$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \\ \Omega_5 & \Omega_6 \end{bmatrix}, \quad L(t) = \begin{bmatrix} L_1(t) & 0 \\ 0 & L_2(t) \\ 0 & L_3(t) \end{bmatrix}$$

Note that A writes

$$A(t) = \frac{\varphi(t)}{y_1(t)} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\eta_c} \\ 0 & 0 & -\dfrac{\kappa}{\eta_c} \end{bmatrix}}_{\tilde{A}},$$

We can choose $L(t) = \dfrac{\varphi(t)}{y_1(t)}\tilde{L}$, where the constant matrix $\tilde{L}$ is such that $\tilde{A} - \tilde{L}C$ is Hurwitz. Finally, the equations of the observer (4) are

$$\dot{\hat{S}} = \hat{k}_1 \left(-\varphi(t) - (1-\varphi(t))K_R(\hat{S}) + k_2 u(t)\right) + \frac{\varphi(t)}{y_1(t)}\tilde{L}_1(y_1(t)-\hat{S}) + \Omega_1\dot{\hat{k}}_1 - \Omega_2\dot{\hat{k}}_3$$

$$\dot{\hat{B}} = \frac{\varphi(t)}{\eta_c y_1(t)}\hat{N} + \frac{\varphi(t)}{y_1(t)}\tilde{L}_2\left(y_2(t)-\hat{B}\right) + \Omega_3\dot{\hat{k}}_1 - \Omega_4\dot{\hat{k}}_3$$

$$\dot{\hat{N}} = -\hat{k}_3\frac{\varphi(t)}{\eta_c y_1}\hat{N} + \frac{\varphi(t)}{y_1(t)}\tilde{L}_3\left(y_2(t)-\hat{B}\right) + \Omega_5\dot{\hat{k}}_1 - \Omega_6\dot{\hat{k}}_3$$

$$\dot{\Omega}_1 = -\frac{\varphi(t)}{y_1(t)}\tilde{L}_1\Omega_1 - \varphi(t) - (1-\varphi(t))K_R(\hat{S}) + k_2 u(t)$$

$$\dot{\Omega}_2 = -\frac{\varphi(t)}{y_1(t)}\tilde{L}_1\Omega_2$$

$$\dot{\Omega}_3 = \frac{\varphi(t)}{y_1(t)}\left(-\tilde{L}_2\Omega_3 + \frac{\Omega_5}{\eta_c}\right)$$

$$\dot{\Omega}_4 = \frac{\varphi(t)}{y_1(t)}\left(-\tilde{L}_2\Omega_4 + \frac{\Omega_6}{\eta_c}\right]$$

$$\dot{\Omega}_5 = \frac{\varphi(t)}{y_1(t)}\left(-\tilde{L}_3\Omega_3 - \frac{\kappa\Omega_5}{\eta_c}\right)$$

$$\dot{\Omega}_6 = \frac{\varphi(t)}{y_1(t)}\left(-\tilde{L}_3\Omega_4 - \frac{\kappa\Omega_6}{\eta_c}\right) + \frac{\varphi(t)}{\eta_c y_1(t)}\hat{N}$$

$$\dot{\hat{k}}_1 = \gamma\left(\Omega_1(y_1(t)-\hat{S}) + \Omega_3(y_2(t)-\hat{B})\right)$$

$$\dot{\hat{k}}_3 = -\gamma\left(\Omega_2(y_1(t)-\hat{S}) + \Omega_4(y_2(t)-\hat{B})\right)$$

Let us underline that this observer allows to reconstruct the parameter k_1 as well, without the need of the former observer.

V. Numerical simulations

For the simulations, we have considered the following set of parameters

k_1	k_2	k_3	k_4	Γ	S^*	S_w	S_h	L	S_0
1.2	5.5	2.5	1.7	3.10^5	0.5	0.2	0.1	1.2	1

with the function

$$\varphi(t) = \frac{2t}{2t+3}$$

and the control

$$u(t) = \max\left(0, \frac{t}{t^2+1} + \frac{\cos(t/2)+\sin(\pi t)}{2} + \frac{\cos(3et)}{4}\right)$$

A. Estimation of k_1, k_2 and S_h

Figure 1 illustrates the adaptive observer given by Proposition 1, with noise.

B. Estimation of N, k_1 and k_3

For the adaptive observer (4), the following tuning has been chosen

C_N^{in}	γ	η_c	$\tilde{L}_1$	$\tilde{L}_2$	$\tilde{L}_3$	Ω_0
1.5	150	0.8	L_1	$0.5\tilde{L}_1$	$0.5\tilde{L}_2$	$0_{3\times 2}$

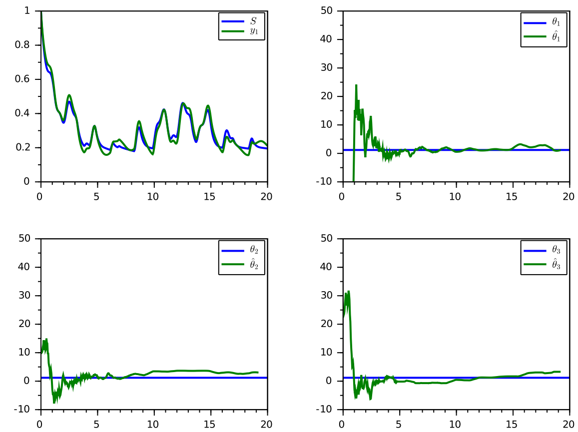

Fig. 1. Adaptive observer for problem A with a 5% noise

$\hat{S}_0$	$\hat{B}_0$	$\hat{N}_0$
$0.5S_0$	$0.5B_0$	$0.5N_0$

Figures 2, 3 show the adaptive observer (4) without and with noise.

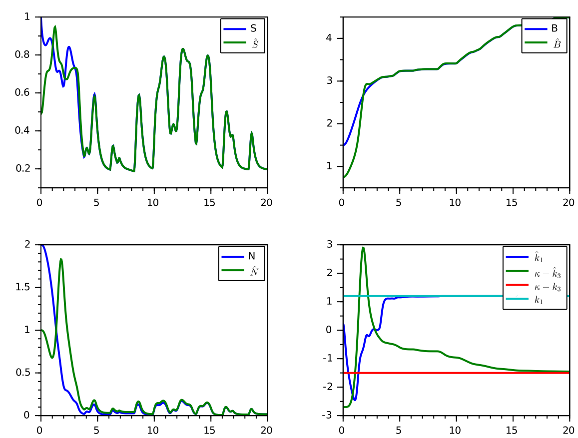

Fig. 2. Adaptive observer for problem B without noise

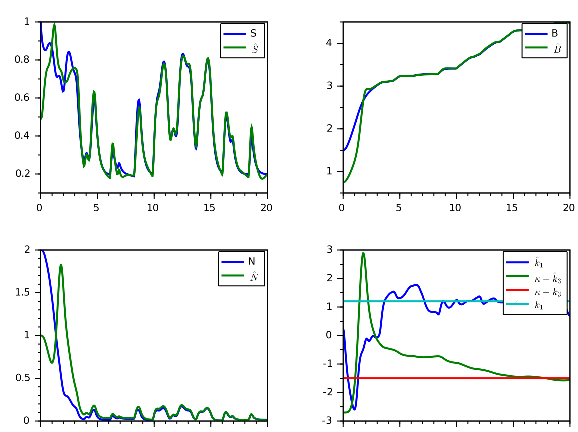

Fig. 3. Adaptive observer for problem B with 5% noise

C. *Comparison with other approaches*

Problem A is a pure identification problem. One can then use a (recursive) least square method to estimate parameters parameters k_1, k_2 and S_h from the measurement y_1. We have use the `lsqrsolve` function of `scilab` based on the Levenberg-Marquardt algorithm. Figures 4, 5 shows that this method works well without noise but is not robust with measurement noise.

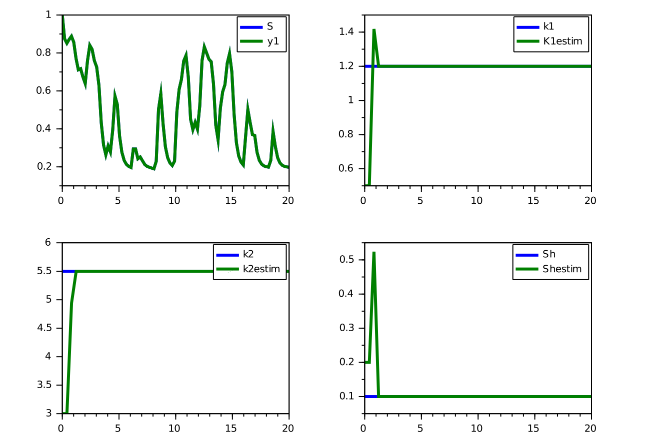

Fig. 4. Least-square estimatuion for problem B without noise

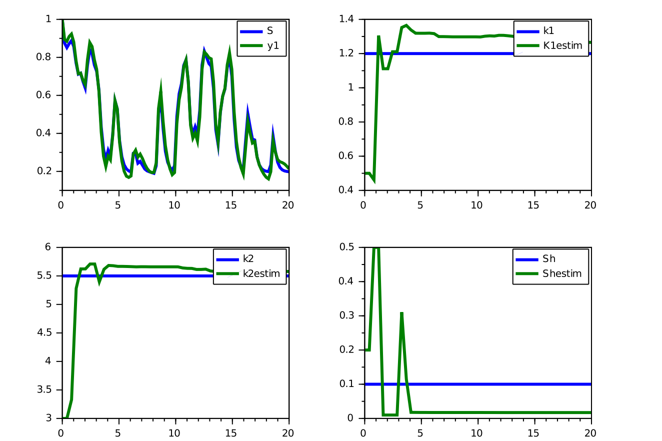

Fig. 5. Least-square estimation for problem B with 5% noise

For Problem B, let us show how to use a classical high-gains observer, assuming that the unknown parameter k_3 belongs to a given interval $[k_3^{min}, k_3^{max}]$. We first note from equation (7) that N cannot reach 0 in finite time. One can then define the variable $Z = -k_3 N$, and write the dynamics of the system (B, N, Z) as follows

$$
\begin{aligned}
\dot{B} &= f(t)N \\
\dot{N} &= f(t)Z + k_4 C_N^{in} u(t) \\
\dot{Z} &= f(t)\left(\frac{Z}{N}\right)Z - \left(\frac{Z}{N}\right) k_4 C_N^{in} u(t)
\end{aligned}
$$

where we posit $f(t) = \frac{\varphi(t)\eta_C}{K_S(y_1(t))y_1(t)}$. Note that this last function is zero only at $t = 0$. Then, for any control $u(\cdot)$, we can define $\tilde{u}(\cdot)$ such that $k_4 C_N^{in} u(t) = f(t)\tilde{u}(t)$ for almost all $t \geq 0$. Note also that along any trajectory one can replace $\frac{Z}{N}$ by a globally bounded function on $\mathbb{R}^2$:

$$
\rho(N, Z) = -\max\left(k_3^{\min}, \min\left(\frac{Z}{\max(N, \varepsilon)}, k_3^{max}\right)\right)
$$

where $\varepsilon > 0$ is an arbitrary small number to ensure $N > \varepsilon$ on the time window. Then, the system takes the form

$$
\begin{bmatrix} \dot{B} \\ \dot{N} \\ \dot{Z} \end{bmatrix} = f(t)\begin{bmatrix} N \\ Z \\ \rho(N,Z)Z \end{bmatrix} + f(t)\tilde{u}(t)\begin{bmatrix} 0 \\ 1 \\ \rho(N,Z) \end{bmatrix}
$$

which is exactly in the nonlinear canonical form in time $\tau = \int_0^t f(s)ds$ (see [13]). Therefore, the observer

$$
\begin{aligned}
\dot{\hat{B}} &= f(t)\hat{N} - 2\chi f(t)(y_2(t) - \hat{B}) \\
\dot{\hat{N}} &= f(t)\hat{Z} - k_4 C_N^{in} u(t) - 3\chi^2 f(t) G_2(y_2(t) - \hat{B}) \\
\dot{\hat{Z}} &= \rho(\hat{N}, \hat{Z})\left(f(t)\hat{Z} - k_4 C_N^{in} u(t)\right) - 3\chi^3 f(t) G_3(y_2(t) - \hat{B}) \\
\hat{k}_3 &= -\rho(\hat{N}, \hat{Z})
\end{aligned}
$$

ensures an exponential convergence provided that the tuning coefficient $\chi > 0$ is sufficiently large. Figure 6 shows good performance of this observer, comparable to the adaptive observer for $\chi = 3$, but unfortunately it is poorly robust with respect to noise (Figure 7), which makes it unreliable in practice. For both problems, we conclude about the superiority of the adaptive observers.

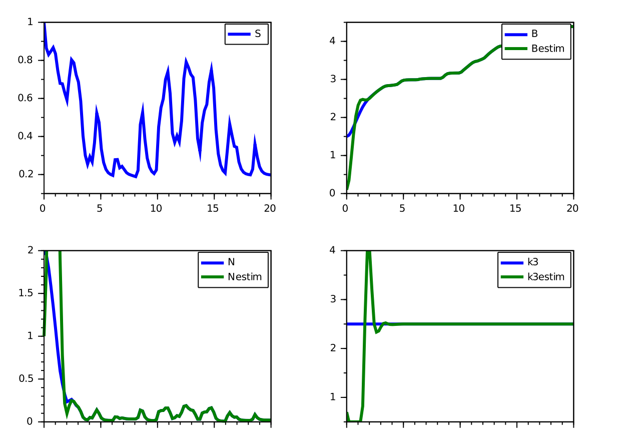

Fig. 6. High gains observer for problem B without noise

VI. Conclusion

In this paper, an adaptive observer has been proposed for a class of nonlinear time-varying systems with disturbances and measurement noises. This one extends the existing results by considering a generic and the most complex scenario. The analysis of convergence and robustness of this type of adaptive observer is specially based on a SIIOS-Lyapunov function method. A crop

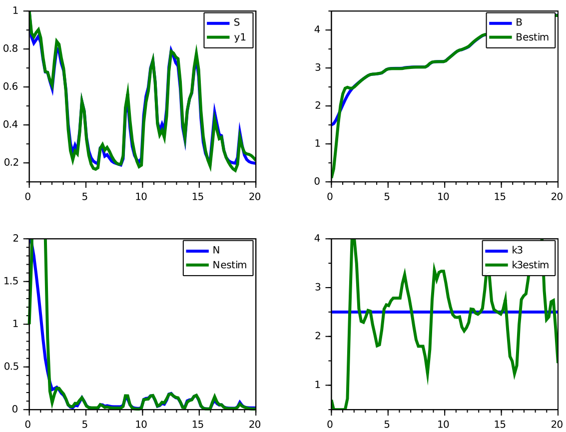

Fig. 7. High gains observer for problem B with 5% noise

irrigation model from the agronomy has been used for illustration of the efficiency of the observer by a numerical simulation.

Appendix

We recall classical results of the literature about stability of non-autonomous linear systems.

Definition 4 (Persistent excitation): Let $\phi : \mathbb{R}_+ \to \mathbb{R}^{m\times n}$ be a continuous bounded function. We say that ϕ is *persistently exciting* if there exists $\overline{\alpha} > \underline{\alpha} > 0$, $T^* > 0$ such that

$$\overline{\alpha} I_m \geq \int_t^{t+T^*} \phi(s)\phi(s)^\top ds \geq \underline{\alpha} I_m, \; \forall t \geq 0.$$

Theorem 3 ([12]): If ϕ is a persistently exciting function, $A(t) \in \mathbb{R}^{n\times n}$ is a bounded piece-wise continuous matrix, and $P(t) = P^\top(t) \succ 0$ a bounded continuous-time matrix in $\mathbb{R}^{n\times n}$ such that

$$\dot{P} + PA + A^\top P \prec 0,$$

for all $t \geq 0$, then the non-autonomous linear system

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} A & -\phi^\top \\ \phi P & 0 \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}$$

is exponentially stable.

References

[1] F.A. Karam, F. Mouneimne, F. Aichouche, A. Rapaport, J. Harmand *Processing Soil Water Content Measurements to Estimate Crop Water - Requirements and Optimization of Irrigation Supply*, CMWRS19, 20-21 September, Delft, the Netherlands, 2019.

[2] G. Bastin, M. R. Gevers, *Stable adaptive observers for nonlinear time-varying systems*, in IEEE Transactions on Automatic Control, vol. 33, no. 7, pp. 650-658, 1988, doi: 10.1109/9.1273.

[3] G. Besançon, *Remarks on nonlinear adaptive observer design*, Systems & Control Letters, vol. 41, no. 4, pp. 271-280, 2000.

[4] N. Brisson, C. Gary, E. Justes, R. Roche, B. Mary, D. Ripoche, D. Zimmer, J. Sierra, P. Bertuzzi, P. Burger et al. *An Overview of the Crop Model STICS.* Eur. J. Agron. Vol. 18, 309–332, 2003.

[5] K. Boumaza, N. Kalboussi, A. Rapaport, S. Roux, C. Sinfort *Optimal control of a crop irrigation model under water scarcity* Optimal Control, Applications and Methods, Vol. 42, N. 6, pp. 1612–1631, 2021.

[6] H. Cox, *On the estimation of state variables and parameters for noisy dynamic systems*, IEEE Transactions on Automatic Control, vol. AC-9, pp. 5-12, 1964.

[7] B. Cheviron, R.W. Vervoort, R. Albasha, R. Dairon, C. le Priol, J.C. Mailhol, *Framework to Use Crop Models for Multi-Objective Constrained Optimization of Irrigation Strategies*, Environ. Model. Softw. Vol 86, pp. 145–157, 2016.

[8] D. Efimov, A. Fradkov, *Design of impulsive adaptive observers for improvement of persistency of excitation*, International Journal of Adaptive Control and Signal Processing, vol. 29, pp. 765–782, 2015.

[9] R. Fernández-Escobar, J.M. García-Novelo, C. Molina-Soria, M.A. Parra, *An approach to nitrogen balance in olive orchards*, Sci. Hortic. Vol 135, pp. 219–226, 2012.

[10] M. Farza, M. M Saad, T. Maatoug, M. Kamoun, *Adaptive observers for nonlinear parameterized class of nonlinear systems*, Automatica, vol. 45, pp. 2292-2299, 2009.

[11] G. Besançon, *editor. Nonlinear observers and applications*, volume 363 of Lecture Notes in Control and Information Sciences. Springer Verlag, Berlin, 2007.

[12] A.P. Morgan and K.S Narendra, *On the stability of nonautonomous differential equations $\dot{x} = [A(t) + B(t)]$ with skew symmetric matrix $B(t)$**, SIAM J. Control and Optimization, Vol. 15, No. 1, 1977.

[13] J.-P. Gauthier, I. Kupka, *Deterministic Observation Thoery and Applications*, Cambridge University Press, 2004.

[14] C.W. Klar, P. Fiener, P. Neuhaus, V.I.S. Lenz-Wiedemann, K. Schneider, *Modelling of Soil Nitrogen Dynamics within the Decision Support System DANUBIA*, Ecological Modelling, Vol. 217, pp. 181–196, 2008.

[15] G. Lüders and K. S. Narendra, *An adaptive observer and identifier for a linear system*, IEEE Transactions on Automatic Control, vol. AC-18, pp. 496-499, 1973.

[16] R. Marino and P. Tomei, *Global adaptive observers for nonlinear systems via filtered transformations*, IEEE Transactions on Automatic Control, vol. 37, no. 8, pp. 1239-1245, 1992, doi: 10.1109/9.151117.

[17] N. Pelak, R. Revelli, A. Porporato, *A Dynamical Systems Framework for Crop Models: Toward Optimal Fertilization and Irrigation Strategies under Climatic Variability.* Ecological Modelling, vol. 365, pp. 80–92, 2017.

[18] Z. Pu, R. Yuan, J. Yi, X. Tan, *A class of adaptive extended state observers for nonlinear disturbed systems.* IEEE Transactions on Industrial Electronics, Vol. 62; pp. 5858-5869, 2015.

[19] P. Steduto, T.C. Hsiao, D. Raes, E. Fereres, *AquaCrop – The FAO Crop Model to Simulate Yield Response to Water: I. Concepts Underlying Principles*, Agron. J. 2009, Vol. 101, pp. 426–437, 2009.

[20] E.D. Sontag, *On the input-to-state stability property*, European Journal of control, Vol. 1, pp. 24–36, 1995.

[21] Sontag, E.D. The ISS philosophy as a unifying framework for stability-like behavior. In: Isidori, A., Lamnabhi-Lagarrigue, F., Respondek, W. (eds) Nonlinear control in the year 2000 volume 2. Lecture Notes in Control and Information Sciences, vol 259. Springer, London, 2001.

[22] Q. Zhang, *Adaptive observer for multiple-input-multiple-output (MIMO) linear time-varying systems*, in IEEE Transactions on Automatic Control, vol. 47, no. 3, pp. 525-529, March 2002, doi: 10.1109/9.989154.