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# An Unified Framework for Measuring Environmentally-adjusted Productivity Change: Theoretical Basis and Empirical Illustration\*

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## Abstract

This paper aims to define an unified framework to analyse environmentally-adjusted productivity change. Equivalence conditions for additive and multiplicative environmentally-adjusted productivity measures are highlighted. Besides, an empirical illustration is provided considering non parametric convex neutral by-production model.

**Keywords:** Data Envelopment Analysis, Convex Neutral Production Model, Distance Functions, Environmentally-adjusted Productivity Indices, Pollution-generating Technology.

**JEL:** D21, D24

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# 1 Introduction

As prominent economic health indicator, productivity measure is usually defined as index number evaluating the performance of economic systems (OECD, 2001). In the literature focusing on productivity studies, the production theoretic approach to index numbers is principally adopted (Prasada Rao, 2020; Caves et al., 1982). In this theoretical area, two approaches are traditionally laid out, namely the multiplicative-based approach and the additive-based one (Briec and Kerstens, 2004; Chambers, 2002; Bjurek, 1996; Färe et al., 1994). Specifically, the multiplicative productivity indices are defined as ratios of multiplicative distance functions (Shephard, 1970; Debreu, 1951; Farrell, 1957) whilst the additive productivity measures are defined as difference-based indicators of directional distance functions (Briec, 1997; Chambers et al., 1996)<sup>1</sup>. Approximation results are provided in the literature establishing connections between the usual multiplicative and additive productivity measures (Briec and Kerstens, 2004; Boussemart et al., 2003). These outcomes are of particular interest to interpret differences of empirical findings that result in employing either additive- or multiplicative-based productivity indices.

The production theoretic approach to index numbers has been mainly designed through unidimensional framework, especially focusing on economic components of the evaluated economic systems (Färe et al., 1994). In the last decades, numerous paper incorporate environmental dimension to appraise productivity changes (Sueyoshi et al., 2017; Zhou et al., 2008). In this line, both additive and multiplicative environmental productivity measures have been defined (Abad and Ravelojaona, 2022, 2021; Azad and Ancev, 2014; Chung et al., 1997). Specifically, multiplicative environmental productivity measures inherit the structure of the Malmquist (Caves et al., 1982) and the Hicks-Moorsteen (Bjurek, 1996) productivity indices. Moreover, additive environmental productivity indicators take the form of the Luenberger (Chambers, 2002) and the Luenberger-Hicks-Moorsteen (Briec and Kerstens, 2004) productivity indicators.

This contribution establishes equivalence conditions for additive and multiplicative

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<sup>1</sup>The main differences between the additive and the multiplicative approaches are presented in Briec and Kerstens (2004), Chambers (1998, 2002) and Diewert (1998), among others.

environmentally-adjusted<sup>2</sup> productivity measures. These equivalence conditions extend the usual approximation results that linked the traditional additive and multiplicative productivity measures (Briec and Kerstens, 2004; Boussemart et al., 2003). Particularly, this contribution highlights specific conditions for which environmentally-adjusted Malmquist and Luenberger productivity measures are equivalent to each other. Moreover, theoretical relation between environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures are introduced. As a result, additive and multiplicative environmentally-adjusted productivity measures are exactly related to each other providing an unified framework for empirical analysis.

Interestingly, this contribution also permits to present additive version of Malmquist and Hicks-Moorsteen environmentally-adjusted productivity indices. This result extends the widely applied Chung et al. (1997) methodology, which provides theoretical background to consider desirable and undesirable components in traditional multiplicative-based productivity measure. Moreover, multiplicative version of Luenberger and Luenberger-Hicks-Moorsteen environmentally-adjusted productivity indicators are provided. This outcome may be understood as a reciprocal result of the Chung et al. (1997) model, handling the issue of joint producing good and bad components in productivity assessment through multiplicative productivity indices.

The applicability of the unified framework presented in this paper is displayed through an empirical illustration based upon the database sourced from the work of Jeon and Sickles (2004). Precisely, the main findings of this paper are illustrated by considering a non parametric convex neutral by-production model. In this line, multiplicative version of the by-production model (Abad and Briec, 2019; Murty et al., 2012; Banker and Maindiratta, 1986) is considered to provide non parametric estimation of the main results highlighted in this paper.

The remainder of this paper unfolds as follows. Section 2 introduces theoretical preliminaries. The pollution-generating production process and the distance functions are also presented in this section. Section 3 displays multiplicative and additive environmentally-

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<sup>2</sup>To distinguish from the measures existing in the literature (environmental measures), the measures proposed in this paper are designated as “environmentally-adjusted” ones.

adjusted productivity measures and further, it provides equivalence conditions for the additive and the multiplicative environmentally-adjusted productivity measures. An empirical illustration is provided in Section 4, highlighting the applicability of the unified framework introduced in this paper. Finally, section 5 concludes.

## 2 Background

In this section, the properties of the pollution-generating production process are presented. Based upon this theoretical background, additive and multiplicative distance functions are displayed and further, equivalence condition between the proposed distance functions is laid out.

### 2.1 Technology definition and properties

Let  $\mathbf{x}_t = (x_t^{\text{ec}}, x_t^{\text{em}}) \in \mathbb{R}_+^n$  denotes the economic factors and the emission-generating inputs used to produce both economic (*i.e.*, intended) and emission-generating (*i.e.*, unintended) outputs  $\mathbf{y}_t = (y_t^{\text{ec}}, y_t^{\text{em}}) \in \mathbb{R}_+^m$ , such that  $n = n_{\text{ec}} + n_{\text{em}}$  and  $m = m_{\text{ec}} + m_{\text{em}}$ .

The pollution-generating technology is defined as follows:

$$T_t := \{(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_+^{n+m} : \mathbf{x}_t \text{ can produce } \mathbf{y}_t\} \quad (2.1)$$

Assume that the production process satisfies the following usual assumptions (Färe et al, 1985):

**T1.** -No free lunch and inaction-  $(0, 0) \in T_t$ ,  $(0, \mathbf{y}_t) \in T_t \Rightarrow \mathbf{y}_t = 0$ ;

**T2.** -Boundedness-  $T_{\mathbf{y}_t} := \{(\mathbf{u}_t, \mathbf{v}_t) \in T_t : \mathbf{v}_t \leq \mathbf{y}_t\}$  is bounded for any  $\mathbf{y}_t \in \mathbb{R}_+^m$ ;

**T3.** -Closedness-  $T_t$  is closed;

Moreover, suppose that for any free disposal cone  $K := \mathbb{R}_+^n \times -\mathbb{R}_+^m$ , the production set satisfies the generalised  $B$ -disposal property:

**T4.** -Generalised  $B$ -disposability-  $T_t := \left( (T_t + K) \cap (T_t + (\mathbb{R}_+^{n_{\text{ec}} + m_{\text{em}}} \times -\mathbb{R}_+^{n_{\text{em}} + m_{\text{ec}}})) \right) \cap (\mathbb{R}_+^n \times \mathbb{R}_+^m)$ .

The axiomatic framework  $T1-T4$  permits to define the production process as an intersection of sub-technologies (Abad and Briec, 2019; Murty et al., 2012). Specifically, the economic (*i.e.*, intended) production activities satisfy the usual free disposal property *-i.e.*,  $(T_t + (\mathbb{R}_+^n \times -\mathbb{R}_+^m)) \cap (\mathbb{R}_+^n \times \mathbb{R}_+^m)$  whilst partially reversed free disposal axiom applies for the polluting (*i.e.*, unintended) production activities; *i.e.*,  $(T_t + (\mathbb{R}_+^{n_{ec}+m_{em}} \times -\mathbb{R}_+^{n_{em}+m_{ec}})) \cap (\mathbb{R}_+^n \times \mathbb{R}_+^m)$ . Remarkably, the theoretical model  $T1-T4$  is fairly weak and do not impose any convexity assumption such that convex neutral production model may be considered.

## 2.2 Technology characterisation: distance functions

In this section, additive and multiplicative distance functions are considered as functional representation of the production process (Chambers and Färe, 2020).

The next result presents additive and multiplicative environmentally-adjusted distance functions.

**Definition 2.1** *Let  $T_t$  be a production technology that satisfies properties  $T1-T4$ . For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_+^{n+m}$ :*

**i.** *The multiplicative environmentally-adjusted distance function is defined as follows,*

$$D_t^{\alpha;\beta}(\mathbf{x}_t, \mathbf{y}_t) := \sup_{\lambda} \left\{ \lambda \geq 1 : (\lambda^{-\alpha} \mathbf{x}'_t, \lambda^{\beta} \mathbf{y}'_t) \in T_t \right\} \quad (2.2)$$

where  $\alpha = (\alpha^{ec}, \alpha^{em}) \in \mathbb{R}_+^n$  and  $\beta = (\beta^{ec}, \beta^{em}) \in \mathbb{R}_+^{m_{ec}} \times \mathbb{R}_-^{m_{em}}$ , such that  $\lambda^{-\alpha} = \text{diag}(\lambda^{-\alpha^{ec}}, \lambda^{-\alpha^{em}})$  and  $\lambda^{\beta} = \text{diag}(\lambda^{\beta^{ec}}, \lambda^{\beta^{em}})$ .

**ii.** *The additive environmentally-adjusted distance function is defined as follows,*

$$\vec{D}_t^{\gamma;\sigma}(\mathbf{x}_t, \mathbf{y}_t) := \sup_{\delta} \left\{ \delta \geq 0 : ((I_n - \delta\gamma) \mathbf{x}'_t, (I_m + \delta\sigma) \mathbf{y}'_t) \in T_t \right\} \quad (2.3)$$

where  $\gamma = (\gamma^{ec}, \gamma^{em}) \in \mathbb{R}_+^n$  and  $\sigma = (\sigma^{ec}, \sigma^{em}) \in \mathbb{R}_+^{m_{ec}} \times \mathbb{R}_-^{m_{em}}$ , with  $\gamma = \text{diag}(\gamma^{ec}, \gamma^{em})$  and  $\sigma = \text{diag}(\sigma^{ec}, \sigma^{em})$ .

In statements **i.** and **ii.**,  $\lambda^{-\alpha}$ ,  $\lambda^{\beta}$ ,  $\gamma$  and  $\sigma$  are such that  $\dim(\lambda^{-\alpha}) = \dim(\gamma) = n \times n$  and  $\dim(\lambda^{\beta}) = \dim(\sigma) = m \times m$ . Moreover,  $I_n$  and  $I_m$  are to the identity matrix of order  $n$  and  $m$ , respectively.

The additive and multiplicative environmentally-adjusted distance functions fully characterise the production process such that<sup>3</sup>:

$$D_t^{\alpha;\beta}(\mathbf{x}_t, \mathbf{y}_t) \geq 1 \Leftrightarrow (\mathbf{x}_t, \mathbf{y}_t) \in T_t \Leftrightarrow \overrightarrow{D}_t^{\gamma;\sigma}(\mathbf{x}_t, \mathbf{y}_t) \geq 0.$$

Interestingly, the general shape of the additive and multiplicative environmentally-adjusted distance functions allows comparisons with existing environmental distance functions (Abad and Ravelojaona, 2022; Picazo-Tadeo et al., 2014; Färe et al., 2004; Chung et al., 1997; Färe et al., 1989). These connections are presented in the Proposition 5.1 (see Appendix I).

### 2.3 Environmentally-adjusted distance functions: equivalence condition

Let  $T_t^{++} := T_t \cap \mathbb{R}_{++}^{n+m}$  be the strictly positive production set. The next result defines equivalence condition for the additive and multiplicative environmentally-adjusted distance functions.

**Proposition 2.2** *For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m}$ , the equivalence condition for the additive and multiplicative environmentally-adjusted distance functions is defined as follows:*

$$\overrightarrow{D}_t^{\gamma;\sigma}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)) \equiv \ln(D_t^{\alpha;\beta}(\mathbf{x}_t, \mathbf{y}_t)) \quad (2.4)$$

where  $\overrightarrow{D}_t^{\gamma;\sigma}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)), \ln(D_t^{\alpha;\beta}(\mathbf{x}_t, \mathbf{y}_t)) \in \ln(T_t^{++})$ , such that  $\gamma = \left( \frac{\alpha^{ec}}{\ln(x_t^{ec})}, \frac{\alpha^{em}}{\ln(x_t^{em})} \right)$  and  $\sigma = \left( \frac{\beta^{ec}}{\ln(y_t^{ec})}, \frac{\beta^{em}}{\ln(y_t^{em})} \right)$ .

See Appendix II for the proof.

The equivalence result highlighted in Proposition 2.2 permits to exactly relate the additive and multiplicative environmentally-adjusted distance functions to one another. Interestingly, this outcome is applied by considering existing environmental distance functions (Abad and Ravelojaona, 2022; Picazo-Tadeo et al., 2014; Färe et al., 2004; Chung et al., 1997; Färe et al., 1989); see Proposition 5.2 in Appendix I.

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<sup>3</sup>The basic properties of the additive- and multiplicative-based distance functions are laid out in Mehdiloozad et al. (2014), Peyrache and Coelli (2009) and Briec (1997).

### 3 Environmentally-adjusted productivity measures

This section displays equivalence conditions between additive- and multiplicative-based environmentally-adjusted productivity measures. In addition, additive versions of multiplicative-based environmentally-adjusted productivity measures are introduced. Reciprocal results laying out multiplicative versions of additive-based environmentally-adjusted productivity measures are also provided extending the widely applied Chung *et al.* (1997) methodology.

#### 3.1 Malmquist and Luenberger productivity measures: equivalence condition

The next result presents the environmentally-adjusted Malmquist and Luenberger productivity measures.

**Definition 3.1** *Assume that  $T_{t,t+1}$  is a production process that satisfies properties T1–T4. For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_+^{n+m}$  and for any  $i, k = \{\text{ec}, \text{em}\}$  with  $k \neq i$ ,*

i. *The environmentally-adjusted Malmquist productivity index is defined as follows,*

$$\text{EAM}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) = \left[ \prod_{i=\{\text{ec}, \text{em}\}} \frac{D_t^{\alpha^i; \beta^i}(\mathbf{x}_t, \mathbf{y}_t)}{D_t^{\alpha^i; \beta^i}(x_{t+1}^i, x_t^k, y_{t+1}^i, y_t^k)} \times \frac{D_{t+1}^{\alpha^i; \beta^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1})}{D_{t+1}^{\alpha^i; \beta^i}(x_t^i, x_{t+1}^k, y_t^i, y_{t+1}^k)} \right]^{\frac{1}{2}} \quad (3.1)$$

where  $\alpha = (\alpha^{\text{ec}}, \alpha^{\text{em}}) \in \mathbb{R}_+^n$  and  $\beta = (\beta^{\text{ec}}, \beta^{\text{em}}) \in \mathbb{R}_+^{m_{\text{ec}}} \times \mathbb{R}_-^{m_{\text{em}}}$ .

ii. *The environmentally-adjusted Luenberger productivity indicator is defined as follows,*

$$\text{EAL}_{t,t+1}^{\gamma;\sigma}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) = \frac{1}{2} \left[ \sum_{i=\{\text{ec}, \text{em}\}} \left( \overrightarrow{D}_t^{\gamma^i; \sigma^i}(\mathbf{x}_t, \mathbf{y}_t) - \overrightarrow{D}_t^{\gamma^i; \sigma^i}(x_{t+1}^i, x_t^k, y_{t+1}^i, y_t^k) \right) + \left( \overrightarrow{D}_{t+1}^{\gamma^i; \sigma^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}) - \overrightarrow{D}_{t+1}^{\gamma^i; \sigma^i}(x_t^i, x_{t+1}^k, y_t^i, y_{t+1}^k) \right) \right] \quad (3.2)$$

where  $\gamma = (\gamma^{\text{ec}}, \gamma^{\text{em}}) \in \mathbb{R}_+^n$  and  $\sigma = (\sigma^{\text{ec}}, \sigma^{\text{em}}) \in \mathbb{R}_+^{m_{\text{ec}}} \times \mathbb{R}_-^{m_{\text{em}}}$ .

The proposition below introduces equivalence condition for the environmentally-adjusted Malmquist and Luenberger productivity measures.



**Proposition 3.2** For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_{++}^{n+m}$ , equivalence condition for the environmentally-adjusted Malmquist and Luenberger productivity measures is as follows:

$$\ln \left( \text{EAM}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \right) \equiv \text{EAL}_{t,t+1}^{\gamma;\sigma} \left( \ln(\mathbf{x}_{t,t+1}), \ln(\mathbf{y}_{t,t+1}) \right), \quad (3.3)$$

with  $\ln \left( \text{EAM}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \right), \text{EAL}_{t,t+1}^{\gamma;\sigma} \left( \ln(\mathbf{x}_{t,t+1}), \ln(\mathbf{y}_{t,t+1}) \right) \in \ln(T_{t,t+1}^{++})$  such that  $\gamma = \left( \frac{\alpha^{\text{ec}}}{\ln(x_{t,t+1}^{\text{ec}})}, \frac{\alpha^{\text{em}}}{\ln(x_{t,t+1}^{\text{em}})} \right)$  and  $\sigma = \left( \frac{\beta^{\text{ec}}}{\ln(y_{t,t+1}^{\text{ec}})}, \frac{\beta^{\text{em}}}{\ln(y_{t,t+1}^{\text{em}})} \right)$ .

In the next statement, additive (respectively, multiplicative) version of the environmentally-adjusted Malmquist (respectively, Luenberger) productivity measure is proposed. The additive version of the environmentally-adjusted Malmquist productivity index is defined through additive distance functions. This productivity measure is named the environmentally-adjusted Malmquist-Luenberger productivity index based upon the initial work of Chung *et al.* (1997). Besides, the reciprocal environmentally-adjusted Malmquist-Luenberger productivity measure proposes a multiplicative version of the environmentally-adjusted Luenberger productivity indicator.

**Corollary 3.3** For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_{++}^{n+m}$ , and any  $\gamma^i = \frac{\alpha^i}{\ln(x_{t,t+1}^i)}$  and  $\sigma^i = \frac{\beta^i}{\ln(y_{t,t+1}^i)}$  with  $i, k = \{\text{ec}, \text{em}\}$  and  $i \neq k$  :

i. The environmentally-adjusted Malmquist-Luenberger productivity index is defined as follows,

$$\text{EAML}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) = \left[ \prod_{i=\{\text{ec}, \text{em}\}} \frac{\exp \left( \vec{\text{D}}_t^{\gamma^i; \sigma^i}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)) \right)}{\exp \left( \vec{\text{D}}_t^{\gamma^i; \sigma^i}(\ln(x_t^i), \ln(x_t^k), \ln(y_t^i), \ln(y_t^k)) \right)} \times \frac{\exp \left( \vec{\text{D}}_{t+1}^{\gamma^i; \sigma^i}(\ln(\mathbf{x}_{t+1}), \ln(\mathbf{y}_{t+1})) \right)}{\exp \left( \vec{\text{D}}_{t+1}^{\gamma^i; \sigma^i}(\ln(x_t^i), \ln(x_{t+1}^k), \ln(y_t^i), \ln(y_{t+1}^k)) \right)} \right]^{\frac{1}{2}} \quad (3.4)$$

ii. The reciprocal environmentally-adjusted Malmquist-Luenberger productivity measure is defined as follows,

$$\overline{\text{EAML}}_{t,t+1}^{\gamma;\sigma}(\ln(\mathbf{x}_{t,t+1}), \ln(\mathbf{y}_{t,t+1})) = \frac{1}{2} \left[ \sum_{i=\{\text{ec}, \text{em}\}} \left( \ln \left( \text{D}_t^{\alpha^i; \beta^i}(\mathbf{x}_t, \mathbf{y}_t) \right) - \ln \left( \text{D}_t^{\alpha^i; \beta^i}(x_{t+1}^i, x_t^k, y_{t+1}^i, y_t^k) \right) \right) + \left( \ln \left( \text{D}_{t+1}^{\alpha^i; \beta^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}) \right) - \ln \left( \text{D}_{t+1}^{\alpha^i; \beta^i}(x_t^i, x_{t+1}^k, y_t^i, y_{t+1}^k) \right) \right) \right] \quad (3.5)$$

## 3.2 Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures: equivalence condition

The upcoming statement displays the environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures.

**Definition 3.4** Let  $T_{t,t+1}$  be a production process that satisfies properties T1-T4. For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_+^{n+m}$  and for any  $i, k = \{\text{ec}, \text{em}\}$  with  $i \neq k$ ,

i. The environmentally-adjusted Hicks-Moorsteen productivity index is defined as follows,

$$\begin{aligned} \text{EAHM}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) &= \left[ \prod_{i=\{\text{ec}, \text{em}\}} \frac{D_t^{\beta^i}(\mathbf{x}_t, \mathbf{y}_t)}{D_t^{\beta^i}(\mathbf{x}_t, y_{t+1}^i, y_t^k)} \times \frac{D_t^{\alpha^i}(\mathbf{x}_t, \mathbf{y}_t)}{D_t^{\alpha^i}(x_{t+1}^i, x_t^k, \mathbf{y}_t)} \right. \\ &\quad \left. \times \frac{D_{t+1}^{\beta^i}(\mathbf{x}_{t+1}, y_t^i, y_{t+1}^k)}{D_{t+1}^{\beta^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1})} \times \frac{D_{t+1}^{\alpha^i}(x_t^i, x_{t+1}^k, \mathbf{y}_{t+1})}{D_{t+1}^{\alpha^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1})} \right]^{\frac{1}{2}} \end{aligned} \quad (3.6)$$

where  $\alpha = (\alpha^{\text{ec}}, \alpha^{\text{em}}) \in \mathbb{R}_+^n$  and  $\beta = (\beta^{\text{ec}}, \beta^{\text{em}}) \in \mathbb{R}_+^{m_{\text{ec}}} \times \mathbb{R}_-^{m_{\text{em}}}$ .

ii. The environmentally-adjusted Luenberger-Hicks-Moorsteen productivity indicator is defined as,

$$\begin{aligned} \text{EALHM}_{t,t+1}^{\gamma;\sigma}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) &= \frac{1}{2} \left[ \sum_{i=\{\text{ec}, \text{em}\}} \left( \vec{D}_t^{\sigma^i}(\mathbf{x}_t, \mathbf{y}_t) - \vec{D}_t^{\sigma^i}(\mathbf{x}_t, y_{t+1}^i, y_t^k) \right) \right. \\ &\quad + \left( \vec{D}_t^{\gamma^i}(\mathbf{x}_t, \mathbf{y}_t) - \vec{D}_t^{\gamma^i}(x_{t+1}^i, x_t^k, \mathbf{y}_t) \right) \\ &\quad + \left( \vec{D}_{t+1}^{\sigma^i}(\mathbf{x}_{t+1}, y_t^i, y_{t+1}^k) - \vec{D}_{t+1}^{\sigma^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}) \right) \\ &\quad \left. + \left( \vec{D}_{t+1}^{\gamma^i}(x_t^i, x_{t+1}^k, \mathbf{y}_{t+1}) - \vec{D}_{t+1}^{\gamma^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}) \right) \right] \end{aligned} \quad (3.7)$$

where  $\gamma = (\gamma^{\text{ec}}, \gamma^{\text{em}}) \in \mathbb{R}_+^n$  and  $\sigma = (\sigma^{\text{ec}}, \sigma^{\text{em}}) \in \mathbb{R}_+^{m_{\text{ec}}} \times \mathbb{R}_-^{m_{\text{em}}}$ .

Equivalence condition for the environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures is defined in the next result.

**Proposition 3.5** For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_+^{n+m}$ , the equivalence condition for the environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures is defined as:

$$\ln \left( \text{EAHM}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \right) \equiv \text{EALHM}_{t,t+1}^{\gamma;\sigma} \left( \ln(\mathbf{x}_{t,t+1}), \ln(\mathbf{y}_{t,t+1}) \right), \quad (3.8)$$

where  $\ln \left( \text{EAHM}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \right), \text{EALHM}_{t,t+1}^{\gamma;\sigma} \left( \ln(\mathbf{x}_{t,t+1}), \ln(\mathbf{y}_{t,t+1}) \right) \in \ln(T_{t,t+1}^{++})$  such that  $\gamma = \left( \frac{\alpha^{\text{ec}}}{\ln(x_t^{\text{ec}})}, \frac{\alpha^{\text{em}}}{\ln(x_t^{\text{em}})} \right)$  and  $\sigma = \left( \frac{\beta^{\text{ec}}}{\ln(y_t^{\text{ec}})}, \frac{\beta^{\text{em}}}{\ln(y_t^{\text{em}})} \right)$ .

Additive version of the environmentally-adjusted Hicks-Moorsteen productivity index and its reciprocal are presented in the statement below. These productivity measures are named the Hicks-Moorsteen-Luenberger index and the reciprocal Hicks-Moorsteen-Luenberger indicator, respectively.

**Corollary 3.6** For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_{++}^{n+m}$ , and any  $\gamma^i = \frac{\alpha^i}{\ln(x_{t,t+1}^i)}$  and  $\sigma^i = \frac{\beta^i}{\ln(y_{t,t+1}^i)}$  with  $i, k = \{\text{ec}, \text{em}\}$  and  $i \neq k$ :

i. The environmentally-adjusted Hicks-Moorsteen-Luenberger productivity index is defined as follows,

$$\begin{aligned} \text{EAHML}_{t,t+1}^{\alpha;\beta}(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) = & \left[ \prod_{i=\{\text{ec}, \text{em}\}} \frac{\exp\left(\vec{D}_t^{\sigma^i}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t))\right)}{\exp\left(\vec{D}_t^{\sigma^i}(\ln(\mathbf{x}_t), \ln(y_{t+1}^i), \ln(y_t^k))\right)} \times \frac{\exp\left(\vec{D}_t^{\gamma^i}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t))\right)}{\exp\left(\vec{D}_t^{\gamma^i}(\ln(x_{t+1}^i), \ln(x_t^k), \ln(\mathbf{y}_t))\right)} \right. \\ & \left. \times \frac{\exp\left(\vec{D}_{t+1}^{\sigma^i}(\ln(\mathbf{x}_{t+1}), \ln(y_t^i), \ln(y_{t+1}^k))\right)}{\exp\left(\vec{D}_{t+1}^{\sigma^i}(\ln(\mathbf{x}_{t+1}), \ln(\mathbf{y}_{t+1}))\right)} \times \frac{\exp\left(\vec{D}_{t+1}^{\gamma^i}(\ln(x_t^i), \ln(x_{t+1}^k), \ln(\mathbf{y}_{t+1}))\right)}{\exp\left(\vec{D}_{t+1}^{\gamma^i}(\ln(\mathbf{x}_{t+1}), \ln(\mathbf{y}_{t+1}))\right)} \right]^{\frac{1}{2}} \end{aligned} \quad (3.9)$$

ii. The reciprocal environmentally-adjusted Hicks-Moorsteen-Luenberger productivity indicator is defined as follows,

$$\begin{aligned} \overline{\text{EAHML}}_{t,t+1}^{\gamma;\sigma}(\ln(\mathbf{x}_{t,t+1}), \ln(\mathbf{y}_{t,t+1})) = & \frac{1}{2} \left[ \sum_{i=\{\text{ec}, \text{em}\}} \left( \ln\left(D_t^{\beta^i}(\mathbf{x}_t, \mathbf{y}_t)\right) - \ln\left(D_t^{\beta^i}(\mathbf{x}_t, y_{t+1}^i, y_t^k)\right) \right) \right. \\ & + \left( \ln\left(D_t^{\alpha^i}(\mathbf{x}_t, \mathbf{y}_t)\right) - \ln\left(D_t^{\alpha^i}(x_{t+1}^i, x_t^k, \mathbf{y}_t)\right) \right) \\ & + \left( \ln\left(D_{t+1}^{\beta^i}(\mathbf{x}_{t+1}, y_t^i, y_{t+1}^k)\right) - \ln\left(D_{t+1}^{\beta^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1})\right) \right) \\ & \left. + \left( \ln\left(D_{t+1}^{\alpha^i}(x_t^i, x_{t+1}^k, \mathbf{y}_{t+1})\right) - \ln\left(D_{t+1}^{\alpha^i}(\mathbf{x}_{t+1}, \mathbf{y}_{t+1})\right) \right) \right] \end{aligned} \quad (3.10)$$

### 3.3 A class of examples

The general shape of the additive and multiplicative environmentally-adjusted productivity measures allows to provide equivalences with classical additive- and multiplicative-based productivity measures as displayed in the next statements.

**Proposition 3.7** For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_{++}^{n+m}$  and a logarithmically transformed production technology  $\ln(T_{t,t+1}^{++})$ , the environmentally-adjusted Malmquist and Luenberger productivity measures are respectively equivalent to classical Malmquist and Luenberger productivity measures, under some conditions.

Table 1 displays the equivalence conditions between the environmentally-adjusted Malmquist and Luenberger productivity measures and classical Malmquist and Luenberger productivity measures, respectively<sup>4</sup>.

Productivity measures	Parameters				References
$EAL_{t,t+1}^{\gamma;\sigma}(\ln(\cdot)) \equiv$	$\gamma^{ec}$	$\gamma^{en}$	$\sigma^{ec}$	$\sigma^{en}$	
$-\ln(M_{t,t+1}^{ec}(\cdot))$	0	0	$\frac{1}{\ln(y_{t,t+1}^{ec})}$	0	Caves et al. (1982)
$\ln(M_{t,t+1}^{em}(\cdot))$	0	0	0	$-\frac{1}{\ln(y_{t,t+1}^{em})}$	
$\ln(EAM_{t,t+1}^{\alpha;\beta}(\cdot)) \equiv$	$\alpha^{ec}$	$\alpha^{en}$	$\beta^{ec}$	$\beta^{en}$	
$L_{t,t+1}^{ec}(\cdot)$	0	0	$\ln(y_{t,t+1}^{ec})$	0	Chambers (2002)
$L_{t,t+1}^{em}(\cdot)$	0	0	0	$-\ln(y_{t,t+1}^{em})$	

Table 1: Environmentally-adjusted and classical Malmquist and Luenberger productivity measures equivalences.

**Proposition 3.8** *For any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_{++}^{n+m}$  and a transformed production technology  $\ln(T_{t,t+1}^{++})$ , the environmentally-adjusted Luenberger-Hicks-Moorsteen and Hicks-Moorsteen productivity measures are respectively equivalent to classical Luenberger-Hicks-Moorsteen and Hicks-Moorsteen productivity measures, under some conditions.*

Table 2 provides the equivalence conditions between the environmentally-adjusted Total Factor Productivity (TFP) measures and the classical Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures.

<sup>4</sup>Note that the outcomes provided in Table 1 are based upon output oriented Malmquist and Luenberger productivity measures. However, equivalence results can also be defined through input orientation.

Productivity measures	Parameters				References
$EALHM_{t,t+1}^{\gamma;\sigma}(\ln(\cdot)) \equiv$	$\gamma^{ec}$	$\gamma^{en}$	$\sigma^{ec}$	$\sigma^{en}$	
$-\ln(HM_{t,t+1}^{ec}(\cdot))$	$\frac{1}{\ln(x_{t,t+1}^{ec})}$	0	$\frac{1}{\ln(y_{t,t+1}^{ec})}$	0	Bjurek (1996)
$\ln(HM_{t,t+1}^{em}(\cdot))$	0	$\frac{1}{\ln(x_{t,t+1}^{em})}$	0	$-\frac{1}{\ln(y_{t,t+1}^{em})}$	
$\ln(EAHM_{t,t+1}^{\alpha;\beta}(\cdot)) \equiv$	$\alpha^{ec}$	$\alpha^{en}$	$\beta^{ec}$	$\beta^{en}$	
$LHM_{t,t+1}^{ec}(\ln(\cdot))$	$\ln(x_{t,t+1}^{ec})$	0	$\ln(y_{t,t+1}^{ec})$	0	Briec and Kerstens (2004)
$LHM_{t,t+1}^{em}(\ln(\cdot))$	0	$\ln(x_{t,t+1}^{em})$	0	$-\ln(y_{t,t+1}^{em})$	

Table 2: Environmentally-adjusted TFP and classical TFP equivalences.

Remark that the outcomes displayed in Table 1 provide as an immediate result the equivalence between the environmentally-adjusted Luenberger productivity indicator and the hyperbolic-based Malmquist productivity index  $M_{t,t+1}^{Ho}(\cdot)$  by considering any  $(\mathbf{x}_{t,t+1}, \mathbf{y}_{t,t+1}) \in \mathbb{R}_{++}^{n+m}$  and a transformed production technology  $\ln(T_{t,t+1}^{++})$ . Reciprocally, the environmental-adjusted Malmquist productivity measure is equivalent to the environmental Luenberger productivity indicator  $EL_{t,t+1}(\cdot)$  (Azad and Ancev, 2014). These equivalences are displayed as follows:

$$\ln(M_{t,t+1}^{Ho}(\cdot)) \equiv EAL_{t,t+1}^{\gamma;\sigma}(\ln(\cdot)) \text{ with } \gamma = 0 \text{ and } \sigma = \left( \frac{1}{\ln(y_t^{ec})}, -\frac{1}{\ln(y_t^{em})} \right), \quad (3.11)$$

$$EL_{t,t+1}(\cdot) \equiv \ln(EAM_t^{\alpha;\beta}(\cdot)) \text{ with } \alpha = 0 \text{ and } \beta = (\ln(y_t^{ec}), -\ln(y_t^{em})). \quad (3.12)$$

## 4 Empirical illustration

This section aims to propose an empirical illustration highlighting the equivalence conditions stated in the previous sections. To do so, both production technology and distance functions are non parametrically specified in the upcoming section.

### 4.1 Non parametric specifications

To propose non parametric specifications of the production technology and of the distance functions, we use the data envelopment analysis (DEA) approach.

In this empirical illustration, the non convex multiplicative production set of Banker and Maindiratta (1986) is considered. Indeed, non convexity allows to take account for local strictly increasing returns-to-scale contrary to the variable returns-to-scale production technology of Banker et al. (1984), which is well known as the BCC model. Moreover, non convexity shows that there exists non-linear relationship between the inputs and the outputs and hence, there can be indivisibilities in the production process (Sahoo and Tone, 2013). Specifically, multiplicative version of the by-production model satisfies  $T1 - T4$  and is considered to display the applicability of the unified framework presented in this paper (Abad and Briec, 2019; Murty et al., 2012).

For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_+^{n+m}$  and any set of observations  $\mathcal{J} = \{1, \dots, J\}$  with  $j \in \mathcal{J}$ , the non parametric multiplicative by-production technology is defined as:

$$T_t^M := \left\{ (\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_+^{n+m} : \mathbf{x}_t \geq \prod_{j \in \mathcal{J}} (\mathbf{x}_t^j)^{\lambda_j}, \mathbf{y}_t \leq \prod_{j \in \mathcal{J}} (\mathbf{y}_t^j)^{\lambda_j}, x_t^{\text{em}} \leq \prod_{j \in \mathcal{J}} (x_t^{\text{em},j})^{\mu_j}, \right. \\ \left. y_t^{\text{em}} \geq \prod_{j \in \mathcal{J}} (y_t^{\text{em},j})^{\mu_j}; \lambda, \mu \geq 0; \sum_{j \in \mathcal{J}} \lambda_j = \mu_j = 1 \right\}. \quad (4.1)$$

Now, suppose that we consider strictly positive input-output vectors such that  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m}$ . Hence, the natural logarithmic transformation of the multiplicative by-production technology yields a Napierian by-production set as follows:

$$\ln(T_t^M) := \left\{ (\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m} : \ln(\mathbf{x}_t) \geq \sum_{j \in \mathcal{J}} \lambda_j \ln(\mathbf{x}_t^j), \ln(\mathbf{y}_t) \leq \sum_{j \in \mathcal{J}} \lambda_j \ln(\mathbf{y}_t^j), \right. \\ \left. \ln(x_t^{\text{em}}) \leq \sum_{j \in \mathcal{J}} \mu_j \ln(x_t^{\text{em},j}), \ln(y_t^{\text{em}}) \geq \sum_{j \in \mathcal{J}} \mu_j \ln(y_t^{\text{em},j}); \lambda, \mu \geq 0; \sum_{j \in \mathcal{J}} \lambda_j = \mu_j = 1 \right\}. \quad (4.2)$$

It is worth noting that the Napierian technology has similar structure with the BCC production set. In such case, the multiplicative by-production set is a log-additive and a log-convex set.

See Appendix III for the non parametric specifications of the distance functions.

## 4.2 Data

In this paper, we use a dataset sourced from the work of Jeon and Sickles (2004). The considered decision making units are 17 OECD countries: (1) Canada; (2) the United States; (3) Japan; (4) Austria; (5) Belgium; (6) Denmark; (7) Finland; (8) France; (9) Germany; (10) Greece; (11) Ireland; (12) Italy; (13) Norway; (14) Spain; (15) Sweden; (16) United Kingdoms; (17) Australia, for the years 1989 and 1990. The dataset includes two no polluting inputs (i) capital and (ii) labour; one polluting input (iii) energy; one polluting output (iv) carbon dioxide emission and one no polluting output (v) gross domestic product.

## 4.3 Results

Observations	(1a) EAM	(1b) ln(EAM)	(2) EAL	(3a) EAHM	(3b) ln(EAHM)	(4) EALHM
Canada	$\infty$	$\infty$	$\infty$	1.0474	0.0464	0.0464
United States	$\infty$	$\infty$	$\infty$	0.9646	-0.0360	-0.0360
Japan	1.0154	0.0152	0.0152	0.9490	-0.0523	-0.0523
Austria	1.0002	0.0002	0.0002	0.9309	-0.0716	-0.0716
Belgium	$\infty$	$\infty$	$\infty$	0.9470	-0.0545	-0.0545
Denmark	$\infty$	$\infty$	$\infty$	1.0806	0.0775	0.0775
Finland	$\infty$	$\infty$	$\infty$	0.9471	-0.0543	-0.0543
France	1.0001	0.0001	0.0001	0.9787	-0.0216	-0.0216
Germany	0.9989	-0.0011	-0.0011	0.9817	-0.0184	-0.0184
Greece	$\infty$	$\infty$	$\infty$	0.8927	-0.1135	-0.1135
Ireland	$\infty$	$\infty$	$\infty$	0.9489	-0.0524	-0.0524
Italy	1.0113	0.0113	0.0113	1.0019	0.0019	0.0019
Norway	$\infty$	$\infty$	$\infty$	1.0321	0.0316	0.0316
Spain	1.0018	0.0018	0.0018	1.0987	0.0941	0.0941
Sweden	0.9991	-0.0009	-0.0009	1.0596	0.0579	0.0579
United Kingdoms	$\infty$	$\infty$	$\infty$	0.9494	-0.0519	-0.0519
Australia	$\infty$	$\infty$	$\infty$	0.8694	-0.1399	-0.1399

Table 3: Environmentally-adjusted productivity scores over the period 1989-1990.

The results in Table 3 are illustrated in Figure 1 for the environmentally-adjusted Malmquist and Luenberger productivity measures and in Figure 2 for the environmentally-adjusted Hicks-Moorsteen and, Luenberger-Hicks-Moorsteen productivity measures.

Results in Table 3 show that 5 countries have productivity improvement through measures (1a-1b) and (2) whereas 2 countries are facing productivity loss over the considered period.

It is worth noting that EAM and EAL are defined with complete cross-sectional distance measures such that infeasibilities may occur (Abad, 2015). In such case, some results are indeterminate and cannot be interpreted. Specifically, in this empirical illustration, 10 out of 17 observations have indeterminate productivity measure.

Total factor productivity measures such as (3a-3b) and (4) allow to overcome the issue of indeterminateness since these measures are always assessed towards feasible directions. Thus, results of productivity measures (3a-3b) and (4) cannot be infeasible and can always be interpreted as show in Table 3. These results show that 7 out of 17 observations have productivity gains.

As visible in Figure 1 and Figure 2, if an observation faces a productivity loss under an EAM productivity measure then it also deals with a productivity deterioration through an EAL productivity indicator. This outcome is a result of the connection between EAM and EAL productivity measures. The same reasoning holds for the EAHM and EALHM total factor productivity measures. Regarding the additive and multiplicative classical families of productivity measures, namely Hicks-Moorsteen-, Luenberger-Hicks-Moorsteen-, Malmquist- and Luenberger-based indicators, similar outcomes are provided in Appendix IV.

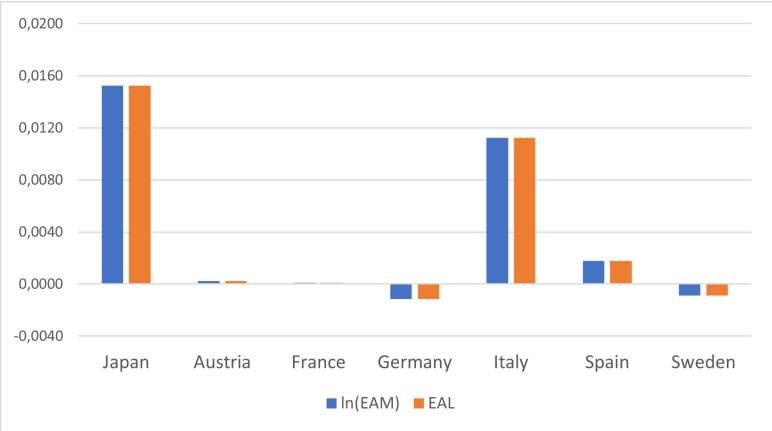


Figure 1: Environmentally-adjusted Luenberger and logarithmically transformed Malmquist productivity measures.



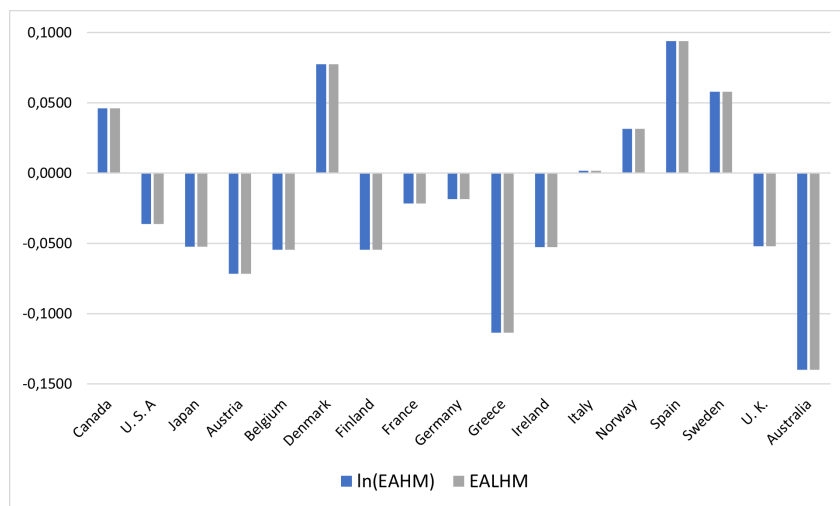


Figure 2: Environmentally-adjusted Luenberger-Hicks-Moorsteen and logarithmically transformed Hicks-Moorsteen productivity measures.

## 5 Concluding Comments

This paper presents a general method comparing additive- and multiplicative-based productivity measures. Specifically, this contribution lays out equivalence conditions for the additive and multiplicative environmentally-adjusted productivity measures. Therefore, an unified framework allowing to compare additive and multiplicative environmentally-adjusted productivity measures is provided.

Using the unified framework defined in this paper, researchers would be able to analyse environmentally-adjusted productivity variation through a general method. Indeed, the proposed approach allows to remove differences in magnitudes of empirical results arising from the use of different families of productivity measures. As a result, the proposed approach would provide a theoretical framework that is neutral as regard to the class of selected productivity measures.

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# Appendix I

The following results highlight connections between the environmentally-adjusted multiplicative distance function (2.2), the intended ( $D_t^{\text{ec}}$ ) and unintended ( $D_t^{\text{em}}$ ) outputs Shephard distance functions (Färe et al., 2004), the hyperbolic output ( $H_t^o$ ) distance function (Färe et al., 1989) and the hyperbolic environmental efficiency ( $\Psi_t$ ) measure (Abad and Ravelojaona, 2022). Moreover, the next outcomes also permit comparing the environmentally-adjusted additive distance function (2.3) with the environmental directional ( $\vec{D}_t$ ) distance function (Chung et al., 1997), the intended ( $\vec{D}_t^{\text{ec}}$ ) and unintended ( $\vec{D}_t^{\text{em}}$ ) sub-vector directional distance functions (Picazo-Tadeo et al., 2014) as well as the environmental disaggregated directional ( $\Xi_t$ ) distance function (Abad and Ravelojaona, 2022).

**Proposition 5.1** *For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_+^{n+m}$ , the environmentally-adjusted additive- and multiplicative-based distance function allows to provide classical additive- and multiplicative-based efficiency measures, as follows:*

Efficiency measures		Parameters				References
$D_t^{\alpha;\beta}(\cdot) \equiv$		$\alpha^{\text{ec}}$	$\alpha^{\text{en}}$	$\beta^{\text{ec}}$	$\beta^{\text{en}}$	
a.	$[D_t^{\text{ec}}(\cdot)]^{-1}$	0	0	1	0	Färe et al. (2004)
b.	$D_t^{\text{em}}(\cdot)$	0	0	0	-1	
c.	$H_t^o(\cdot)$	0	0	1	-1	Färe et al. (1989)
d.	$\Psi_t(\cdot)$	1	1	1	-1	Abad and Ravelojaona (2022)
$\vec{D}_t^{\gamma;\sigma}(\cdot) \equiv$		$\gamma^{\text{ec}}$	$\gamma^{\text{en}}$	$\sigma^{\text{ec}}$	$\sigma^{\text{en}}$	
e.	$\vec{D}_t^{\text{ec}}(\cdot)$	0	0	1	0	Picazo-Tadeo et al. (2014)
f.	$\vec{D}_t^{\text{em}}(\cdot)$	0	0	0	-1	
g.	$\vec{D}_t(\cdot)$	0	0	1	-1	Chung et al. (1997)
h.	$\Xi_t(\cdot)$	1	1	1	-1	Abad and Ravelojaona (2022)

Table 4: Environmentally-adjusted and classical distance functions equivalence

The upcoming statements display equivalence conditions for the proposed environmentally-

adjusted distance functions (2.2)-(2.3) and the aforementioned additive- and multiplicative-based distance functions.

**Proposition 5.2** *For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m}$  and any logarithmically transformed production set  $\ln(T_t^{++})$ , the environmentally-adjusted additive and multiplicative distance functions are equivalent to classical additive- and multiplicative-based efficiency measures, as follows:*

Efficiency measures		Parameters				Reference
	$\ln(D_t^{\alpha;\beta}(\cdot)) \equiv$	$\alpha^{ec}$	$\alpha^{en}$	$\beta^{ec}$	$\beta^{en}$	
<b>i.</b>	$\vec{D}_t^{ec}(\ln(\cdot))$	0	0	$\ln(y_t^{ec})$	0	Picazo-Tadeo et al. (2014)
<b>ii.</b>	$\vec{D}_t^{em}(\ln(\cdot))$	0	0	0	$-\ln(y_t^{em})$	
<b>iii.</b>	$\vec{D}_t(\ln(\cdot))$	0	0	$\ln(y_t^{ec})$	$-\ln(y_t^{em})$	Chung et al. (1997)
<b>iv.</b>	$\Xi_t(\ln(\cdot))$	$\ln(x_t^{ec})$	$\ln(x_t^{em})$	$\ln(y_t^{ec})$	$-\ln(y_t^{em})$	Abad and Ravelojaona (2022)
$\vec{D}_t^{\gamma;\sigma}(\ln(\cdot)) \equiv$		$\gamma^{ec}$	$\gamma^{en}$	$\sigma^{ec}$	$\sigma^{en}$	
<b>v.</b>	$-\ln(D_t^{ec}(\cdot))$	0	0	$\frac{1}{\ln(y_t^{ec})}$	0	Färe et al. (2004)
<b>vi.</b>	$\ln(D_t^{em}(\cdot))$	0	0	0	$-\frac{1}{\ln(y_t^{em})}$	
<b>vii.</b>	$\ln(H_t^o(\cdot))$	0	0	$\frac{1}{\ln(y_t^{ec})}$	$-\frac{1}{\ln(y_t^{em})}$	Färe et al. (1989)
<b>viii.</b>	$\ln(\Psi_t(\cdot))$	$\frac{1}{\ln(x_t^{ec})}$	$\frac{1}{\ln(x_t^{em})}$	$\frac{1}{\ln(y_t^{ec})}$	$-\frac{1}{\ln(y_t^{em})}$	Abad and Ravelojaona (2022)

Table 5: Equivalence conditions: environmentally-adjusted and classical efficiency measures.

Remark that the hyperbolic-based distance function (c.) can be approximated through a Taylor's expansion as proposed by Färe et al. (2016), with respect to some constraints. However, this paper assesses the hyperbolic efficiency measure by using the equivalence (vi.) in Proposition 5.2.

## Appendix II

*Proof of Proposition 2.2:* Assume that  $T_t$  is a production technology that satisfies properties  $T1$ - $T4$ . For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m}$ , the environmentally-adjusted additive distance function (2.3) is defined as follows,

$$\vec{D}_t^{\gamma;\sigma}(\mathbf{x}_t, \mathbf{y}_t) := \sup_{\delta} \left\{ \delta \geq 0 : \left( (I_n - \delta\gamma)\mathbf{x}'_t, (I_m + \delta\sigma)\mathbf{y}'_t \right) \in T_t \right\} \quad (5.1)$$

Therefore,

$$\vec{D}_t^{\gamma;\sigma}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)) := \sup_{\delta} \left\{ \delta \geq 0 : \left( (I_n - \delta\gamma)\ln(\mathbf{x}_t)', (I_m + \delta\sigma)\ln(\mathbf{y}_t)' \right) \in T_t^{\ln} \right\}, \quad (5.2)$$

such that  $\ln(\mathbf{x}_t) = (\ln(x_t^{\text{ec}}), \ln(x_t^{\text{em}}))$ ,  $\ln(\mathbf{y}_t) = (\ln(y_t^{\text{ec}}), \ln(y_t^{\text{em}}))$  and  $T_t^{\ln} := \{(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)) : (\mathbf{x}_t, \mathbf{y}_t) \in T_t^{++}\} =: \ln(T_t^{++})$ .

Moreover, according to the definition of the environmentally-adjusted multiplicative distance function (2.2),

$$\ln(D_t^{\alpha;\beta}(\mathbf{x}_t, \mathbf{y}_t)) := \sup_{\lambda} \left\{ \lambda \geq 1 : (-\alpha \ln(\lambda) + \ln(\mathbf{x}_t)', \beta \ln(\lambda) + \ln(\mathbf{y}_t)') \in \ln(T_t^{++}) \right\}. \quad (5.3)$$

As a result, since  $T_t^{\ln} \equiv \ln(T_t^{++})$  then,  $\vec{D}_t^{\gamma;\sigma}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)) \equiv \ln(D_t^{\alpha;\beta}(\mathbf{x}_t, \mathbf{y}_t))$  with  $\gamma = \left( \frac{\alpha^{\text{ec}}}{\ln(x_t^{\text{ec}})}, \frac{\alpha^{\text{em}}}{\ln(x_t^{\text{em}})} \right)$  and  $\sigma = \left( \frac{\beta^{\text{ec}}}{\ln(y_t^{\text{ec}})}, \frac{\beta^{\text{em}}}{\ln(y_t^{\text{em}})} \right)$ .  $\square$



## Appendix III

Non parametric specification of the distance functions.

### Multiplicative environmentally-adjusted distance function

The multiplicative environmentally-adjusted distance function is defined within the multiplicative technology  $T_t^M$ . Let  $\alpha = (\alpha^{ec}, \alpha^{em}) \in \mathbb{R}_+^n$  and  $\beta = (\beta^{ec}, \beta^{em}) \in \mathbb{R}_+^{m_{ec}} \times \mathbb{R}_-^{m_{em}}$ , for any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m}$ :

$$\begin{aligned}
 D_t^{\alpha; \beta}(\mathbf{x}_t, \mathbf{y}_t) &= \min \lambda \\
 \text{s.t.} \quad &\lambda^\alpha \mathbf{x}_t \geq \prod_{j \in \mathcal{J}} (\mathbf{x}_t^j)^{\lambda_j} \\
 &\lambda^\beta \mathbf{y}_t \leq \prod_{j \in \mathcal{J}} (\mathbf{y}_t^j)^{\lambda_j} \\
 &\lambda^{\alpha^{em}} x_t^{em} \leq \prod_{j \in \mathcal{J}} (x_t^{em, j})^{\mu_j} \\
 &\lambda^{\beta^{em}} y_t^{em} \geq \prod_{j \in \mathcal{J}} (y_t^{em, j})^{\mu_j} \\
 &\sum_{j \in \mathcal{J}} \lambda_j = 1 \\
 &\sum_{j \in \mathcal{J}} \mu_j = 1 \\
 &\lambda, \mu \geq 0.
 \end{aligned}$$

### Additive environmentally-adjusted distance function

The additive environmentally-adjusted distance function is defined within the natural logarithmic transformed multiplicative production technology  $\ln(T_t^M)$ . For any  $(\mathbf{x}_t, \mathbf{y}_t) \in \mathbb{R}_{++}^{n+m}$  and any  $\gamma = (\gamma^{ec}, \gamma^{em})$  and  $\sigma = (\sigma^{ec}, \sigma^{em})$  where  $\gamma^{ec}, \gamma^{em} \in \mathbb{R}_+^n$ ,  $\sigma^{ec} \in \mathbb{R}_+^{m_{ec}}$  and  $\sigma^{em} \in \mathbb{R}_-^{m_{em}}$ ,

$$\begin{aligned}
\overrightarrow{D}_t^{\gamma;\sigma}(\ln(\mathbf{x}_t), \ln(\mathbf{y}_t)) &= \max \delta \\
\text{s.t.} \quad & (1 - \delta\gamma)\ln(\mathbf{x}_t) \geq \sum_{j \in \mathcal{J}} \lambda_j \ln(\mathbf{x}_t^j) \\
& (1 + \delta\sigma)\ln(\mathbf{y}_t) \leq \sum_{j \in \mathcal{J}} \lambda_j \ln(\mathbf{y}_t^j) \\
& (1 - \delta\gamma^{\text{em}})\ln(x_t^{\text{em}}) \leq \sum_{j \in \mathcal{J}} \mu_j \ln(x_t^{\text{em},j}) \\
& (1 + \delta\sigma^{\text{em}})\ln(y_t^{\text{em}}) \geq \sum_{j \in \mathcal{J}} \mu_j \ln(y_t^{\text{em},j}) \\
& \sum_{j \in \mathcal{J}} \lambda_j = 1 \\
& \sum_{j \in \mathcal{J}} \mu_j = 1 \\
& \lambda, \mu \geq 0.
\end{aligned}$$

## Appendix IV

<b>Observations</b>	$M^{ec}$	$\ln(M^{ec})$	$L^{ec}$	$HM^{ec}$	$\ln(HM^{ec})$	$LHM^{ec}$
<b>Canada</b>	1.0053	0.0053	0.0053	0.974	-0.0263	-0.0263
<b>U. S. A</b>	1.0245	0.0242	0.0242	0.9811	-0.0191	-0.0191
<b>Japan</b>	1.0131	0.0130	0.013	1.0116	0.0115	0.0115
<b>Austria</b>	1.0002	0.0002	0.0002	1.0256	0.0253	0.0252
<b>Belgium</b>	1.0224	0.0222	0.0221	1.0205	0.0203	0.0203
<b>Denmark</b>	1.0049	0.0049	0.0049	1.0063	0.0063	0.0063
<b>Finland</b>	1	0	0	0.9852	-0.0149	-0.015
<b>France</b>	1.0001	0.0001	0.0001	1.0159	0.0158	0.0158
<b>Germany</b>	1.0003	0.0003	0.0003	1.0351	0.0345	0.0345
<b>Greece</b>	1.0179	0.0177	0.0177	0.9982	-0.0018	-0.0018
<b>Ireland</b>	1.048	0.0469	0.0469	1.0759	0.0732	0.0731
<b>Italy</b>	1.0098	0.0098	0.0098	1.0002	0.0002	0.0002
<b>Norway</b>	1.0016	0.0016	0.0016	1.0122	0.0121	0.0122
<b>Spain</b>	1.0018	0.0018	0.0018	0.9878	-0.0123	-0.0122
<b>Sweden</b>	0.9991	-0.0009	-0.0009	1.0006	0.0006	0.0006
<b>U. K.</b>	1.0303	0.0299	0.0299	0.9698	-0.0307	-0.0307
<b>Australia</b>	1.0183	0.0181	0.0181	0.9537	-0.0474	-0.0474

Table 6: Economic-based productivity scores over the period 1989-1990.

<b>Observations</b>	$M^{\text{em}}$	$\ln(M^{\text{em}})$	$L^{\text{em}}$	$HM^{\text{em}}$	$\ln(HM^{\text{em}})$	$LHM^{\text{em}}$
<b>Canada</b>	$\infty$	$\infty$	$\infty$	1.0754	0.0727	0.0727
<b>U. S. A</b>	$\infty$	$\infty$	$\infty$	0.9832	-0.0169	-0.017
<b>Japan</b>	1.002	0.0020	0.002	0.9382	-0.0638	-0.0638
<b>Austria</b>	1	0	0	0.9077	-0.0968	-0.0968
<b>Belgium</b>	$\infty$	$\infty$	$\infty$	0.928	-0.0747	-0.0747
<b>Denmark</b>	$\infty$	$\infty$	$\infty$	1.0738	0.0712	0.0712
<b>Finland</b>	$\infty$	$\infty$	$\infty$	0.9614	-0.0394	-0.0394
<b>France</b>	1	0	0	0.9634	-0.0373	-0.0373
<b>Germany</b>	0.9986	-0.0014	-0.0014	0.9484	-0.0530	-0.053
<b>Greece</b>	$\infty$	$\infty$	$\infty$	0.8943	-0.1117	-0.1117
<b>Ireland</b>	$\infty$	$\infty$	$\infty$	0.882	-0.1256	-0.1256
<b>Italy</b>	1.0015	0.0015	0.0015	1.0018	0.0018	0.0018
<b>Norway</b>	$\infty$	$\infty$	$\infty$	1.0197	0.0195	0.0195
<b>Spain</b>	1	0	0	1.1122	0.1063	0.1064
<b>Sweden</b>	1	0	0	1.0589	0.0572	0.0572
<b>U. K.</b>	$\infty$	$\infty$	$\infty$	0.979	-0.0212	-0.0212
<b>Australia</b>	$\infty$	$\infty$	$\infty$	0.9116	-0.0926	-0.0925

Table 7: Polluting-based productivity scores over the period 1989-1990.