An Unified Framework for Measuring Environmentally-adjusted Productivity Change: Theoretical Basis and Empirical Illustration *

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An Unified Framework for Measuring Environmentally-adjusted Productivity Change: Theoretical Basis and Empirical Illustration*

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Abstract

This paper aims to define an unified framework to analyse environmentally-adjusted productivity change. Equivalence conditions for additive and multiplicative environmentally-adjusted productivity measures are highlighted. Besides, an empirical illustration is provided considering non parametric convex neutral by-production model.

Keywords: Data Envelopment Analysis, Convex Neutral Production Model, Distance Functions, Environmentally-adjusted Productivity Indices, Pollution-generating Technology.

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1 Introduction

As prominent economic health indicator, productivity measure is usually defined as index number evaluating the performance of economic systems (OECD, 2001). In the literature focusing on productivity studies, the production theoretic approach to index numbers is principally adopted (Prasada Rao, 2020; Caves et *al.*, 1982). In this theoretical area, two approaches are traditionally laid out, namely the multiplicative-based approach and the additive-based one (Briec and Kerstens, 2004; Chambers, 2002; Bjurek, 1996; Färe et *al.*, 1994). Specifically, the multiplicative productivity indices are defined as ratios of multiplicative distance functions (Shephard, 1970; Debreu, 1951; Farrell, 1957) whilst the additive productivity measures are defined as difference-based indicators of directional distance functions (Briec, 1997; Chambers et *al.*, 1996)[1]. Approximation results are provided in the literature establishing connections between the usual multiplicative and additive productivity measures (Briec and Kerstens, 2004; Boussemart et *al.*, 2003). These outcomes are of particular interest to interpret differences of empirical findings that result in employing either additive- or multiplicative-based productivity indices.

The production theoretic approach to index numbers has been mainly designed through unidimensional framework, especially focusing on economic components of the evaluated economic systems (Färe et *al.*, 1994). In the last decades, numerous paper incorporate environmental dimension to appraise productivity changes (Sueyoshi et *al.*, 2017; Zhou et *al.*, 2008). In this line, both additive and multiplicative environmental productivity measures have been defined (Abad and Ravelojaona, 2022, 2021; Azad and Ancev, 2014; Chung et *al.*, 1997). Specifically, multiplicative environmental productivity measures inherit the structure of the Malmquist (Caves et *al.*, 1982) and the Hicks-Moorsteen (Bjurek, 1996) productivity indices. Moreover, additive environmental productivity indicators take the form of the Luenberger (Chambers, 2002) and the Luenberger-Hicks-Moorsteen (Briec and Kerstens, 2004) productivity indicators.

This contribution establishes equivalence conditions for additive and multiplicative

[1]The main differences between the additive and the multiplicative approaches are presented in Briec and Kerstens (2004), Chambers (1998, 2002) and Diewert (1998), among others.

environmentally-adjusted[2] productivity measures. These equivalence conditions extend the usual approximation results that linked the traditional additive and multiplicative productivity measures (Briec and Kerstens, 2004; Boussemart et *al.*, 2003). Particularly, this contribution highlights specific conditions for which environmentally-adjusted Malmquist and Luenberger productivity measures are equivalent to each other. Moreover, theoretical relation between environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures are introduced. As a result, additive and multiplicative environmentally-adjusted productivity measures are exactly related to each other providing an unified framework for empirical analysis.

Interestingly, this contribution also permits to present additive version of Malmquist and Hicks-Moorsteen environmentally-adjusted productivity indices. This result extends the widely applied Chung et *al.* (1997) methodology, which provides theoretical background to consider desirable and undesirable components in traditional multiplicative-based productivity measure. Moreover, multiplicative version of Luenberger and Luenberger-Hicks-Moorsteen environmentally-adjusted productivity indicators are provided. This outcome may be understood as a reciprocal result of the Chung et *al.* (1997) model, handling the issue of joint producing good and bad components in productivity assessment through multiplicative productivity indices.

The applicability of the unified framework presented in this paper is displayed through an empirical illustration based upon the database sourced from the work of Jeon and Sickles (2004). Precisely, the main findings of this paper are illustrated by considering a non parametric convex neutral by-production model. In this line, multiplicative version of the by-production model (Abad and Briec, 2019; Murty et *al.*, 2012; Banker and Maindiratta, 1986) is considered to provide non parametric estimation of the main results highlighted in this paper.

The remainder of this paper unfolds as follows. Section 2 introduces theoretical preliminaries. The pollution-generating production process and the distance functions are also presented in this section. Section 3 displays multiplicative and additive environmentally-

[2] To distinguish from the measures existing in the literature (environmental measures), the measures proposed in this paper are designated as "environmentally-adjusted" ones.

adjusted productivity measures and further, it provides equivalence conditions for the additive and the multiplicative environmentally-adjusted productivity measures. An empirical illustration is provided in Section 4, highlighting the applicability of the unified framework introduced in this paper. Finally, section 5 concludes.

2 Background

In this section, the properties of the pollution-generating production process are presented. Based upon this theoretical background, additive and multiplicative distance functions are displayed and further, equivalence condition between the proposed distance functions is laid out.

2.1 Technology definition and properties

Let $\mathbf{x}_{\mathtt{t}} = (x_{\mathtt{t}}^{\mathtt{ec}}, x_{\mathtt{t}}^{\mathtt{em}}) \in \mathbb{R}_+^n$ denotes the economic factors and the emission-generating inputs used to produce both economic (*i.e.*, intended) and emission-generating (*i.e.*, unintended) outputs $\mathbf{y}_{\mathtt{t}} = (y_{\mathtt{t}}^{\mathtt{ec}}, y_{\mathtt{t}}^{\mathtt{em}}) \in \mathbb{R}_+^m$, such that $n = n_{\mathtt{ec}} + n_{\mathtt{em}}$ and $m = m_{\mathtt{ec}} + m_{\mathtt{em}}$.

The pollution-generating technology is defined as follows:

$$T_{\mathtt{t}} := \left\{ (\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}_+^{n+m} : \mathbf{x}_{\mathtt{t}} \, \text{can produce} \, \mathbf{y}_{\mathtt{t}} \right\} \tag{2.1}$$

Assume that the production process satisfies the following usual assumptions (Färe et al, 1985):

T1. -*No free lunch and inaction*- $(0, 0) \in T_{\mathtt{t}}$, $(0, \mathbf{y}_{\mathtt{t})} \in T_{\mathtt{t}} \Rightarrow \mathbf{y}_{\mathtt{t}} = 0$;

T2. -*Boundedness*- $T_{\mathbf{y}_{\mathtt{t}}} := \{(\mathbf{u}_{\mathtt{t}}, \mathbf{v}_{\mathtt{t}}) \in T_{\mathtt{t}} : \mathbf{v}_{\mathtt{t}} \leq \mathbf{y}_{\mathtt{t}}\}$ is bounded for any $\mathbf{y}_{\mathtt{t}} \in \mathbb{R}_+^m$;

T3. -*Closedness*- $T_{\mathtt{t}}$ is closed;

Moreover, suppose that for any free disposal cone $K := \mathbb{R}_+^n \times -\mathbb{R}_+^m$, the production set satisfies the generalised B-disposal property:

T4. -*Generalised B-disposability*- $T_{\mathtt{t}} := \Big(\big(T_{\mathtt{t}} + K\big) \cap \big(T_{\mathtt{t}} + (\mathbb{R}_+^{n_{\mathtt{ec}}+m_{\mathtt{em}}} \times -\mathbb{R}_+^{n_{\mathtt{em}}+m_{\mathtt{ec}}})\big) \Big) \cap (\mathbb{R}_+^n \times \mathbb{R}_+^m)$.

The axiomatic framework *T1–T4* permits to define the production process as an intersection of sub-technologies (Abad and Briec, 2019; Murty et *al.*, 2012). Specifically, the economic (*i.e.*, intended) production activities satisfy the usual free disposal property –*i.e.*, $\left(T_{\mathtt{t}} + (\mathbb{R}^n_+ \times -\mathbb{R}^m_+)\right) \cap \left(\mathbb{R}^n_+ \times \mathbb{R}^m_+\right)$– whilst partially reversed free disposal axiom applies for the polluting (*i.e.*, unintended) production activities; *i.e.*, $\left(T_{\mathtt{t}} + (\mathbb{R}^{n_{\mathtt{ec}}+m_{\mathtt{em}}}_+ \times -\mathbb{R}^{n_{\mathtt{em}}+m_{\mathtt{ec}}}_+)\right) \cap \left(\mathbb{R}^n_+ \times \mathbb{R}^m_+\right)$. Remarkably, the theoretical model *T1–T4* is fairly weak and do not impose any convexity assumption such that convex neutral production model may be considered.

2.2 Technology characterisation: distance functions

In this section, additive and multiplicative distance functions are considered as functional representation of the production process (Chambers and Färe, 2020).

The next result presents additive and multiplicative environmentally-adjusted distance functions.

Definition 2.1 *Let $T_{\mathtt{t}}$ be a production technology that satisfies properties T1–T4. For any $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}^{n+m}_+$:*

i. *The multiplicative environmentally-adjusted distance function is defined as follows,*

$$\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) := \sup_{\lambda} \left\{ \lambda \geq 1 : \left(\lambda^{-\alpha} \mathbf{x}'_{\mathtt{t}}, \lambda^{\beta} \mathbf{y}'_{\mathtt{t}} \right) \in T_{\mathtt{t}} \right\} \tag{2.2}$$

where $\alpha = (\alpha^{\mathtt{ec}}, \alpha^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\beta = (\beta^{\mathtt{ec}}, \beta^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$, such that $\lambda^{-\alpha} = \mathrm{diag}(\lambda^{-\alpha^{\mathtt{ec}}}, \lambda^{-\alpha^{\mathtt{em}}})$ and $\lambda^{\beta} = \mathrm{diag}(\lambda^{\beta^{\mathtt{ec}}}, \lambda^{\beta^{\mathtt{em}}})$.

ii. *The additive environmentally-adjusted distance function is defined as follows,*

$$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) := \sup_{\delta} \left\{ \delta \geq 0 : \left((I_n - \delta\gamma)\mathbf{x}'_{\mathtt{t}}, (I_m + \delta\sigma)\mathbf{y}'_{\mathtt{t}} \right) \in T_{\mathtt{t}} \right\} \tag{2.3}$$

where $\gamma = (\gamma^{\mathtt{ec}}, \gamma^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\sigma = (\sigma^{\mathtt{ec}}, \sigma^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$, with $\gamma = \mathrm{diag}(\gamma^{\mathtt{ec}}, \gamma^{\mathtt{em}})$ and $\sigma = \mathrm{diag}(\sigma^{\mathtt{ec}}, \sigma^{\mathtt{em}})$.

In statements **i.** *and* **ii.**, *$\lambda^{-\alpha}$, λ^{β}, γ and σ are such that $\mathrm{dim}(\lambda^{-\alpha}) = \mathrm{dim}(\gamma) = n \times n$ and $\mathrm{dim}(\lambda^{\beta}) = \mathrm{dim}(\sigma) = m \times m$. Moreover, I_n and I_m are to the identity matrix of order n and m, respectively.*

The additive and multiplicative environmentally-adjusted distance functions fully characterise the production process such that[3]:

$$\mathtt{D}_{\mathtt{t}}^{\alpha;\beta}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \geq 1 \Leftrightarrow (\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in T_{\mathtt{t}} \Leftrightarrow \overrightarrow{\mathtt{D}}_{\mathtt{t}}^{\gamma;\sigma}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \geq 0.$$

Interestingly, the general shape of the additive and multiplicative environmentally-adjusted distance functions allows comparisons with existing environmental distance functions (Abad and Ravelojaona, 2022; Picazo-Tadeo et *al.*, 2014; Färe et *al.*, 2004; Chung et *al.*, 1997; Färe et *al.*, 1989). These connections are presented in the Proposition 5.1 (see Appendix I).

2.3 Environmentally-adjusted distance functions: equivalence condition

Let $T_{\mathtt{t}}^{++} := T_{\mathtt{t}} \cap \mathbb{R}_{++}^{n+m}$ be the strictly positive production set. The next result defines equivalence condition for the additive and multiplicative environmentally-adjusted distance functions.

Proposition 2.2 *For any* $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}_{++}^{n+m}$*, the equivalence condition for the additive and multiplicative environmentally-adjusted distance functions is defined as follows:*

$$\overrightarrow{\mathtt{D}}_{\mathtt{t}}^{\gamma;\sigma}\left(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}})\right) \equiv \ln\left(\mathtt{D}_{\mathtt{t}}^{\alpha;\beta}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\right) \tag{2.4}$$

where $\overrightarrow{\mathtt{D}}_{\mathtt{t}}^{\gamma;\sigma}\left(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}})\right), \ln\left(\mathtt{D}_{\mathtt{t}}^{\alpha;\beta}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\right) \in \ln\left(T_{\mathtt{t}}^{++}\right)$*, such that* $\gamma = \left(\frac{\alpha^{\mathtt{ec}}}{\ln(x_{\mathtt{t}}^{\mathtt{ec}})}, \frac{\alpha^{\mathtt{em}}}{\ln(x_{\mathtt{t}}^{\mathtt{em}})}\right)$ *and* $\sigma = \left(\frac{\beta^{\mathtt{ec}}}{\ln(y_{\mathtt{t}}^{\mathtt{ec}})}, \frac{\beta^{\mathtt{em}}}{\ln(y_{\mathtt{t}}^{\mathtt{em}})}\right)$.

See Appendix II for the proof.

The equivalence result highlighted in Proposition 2.2 permits to exactly relate the additive and multiplicative environmentally-adjusted distance functions to one another. Interestingly, this outcome is applied by considering existing environmental distance functions (Abad and Ravelojaona, 2022; Picazo-Tadeo et *al.*, 2014; Färe et *al.*, 2004; Chung et *al.*, 1997; Färe et *al.*, 1989); see Proposition 5.2 in Appendix I.

[3]The basic properties of the additive- and multiplicative-based distance functions are laid out in Mehdiloozad et *al.* (2014), Peyrache and Coelli (2009) and Briec (1997).

3 Environmentally-adjusted productivity measures

This section displays equivalence conditions between additive- and multiplicative-based environmentally-adjusted productivity measures. In addition, additive versions of multiplicative-based environmentally-adjusted productivity measures are introduced. Reciprocal results laying out multiplicative versions of additive-based environmentally-adjusted productivity measures are also provided extending the widely applied Chung et *al.* (1997) methodology.

3.1 Malmquist and Luenberger productivity measures: equivalence condition

The next result presents the environmentally-adjusted Malmquist and Luenberger productivity measures.

Definition 3.1 *Assume that $T_{\mathtt{t},\mathtt{t}+1}$ is a production process that satisfies properties T1–T4. For any $(\mathbf{x}_{\mathtt{t},\mathtt{t}+1}, \mathbf{y}_{\mathtt{t},\mathtt{t}+1}) \in \mathbb{R}_+^{n+m}$ and for any $i, k = \{\mathtt{ec}, \mathtt{em}\}$ with $k \neq i$,*

i. *The environmentally-adjusted Malmquist productivity index is defined as follows,*

$$\mathrm{EAM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t}+1}(\mathbf{x}_{\mathtt{t},\mathtt{t}+1}, \mathbf{y}_{\mathtt{t},\mathtt{t}+1}) = \left[\prod_{i=\{\mathtt{ec},\mathtt{em}\}} \frac{\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})}{\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t}}(x^i_{\mathtt{t}+1}, x^k_{\mathtt{t}}, y^i_{\mathtt{t}+1}, y^k_{\mathtt{t}})} \times \frac{\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t}+1}(\mathbf{x}_{\mathtt{t}+1}, \mathbf{y}_{\mathtt{t}+1})}{\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t}+1}(x^i_{\mathtt{t}}, x^k_{\mathtt{t}+1}, y^i_{\mathtt{t}}, y^k_{\mathtt{t}+1})} \right]^{\frac{1}{2}} \tag{3.1}$$

where $\alpha = (\alpha^{\mathtt{ec}}, \alpha^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\beta = (\beta^{\mathtt{ec}}, \beta^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$.

ii. *The environmentally-adjusted Luenberger productivity indicator is defined as follows,*

$$\begin{aligned}\mathrm{EAL}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t}+1}(\mathbf{x}_{\mathtt{t},\mathtt{t}+1}, \mathbf{y}_{\mathtt{t},\mathtt{t}+1}) = \frac{1}{2} \Bigg[\sum_{i=\{\mathtt{ec},\mathtt{em}\}} &\Big(\overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t}}(x^i_{\mathtt{t}+1}, x^k_{\mathtt{t}}, y^i_{\mathtt{t}+1}, y^k_{\mathtt{t}}) \Big) \\ &+ \Big(\overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t}+1}(\mathbf{x}_{\mathtt{t}+1}, \mathbf{y}_{\mathtt{t}+1}) - \overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t}+1}(x^i_{\mathtt{t}}, x^k_{\mathtt{t}+1}, y^i_{\mathtt{t}}, y^k_{\mathtt{t}+1}) \Big) \Bigg]\end{aligned} \tag{3.2}$$

where $\gamma = (\gamma^{\mathtt{ec}}, \gamma^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\sigma = (\sigma^{\mathtt{ec}}, \sigma^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$.

The proposition below introduces equivalence condition for the environmentally-adjusted Malmquist and Luenberger productivity measures.

Proposition 3.2 *For any* $(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) \in \mathbb{R}^{n+m}_{++}$*, equivalence condition for the environmentally-adjusted Malmquist and Luenberger productivity measures is as follows:*

$$\ln\Big(\mathrm{EAM}^{\alpha;\beta}_{\mathtt{t,t+1}}(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}})\Big) \equiv \mathrm{EAL}^{\gamma;\sigma}_{\mathtt{t,t+1}}\Big(\ln(\mathbf{x}_{\mathtt{t,t+1}}), \ln(\mathbf{y}_{\mathtt{t,t+1}})\Big), \tag{3.3}$$

with $\ln\Big(\mathrm{EAM}^{\alpha;\beta}_{\mathtt{t,t+1}}(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}})\Big), \mathrm{EAL}^{\gamma;\sigma}_{\mathtt{t,t+1}}\Big(\ln(\mathbf{x}_{\mathtt{t,t+1}}), \ln(\mathbf{y}_{\mathtt{t,t+1}})\Big) \in \ln\left(T^{++}_{\mathtt{t,t+1}}\right)$ *such that* $\gamma = \left(\frac{\alpha^{\mathtt{ec}}}{\ln(x^{\mathtt{ec}}_{\mathtt{t,t+1}})}, \frac{\alpha^{\mathtt{em}}}{\ln(x^{\mathtt{em}}_{\mathtt{t,t+1}})}\right)$ *and* $\sigma = \left(\frac{\beta^{\mathtt{ec}}}{\ln(y^{\mathtt{ec}}_{\mathtt{t,t+1}})}, \frac{\beta^{\mathtt{em}}}{\ln(y^{\mathtt{em}}_{\mathtt{t,t+1}})}\right)$.

In the next statement, additive (respectively, multiplicative) version of the environmentally-adjusted Malmquist (respectively, Luenberger) productivity measure is proposed. The additive version of the environmentally-adjusted Malmquist productivity index is defined through additive distance functions. This productivity measure is named the environmentally-adjusted Malmquist-Luenberger productivity index based upon the initial work of Chung et *al.* (1997). Besides, the reciprocal environmentally-adjusted Malmquist-Luenberger productivity measure proposes a multiplicative version of the environmentally-adjusted Luenberger productivity indicator.

Corollary 3.3 *For any* $(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) \in \mathbb{R}^{n+m}_{++}$*, and any* $\gamma^i = \frac{\alpha^i}{\ln(x^i_{\mathtt{t,t+1}})}$ *and* $\sigma^i = \frac{\beta^i}{\ln(y^i_{\mathtt{t,t+1}})}$ *with* $i, k = \{\mathtt{ec}, \mathtt{em}\}$ *and* $i \neq k$ *:*

i. *The environmentally-adjusted Malmquist-Luenberger productivity index is defined as follows,*

$$\begin{aligned}\mathrm{EAML}^{\alpha;\beta}_{\mathtt{t,t+1}}(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) = \Bigg[\prod_{i=\{\mathtt{ec,em}\}} & \frac{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t}}(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}}))\right)}{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t}}(\ln(x^i_{\mathtt{t+1}}), \ln(x^k_{\mathtt{t}}), \ln(y^i_{\mathtt{t+1}}), \ln(y^k_{\mathtt{t}}))\right)} \\ & \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t+1}}(\ln(\mathbf{x}_{\mathtt{t+1}}), \ln(\mathbf{y}_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^i;\sigma^i}_{\mathtt{t+1}}(\ln(x^i_{\mathtt{t}}), \ln(x^k_{\mathtt{t+1}}), \ln(y^i_{\mathtt{t}}), \ln(y^k_{\mathtt{t+1}}))\right)}\Bigg]^{\frac{1}{2}}\end{aligned} \tag{3.4}$$

ii. *The reciprocal environmentally-adjusted Malmquist-Luenberger productivity measure is defined as follows,*

$$\begin{aligned}\overline{\mathrm{EAML}}^{\gamma;\sigma}_{\mathtt{t,t+1}}(\ln(\mathbf{x}_{\mathtt{t,t+1}}), \ln(\mathbf{y}_{\mathtt{t,t+1}})) = \; & \frac{1}{2}\Bigg[\sum_{i=\{\mathtt{ec,em}\}}\left(\ln\left(\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\right) - \ln\left(\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t}}(x^i_{\mathtt{t+1}}, x^k_{\mathtt{t}}, y^i_{\mathtt{t+1}}, y^k_{\mathtt{t}})\right)\right) \\ & + \left(\ln\left(\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\alpha^i;\beta^i}_{\mathtt{t+1}}(x^i_{\mathtt{t}}, x^k_{\mathtt{t+1}}, y^i_{\mathtt{t}}, y^k_{\mathtt{t+1}})\right)\right)\Bigg]\end{aligned} \tag{3.5}$$

3.2 Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures: equivalence condition

The upcoming statement displays the environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures.

Definition 3.4 *Let $T_{\mathtt{t},\mathtt{t+1}}$ be a production process that satisfies properties T1-T4. For any $(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}}) \in \mathbb{R}^{n+m}_{+}$ and for any $i, k = \{\mathtt{ec}, \mathtt{em}\}$ with $i \neq k$,*

i. *The environmentally-adjusted Hicks-Moorsteen productivity index is defined as follows,*

$$\mathrm{EAHM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}}) = \Bigg[\prod_{i=\{\mathtt{ec},\mathtt{em}\}} \frac{\mathtt{D}^{\beta^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})}{\mathtt{D}^{\beta^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, y^i_{\mathtt{t+1}}, y^k_{\mathtt{t}})} \times \frac{\mathtt{D}^{\alpha^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})}{\mathtt{D}^{\alpha^i}_{\mathtt{t}}(x^i_{\mathtt{t+1}}, x^k_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})} \times \frac{\mathtt{D}^{\beta^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, y^i_{\mathtt{t}}, y^k_{\mathtt{t+1}})}{\mathtt{D}^{\beta^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})} \times \frac{\mathtt{D}^{\alpha^i}_{\mathtt{t+1}}(x^i_{\mathtt{t}}, x^k_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})}{\mathtt{D}^{\alpha^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})} \Bigg]^{\frac{1}{2}} \tag{3.6}$$

where $\alpha = (\alpha^{\mathtt{ec}}, \alpha^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\beta = (\beta^{\mathtt{ec}}, \beta^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$.

ii. *The environmentally-adjusted Luenberger-Hicks-Moorsteen productivity indicator is defined as,*

$$\begin{aligned}\mathrm{EALHM}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}}) = \frac{1}{2}\Bigg[\sum_{i=\{\mathtt{ec},\mathtt{em}\}} & \Big(\overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, y^i_{\mathtt{t+1}}, y^k_{\mathtt{t}})\Big) \\ & + \Big(\overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t}}(x^i_{\mathtt{t+1}}, x^k_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\Big) \\ & + \Big(\overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, y^i_{\mathtt{t}}, y^k_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})\Big) \\ & + \Big(\overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t+1}}(x^i_{\mathtt{t}}, x^k_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})\Big)\Bigg]\end{aligned} \tag{3.7}$$

where $\gamma = (\gamma^{\mathtt{ec}}, \gamma^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\sigma = (\sigma^{\mathtt{ec}}, \sigma^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$.

Equivalence condition for the environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures is defined in the next result.

Proposition 3.5 *For any $(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}}) \in \mathbb{R}^{n+m}_{++}$, the equivalence condition for the environmentally-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures is defined as:*

$$\ln\Big(\mathrm{EAHM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}})\Big) \equiv \mathrm{EALHM}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\Big(\ln(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}), \ln(\mathbf{y}_{\mathtt{t},\mathtt{t+1}})\Big), \tag{3.8}$$

where $\ln\Big(\mathrm{EAHM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}})\Big), \mathrm{EALHM}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\Big(\ln(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}), \ln(\mathbf{y}_{\mathtt{t},\mathtt{t+1}})\Big) \in \ln\left(T^{++}_{\mathtt{t},\mathtt{t+1}}\right)$ *such that* $\gamma = \left(\frac{\alpha^{\mathtt{ec}}}{\ln(x^{\mathtt{ec}}_{\mathtt{t}})}, \frac{\alpha^{\mathtt{em}}}{\ln(x^{\mathtt{em}}_{\mathtt{t}})}\right)$ *and* $\sigma = \left(\frac{\beta^{\mathtt{ec}}}{\ln(y^{\mathtt{ec}}_{\mathtt{t}})}, \frac{\beta^{\mathtt{em}}}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}\right)$.

Additive version of the environmentally-adjusted Hicks-Moorsteen productivity index and its reciprocal are presented in the statement below. These productivity measures are named the Hicks-Moorsteen-Luenberger index and the reciprocal Hicks-Moorsteen-Luenberger indicator, respectively.

Corollary 3.6 *For any* $(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) \in \mathbb{R}^{n+m}_{++}$*, and any* $\gamma^i = \dfrac{\alpha^i}{\ln(x^i_{\mathtt{t,t+1}})}$ *and* $\sigma^i = \dfrac{\beta^i}{\ln(y^i_{\mathtt{t,t+1}})}$ *with* $i, k = \{\mathtt{ec}, \mathtt{em}\}$ *and* $i \neq k$*:*

i. *The environmentally-adjusted Hicks-Moorsteen-Luenberger productivity index is defined as follows,*

$$\begin{aligned}\mathrm{EAHML}^{\alpha;\beta}_{\mathtt{t,t+1}}(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) = \Bigg[\prod_{i=\{\mathtt{ec},\mathtt{em}\}} & \frac{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t}}(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}}))\right)}{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t}}(\ln(\mathbf{x}_{\mathtt{t}}), \ln(y^i_{\mathtt{t+1}}), \ln(y^k_{\mathtt{t}}))\right)} \times \frac{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t}}(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}}))\right)}{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t}}(\ln(x^i_{\mathtt{t+1}}), \ln(x^k_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}}))\right)} \\ & \times \frac{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t+1}}(\ln(\mathbf{x}_{\mathtt{t+1}}), \ln(y^i_{\mathtt{t}}), \ln(y^k_{\mathtt{t+1}}))\right)}{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\sigma^i}_{\mathtt{t+1}}(\ln(\mathbf{x}_{\mathtt{t+1}}), \ln(\mathbf{y}_{\mathtt{t+1}}))\right)} \times \frac{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t+1}}(\ln(x^i_{\mathtt{t}}), \ln(x^k_{\mathtt{t+1}}), \ln(\mathbf{y}_{\mathtt{t+1}}))\right)}{\mathtt{exp}\left(\overrightarrow{\mathtt{D}}^{\gamma^i}_{\mathtt{t+1}}(\ln(\mathbf{x}_{\mathtt{t+1}}), \ln(\mathbf{y}_{\mathtt{t+1}}))\right)} \Bigg]^{\frac{1}{2}}\end{aligned} \tag{3.9}$$

ii. *The reciprocal environmentally-adjusted Hicks-Moorsteen-Luenberger productivity indicator is defined as follows,*

$$\begin{aligned}\overline{\mathrm{EAHML}}^{\gamma;\sigma}_{\mathtt{t,t+1}}(\ln(\mathbf{x}_{\mathtt{t,t+1}}), \ln(\mathbf{y}_{\mathtt{t,t+1}})) = \frac{1}{2}\Bigg[& \sum_{i=\{\mathtt{ec},\mathtt{em}\}} \left(\ln\left(\mathtt{D}^{\beta^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\right) - \ln\left(\mathtt{D}^{\beta^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, y^i_{\mathtt{t+1}}, y^k_{\mathtt{t}})\right)\right) \\ & + \left(\ln\left(\mathtt{D}^{\alpha^i}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\right) - \ln\left(\mathtt{D}^{\alpha^i}_{\mathtt{t}}(x^i_{\mathtt{t+1}}, x^k_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\right)\right) \\ & + \left(\ln\left(\mathtt{D}^{\beta^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, y^i_{\mathtt{t}}, y^k_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\beta^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})\right)\right) \\ & + \left(\ln\left(\mathtt{D}^{\alpha^i}_{\mathtt{t+1}}(x^i_{\mathtt{t}}, x^k_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\alpha^i}_{\mathtt{t+1}}(\mathbf{x}_{\mathtt{t+1}}, \mathbf{y}_{\mathtt{t+1}})\right)\right) \Bigg]\end{aligned} \tag{3.10}$$

3.3 A class of examples

The general shape of the additive and multiplicative environmentally-adjusted productivity measures allows to provide equivalences with classical additive- and multiplicative-based productivity measures as displayed in the next statements.

Proposition 3.7 *For any* $(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) \in \mathbb{R}^{n+m}_{++}$ *and a logarithmically transformed production technology* $\ln\left(T^{++}_{\mathtt{t,t+1}}\right)$*, the environmentally-adjusted Malmquist and Luenberger productivity measures are respectively equivalent to classical Malmquist and Luenberger productivity measures, under some conditions.*

Table 1 displays the equivalence conditions between the environmentally-adjusted Malmquist and Luenberger productivity measures and classical Malmquist and Luenberger productivity measures, respectively[4].

Productivity measures	**Parameters**				**References**
$\mathrm{EAL}^{\gamma;\sigma}_{\mathtt{t,t+1}}(\ln(\cdot)) \equiv$	$\gamma^{\mathtt{ec}}$	$\gamma^{\mathtt{en}}$	$\sigma^{\mathtt{ec}}$	$\sigma^{\mathtt{en}}$	
$-\ln(M^{\mathtt{ec}}_{\mathtt{t,t+1}}(\cdot))$	0	0	$\frac{1}{\ln(y^{\mathtt{ec}}_{\mathtt{t,t+1}})}$	0	Caves et *al.* (1982)
$\ln(M^{\mathtt{em}}_{\mathtt{t,t+1}}(\cdot))$	0	0	0	$-\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t,t+1}})}$	
$\ln(\mathrm{EAM}^{\alpha;\beta}_{\mathtt{t,t+1}}(\cdot)) \equiv$	$\alpha^{\mathtt{ec}}$	$\alpha^{\mathtt{en}}$	$\beta^{\mathtt{ec}}$	$\beta^{\mathtt{en}}$	
$L^{\mathtt{ec}}_{\mathtt{t,t+1}}(\cdot)$	0	0	$\ln(y^{\mathtt{ec}}_{\mathtt{t,t+1}})$	0	Chambers (2002)
$L^{\mathtt{em}}_{\mathtt{t,t+1}}(\cdot)$	0	0	0	$-\ln(y^{\mathtt{em}}_{\mathtt{t,t+1}})$	

Table 1: Environmentally-adjusted and classical Malmquist and Luenberger productivity measures equivalences.

Proposition 3.8 *For any* $(\mathbf{x}_{\mathtt{t,t+1}}, \mathbf{y}_{\mathtt{t,t+1}}) \in \mathbb{R}^{n+m}_{++}$ *and a transformed production technology* $\ln\left(T^{++}_{\mathtt{t,t+1}}\right)$*, the environmentally-adjusted Luenberger-Hicks-Moorsteen and Hicks-Moorsteen productivity measures are respectively equivalent to classical Luenberger-Hicks-Moorsteen and Hicks-Moorsteen productivity measures, under some conditions.*

Table 2 provides the equivalence conditions between the environmentally-adjusted Total Factor Productivity (TFP) measures and the classical Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures.

[4]Note that the outcomes provided in Table 1 are based upon output oriented Malmquist and Luenberger productivity measures. However, equivalence results can also be defined through input orientation.

Productivity measures	**Parameters**				**References**
$\mathrm{EALHM}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}(\ln(\cdot)) \equiv$	$\gamma^{\mathtt{ec}}$	$\gamma^{\mathtt{en}}$	$\sigma^{\mathtt{ec}}$	$\sigma^{\mathtt{en}}$	
$-\ln(HM^{\mathtt{ec}}_{\mathtt{t},\mathtt{t+1}}(\cdot))$	$\frac{1}{\ln(x^{\mathtt{ec}}_{\mathtt{t},\mathtt{t+1}})}$	0	$\frac{1}{\ln(y^{\mathtt{ec}}_{\mathtt{t},\mathtt{t+1}})}$	0	Bjurek (1996)
$\ln(HM^{\mathtt{em}}_{\mathtt{t},\mathtt{t+1}}(\cdot))$	0	$\frac{1}{\ln(x^{\mathtt{em}}_{\mathtt{t},\mathtt{t+1}})}$	0	$-\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t},\mathtt{t+1}})}$	
$\ln(\mathrm{EAHM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(\cdot)) \equiv$	$\alpha^{\mathtt{ec}}$	$\alpha^{\mathtt{en}}$	$\beta^{\mathtt{ec}}$	$\beta^{\mathtt{en}}$	
$LHM^{\mathtt{ec}}_{\mathtt{t},\mathtt{t+1}}(\ln(\cdot))$	$\ln(x^{\mathtt{ec}}_{\mathtt{t},\mathtt{t+1}})$	0	$\ln(y^{\mathtt{ec}}_{\mathtt{t},\mathtt{t+1}})$	0	Briec and Kerstens (2004)
$LHM^{\mathtt{em}}_{\mathtt{t},\mathtt{t+1}}(\ln(\cdot))$	0	$\ln(x^{\mathtt{em}}_{\mathtt{t},\mathtt{t+1}})$	0	$-\ln(y^{\mathtt{em}}_{\mathtt{t},\mathtt{t+1}})$	

Table 2: Environmentally-adjusted TFP and classical TFP equivalences.

Remark that the outcomes displayed in Table 1 provide as an immediate result the equivalence between the environmentally-adjusted Luenberger productivity indicator and the hyperbolic-based Malmquist productivity index $M^{H^o}_{\mathtt{t},\mathtt{t+1}}(\cdot)$ by considering any $(\mathbf{x}_{\mathtt{t},\mathtt{t+1}}, \mathbf{y}_{\mathtt{t},\mathtt{t+1}}) \in \mathbb{R}^{n+m}_{++}$ and a transformed production technology $\ln\left(T^{++}_{\mathtt{t},\mathtt{t+1}}\right)$. Reciprocally, the environmetally-adjusted Malmquist productivity measure is equivalent to the environmetal Luenberger productivity indicator $EL_{\mathtt{t},\mathtt{t+1}}(\cdot)$ (Azad and Ancev, 2014). These equivalences are displayed as follows:

$$\ln\left(M^{H^o}_{\mathtt{t},\mathtt{t+1}}(\cdot)\right) \equiv \mathrm{EAL}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\left(\ln(\cdot)\right) \text{ with } \gamma = 0 \text{ and } \sigma = \left(\frac{1}{\ln(y^{\mathtt{ec}}_{\mathtt{t}})}, -\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}\right), \tag{3.11}$$

$$EL_{\mathtt{t},\mathtt{t+1}}(\cdot) \equiv \ln\left(\mathrm{EAM}^{\alpha;\beta}_{\mathtt{t}}(\cdot)\right) \text{ with } \alpha = 0 \text{ and } \beta = \left(\ln(y^{\mathtt{ec}}_{\mathtt{t}}), -\ln(y^{\mathtt{em}}_{\mathtt{t}})\right). \tag{3.12}$$

4 Empirical illustration

This section aims to propose an empirical illustration highlighting the equivalence conditions stated in the previous sections. To do so, both production technology and distance functions are non parametrically specified in the upcoming section.

4.1 Non parametric specifications

To propose non parametric specifications of the production technology and of the distance functions, we use the data envelopment analysis (DEA) approach.

In this empirical illustration, the non convex multiplicative production set of Banker and Maindiratta (1986) is considered. Indeed, non convexity allows to take account for local strictly increasing returns-to-scale contrary to the variable returns-to-scale production technology of Banker et *al.* (1984), which is well known as the BCC model. Moreover, non convexity shows that there exists non-linear relationship between the inputs and the outputs and hence, there can be indivisibilities in the production process (Sahoo and Tone, 2013). Specifically, multiplicative version of the by-production model satisfies $T1 - T4$ and is considered to display the applicability of the unified framework presented in this paper (Abad and Briec, 2019; Murty et *al.*, 2012).

For any $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}_{+}^{n+m}$ and any set of observations $\mathcal{J} = \{1, \cdots, J\}$ with $j \in \mathcal{J}$, the non parametric multiplicative by-production technology is defined as:

$$T_{\mathtt{t}}^{\mathtt{M}} := \Bigg\{ (\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}_{+}^{n+m} : \mathbf{x}_{\mathtt{t}} \geq \prod_{j \in \mathcal{J}} \left(\mathbf{x}_{\mathtt{t}}^{j}\right)^{\lambda_j}, \mathbf{y}_{\mathtt{t}} \leq \prod_{j \in \mathcal{J}} \left(\mathbf{y}_{\mathtt{t}}^{j}\right)^{\lambda_j}, x_{\mathtt{t}}^{\mathtt{em}} \leq \prod_{j \in \mathcal{J}} \left(x_{\mathtt{t}}^{\mathtt{em},j}\right)^{\mu_j},$$

$$y_{\mathtt{t}}^{\mathtt{em}} \geq \prod_{j \in \mathcal{J}} \left(y_{\mathtt{t}}^{\mathtt{em},j}\right)^{\mu_j}; \lambda, \mu \geq 0; \sum_{j \in \mathcal{J}} \lambda_j = \mu_j = 1 \Bigg\}. \quad (4.1)$$

Now, suppose that we consider strictly positive input-output vectors such that $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}_{++}^{n+m}$. Hence, the natural logarithmic transformation of the multiplicative by-production technology yields a Napierian by-production set as follows:

$$\ln\left(T_{\mathtt{t}}^{\mathtt{M}}\right) := \Bigg\{ (\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}_{++}^{n+m} : \ln(\mathbf{x}_{\mathtt{t}}) \geq \sum_{j \in \mathcal{J}} \lambda_j \ln\left(\mathbf{x}_{\mathtt{t}}^{j}\right), \ln(\mathbf{y}_{\mathtt{t}}) \leq \sum_{j \in \mathcal{J}} \lambda_j \ln\left(\mathbf{y}_{\mathtt{t}}^{j}\right),$$

$$\ln(x_{\mathtt{t}}^{\mathtt{em}}) \leq \sum_{j \in \mathcal{J}} \mu_j \ln\left(x_{\mathtt{t}}^{\mathtt{em},j}\right), \ln(y_{\mathtt{t}}^{\mathtt{em}}) \geq \sum_{j \in \mathcal{J}} \mu_j \ln\left(y_{\mathtt{t}}^{\mathtt{em},j}\right); \lambda, \mu \geq 0; \sum_{j \in \mathcal{J}} \lambda_j = \mu_j = 1 \Bigg\}. \quad (4.2)$$

It is worth noting that the Napierian technology has similar structure with the BCC production set. In such case, the multiplicative by-production set is a log-additive and a log-convex set.

See Appendix III for the non parametric specifications of the distance functions.

4.2 Data

In this paper, we use a dataset sourced from the work of Jeon and Sickles (2004). The considered decision making units are 17 OECD countries: (1) Canada; (2) the United States; (3) Japan; (4) Austria; (5) Belgium; (6) Denmark; (7) Finland; (8) France; (9) Germany; (10) Greece; (11) Ireland; (12) Italy; (13) Norway; (14) Spain; (15) Sweden; (16) United Kingdoms; (17) Australia, for the years 1989 and 1990. The dataset includes two no polluting inputs (i) capital and (ii) labour; one polluting input (iii) energy; one polluting output (iv) carbon dioxide emission and one no polluting output (v) gross domestic product.

4.3 Results

Observations	**(1a) EAM**	**(1b) ln(EAM)**	**(2) EAL**	**(3a) EAHM**	**(3b) ln(EAHM)**	**(4) EALHM**
Canada	∞	∞	∞	1.0474	0.0464	0.0464
United States	∞	∞	∞	0.9646	-0.0360	-0.0360
Japan	1.0154	0.0152	0.0152	0.9490	-0.0523	-0.0523
Austria	1.0002	0.0002	0.0002	0.9309	-0.0716	-0.0716
Belgium	∞	∞	∞	0.9470	-0.0545	-0.0545
Denmark	∞	∞	∞	1.0806	0.0775	0.0775
Finland	∞	∞	∞	0.9471	-0.0543	-0.0543
France	1.0001	0.0001	0.0001	0.9787	-0.0216	-0.0216
Germany	0.9989	-0.0011	-0.0011	0.9817	-0.0184	-0.0184
Greece	∞	∞	∞	0.8927	-0.1135	-0.1135
Ireland	∞	∞	∞	0.9489	-0.0524	-0.0524
Italy	1.0113	0.0113	0.0113	1.0019	0.0019	0.0019
Norway	∞	∞	∞	1.0321	0.0316	0.0316
Spain	1.0018	0.0018	0.0018	1.0987	0.0941	0.0941
Sweden	0.9991	-0.0009	-0.0009	1.0596	0.0579	0.0579
United Kingdoms	∞	∞	∞	0.9494	-0.0519	-0.0519
Australia	∞	∞	∞	0.8694	-0.1399	-0.1399

Table 3: Environmentally-adjusted productivity scores over the period 1989-1990.

The results in Table 3 are illustrated in Figure 1 for the environmentally-adjusted Malmquist and Luenberger productivity measures and in Figure 2 for the environmentally-adjusted Hicks-Moorsteen and, Luenberger-Hicks-Moorsteen productivity measures.

Results in Table 3 show that 5 countries have productivity improvement through measures (1a-1b) and (2) whereas 2 countries are facing productivity loss over the considered period.

It is worth noting that EAM and EAL are defined with complete cross-sectional distance measures such that infeasibilities may occur (Abad, 2015). In such case, some results are indeterminate and cannot be interpreted. Specifically, in this empirical illustration, 10 out of 17 observations have indeterminate productivity measure.

Total factor productivity measures such as (3a-3b) and (4) allow to overcome the issue of indeterminateness since these measures are always assessed towards feasible directions. Thus, results of productivity measures (3a-3b) and (4) cannot be infeasible and can always be interpreted as show in Table 3. These results show that 7 out of 17 observations have productivity gains.

As visible in Figure 1 and Figure 2, if an observation faces a productivity loss under an EAM productivity measure then it also deals with a productivity deterioration through an EAL productivity indicator. This outcome is a result of the connection between EAM and EAL productivity measures. The same reasoning holds for the EAHM and EALHM total factor productivity measures. Regarding the additive and multiplicative classical families of productivity measures, namely Hicks-Moorsteen-, Luenberger-Hicks-Moorsteen-, Malmquist- and Luenberger-based indicators, similar outcomes are provided in Appendix IV.

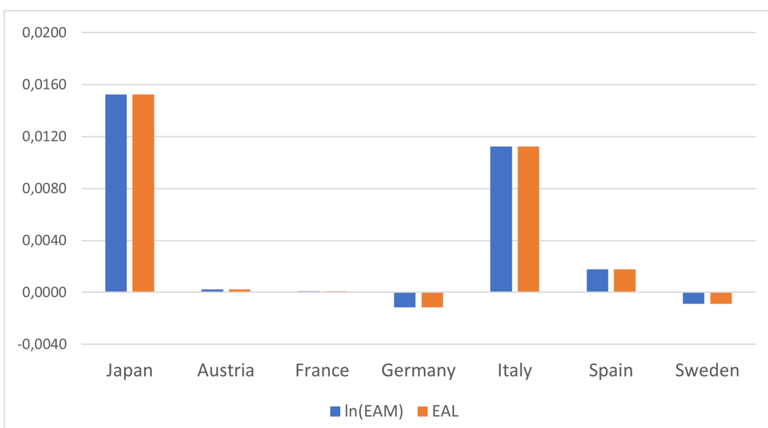

Figure 1: Environmentally-adjusted Luenberger and logarithmically transformed Malmquist productivity measures.

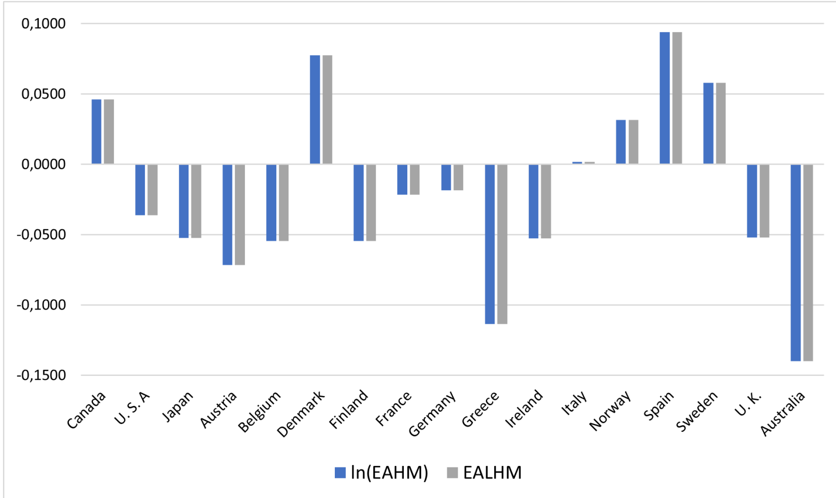

Figure 2: Environmentally-adjusted Luenberger-Hicks-Moorsteen and logarithmically transformed Hicks-Moorsteen productivity measures.

5 Concluding Comments

This paper presents a general method comparing additive- and multiplicative-based productivity measures. Specifically, this contribution lays out equivalence conditions for the additive and multiplicative environmentally-adjusted productivity measures. Therefore, an unified framework allowing to compare additive and multiplicative environmentally-adjusted productivity measures is provided.

Using the unified framework defined in this paper, researchers would be able to analyse environmentally-adjusted productivity variation through a general method. Indeed, the proposed approach allows to remove differences in magnitudes of empirical results arising from the use of different families of productivity measures. As a result, the proposed approach would provide a theoretical framework that is neutral as regard to the class of selected productivity measures.

References

[1] Abad, A., P. Ravelojaona (2022) A Generalization of Environmental Productivity Analysis, *Journal of Productivity Analysis*, 57, 61-78.

[2] Abad, A., P. Ravelojaona (2021) Pollution-adjusted Productivity Analysis: The Use of Malmquist and Luenberger Productivity Measures, *Managerial and Decision Economics*, 42(3), 635-648.

[3] Abad, A., W. Briec (2019) On the Axiomatic of Pollution-generating Technologies: a Non-Parametric Approach, *European Journal of Operational Research*, 277(1), 377-390.

[4] Abad, A. (2015) An environmental generalised Luenberger-Hicks-Moorsteen productivity indicator and an environmental generalised Hicks-Moorsteen productivity index, *Journal of Environmental Management*, 161, 325-334.

[5] Azad, M.A.S., T. Ancev (2014) Measuring environmental efficiency of agricultural water use:A Luenberger environmental indicator, *Journal of Environmental Management*, 145, 314-320.

[6] Banker, R., A. Maindiratta (1986) Piecewise Loglinear Estimation of Efficient Production Surfaces, *Management Science*, 32(1), 126-135.

[7] Banker, R., Charnes, A., and W.W. Cooper (1984) Some Models for Estimating Technical and Scale Efficiency in Data Envelopment Analysis,*Management Science*, 30, 1078-1092.

[8] Bjurek, H. (1996) The Malmquist Total Factor Productivity Index, *Scandinavian Journal of Economics*, 98, 303-313.

[9] Boussemart, J.P., W. Briec, K. Kerstens, J.-C. Poutineau (2003) Luenberger and Malmquist Productivity Indices: Theoretical Comparisons and Empirical Illustration, *Bulletin of Economic Research*, 55(4), 391-405.

[10] Briec, W. (1997) A Graph-Type Extension of Farrell Technical Efficiency Measure, *Journal of Productivity Analysis*, 8, 95-110.

[11] Briec, W., K. Kerstens (2004) A Luenberger-Hicks-Moorsteen Productivity Indicator: Its Relation to the Hicks-Moorsteen Productivity Index and theLuenberger Productivity Indicator, *Economic Theory*, 23(4), 925-939.

[12] Caves, D.W., L.R. Christensen, W.E. Diewert (1982) The Economic Theory of Index Numbers and the Measurement of Inputs, Outputs and Productivity, *Econometrica*, 50, 1393-1414.

[13] Chambers, R.G. (2002) Exact Nonradial Input, Output, and Productivity Measurement, *Economic Theory*, 20, 751-765.

[14] Chambers, R.G., R. Färe (2020) Distance Functions in Production Economics, in Ray S.C., Chambers R., Kumbhakar S. (eds) *Handbook of Production Economics*, Springer, Singapore.

[15] Chambers, R.G., Chung, Y., R. Färe (1998) Profit, Directional Distance Functions, and Nerlovian Efficiency, *Journal of Optimization Theory and Applications*, 98, 351-364.

[16] Chambers, R.G., Färe, R., S. Grosskopf (1996) Productivity Growth in APEC Countries, *Pacific Economic Review*, 1, 181-190.

[17] Chung, Y.H., Färe, R., S. Grosskopf (1997) Productivity and undesirable outputs: A directional distance function approach, *Journal of Environmental Management*, 51, 229-240.

[18] Debreu, G. (1951) The coefficient of ressource utilisation, *Econometrica*, 19, 273-292.

[19] Diewert, W.E. (1998) *Index Number Theory Using Differences rather than Ratios*, Vancouver, University of British Columbia (Department of Economics: DP 98-10).

[20] Farrell, M.J. (1957) The measurement of technical efficiency, *Journal of the Royal Statistical Society*, 120(3), 253-290.

[21] Färe, R., Grosskopf, S., C.A.K. Lovell (1985) *The Measurement of Efficiency of Production*, Springer.

[22] Färe, R., Grosskopf, S., Lovell, C.A.K., C. Pasurka (1989) Multilateral productivity comparisons when some outputs are undesirable: A non parametric approach, *The Review of Economics and Statistics*, 71, 90-98.

[23] Färe, R., Grosskopf, S., Norris, M., Z. Zhang (1994) Productivity growth, technical progress, and efficiency change in industrialized countries, *The American Economic Review*, 84, 66-83.

[24] Färe, R., Grosskopf, S., F. Hernandez-Sancho (2004) Environmental performance: an index number approach, *Resource and Energy Economics*, 26, 343-352.

[25] Färe, R., Margaritis, D., Rouse, P., I. Roshdi (2016) Estimating the hyperbolic distance function: A directional distance function approach, *European Journal of Operational Research*, 254, 312-319.

[26] Jeon, B. M., R. C. Sickles (2004) The Role of Environmental Factors in Growth Accounting: a Nonparametric Analysis, *Journal of Applied Econometrics*, 19(5), 567-591.

[27] Mehdiloozad, M., B.K. Sahoo, I. Roshdi (2014) A Generalized Multiplicative Directional Distance Function for Efficiency Measurement in DEA, *European Journal of Operational Research*, 232, 679-688.

[28] Murty, S. , R. R. Russell, S. B. Levkoff (2012) On Modeling Pollution-Generating Technologies, *Journal of Environmental Economics and Management*, 64, 117-135.

[29] Peyrache, A., T.J. Coelli (2009) *Multiplicative directional distance function*, CEPA working paper, School of Economics, University of Queensland, Australia.

[30] Picazo-Tadeo, A.J., J. Castillo-Giménez, M. Beltrán-Esteve (2014) An intertemporal approach to measuring environmental performance with directional distance functions: Greenhouse gas emissions in theEuropean Union, *Ecological Economics* , 100, 173-182.

[31] Prasada Rao, D.S. (2020) Index Numbers and Productivity Measurement, in Ray S.C., Chambers R., Kumbhakar S. (eds) *Handbook of Production Economics*, Springer, Singapore.

[32] Sahoo, B. K., K. Tone (2013) Non-parametric Measurement of Economies of Scale and Scope in Non-competitive Environment with Price Uncertainty, *Omega*, 41(1), 97-111.

[33] Shephard, R.W. (1970) *Theory of Cost and Production Functions*,Princeton: Princeton University Press.

[34] Sueyoshi, T, Yuan, Y, M. Goto (2017) A literature study for DEA applied to energy and environment, *Energy Economics*, 62, 102-124.

[35] Zhou, P., Ang, B. W., and K. L. Poh (2008) A survey of data envelopment analysis in energy and environmental studies, *European Journal of Operational Research*, 189, 1-18.

Appendix I

The following results highlight connections between the environmentally-adjusted multiplicative distance function (2.2), the intended ($D^{\texttt{ec}}_{\texttt{t}}$) and unintended ($D^{\texttt{em}}_{\texttt{t}}$) outputs Shephard distance functions (Färe et *al.*, 2004), the hyperbolic output ($H^{o}_{\texttt{t}}$) distance function (Färe et *al.*, 1989) and the hyperbolic environmental efficiency ($\Psi_{\texttt{t}}$) measure (Abad and Ravelojaona, 2022). Moreover, the next outcomes also permit comparing the environmentally-adjusted additive distance function (2.3) with the environmental directional ($\overrightarrow{D}_{\texttt{t}}$) distance function (Chung et *al.*, 1997), the intended ($\overrightarrow{D}^{\texttt{ec}}_{\texttt{t}}$) and unintended ($\overrightarrow{D}^{\texttt{em}}_{\texttt{t}}$) sub-vector directional distance functions (Picazo-Tadeo et *al.*, 2014) as well as the environmental disaggregated directional ($\Xi_{\texttt{t}}$) distance function (Abad and Ravelojaona, 2022).

Proposition 5.1 *For any* $(\mathbf{x}_{\texttt{t}}, \mathbf{y}_{\texttt{t}}) \in \mathbb{R}^{n+m}_{+}$*, the environmentally-adjusted additive- and multiplicative-based distance function allows to provide classical additive- and multiplicative-based efficiency measures, as follows:*

	Efficiency measures	**Parameters**				**References**
	$\mathtt{D}^{\alpha;\beta}_{\texttt{t}}(\cdot) \equiv$	$\alpha^{\texttt{ec}}$	$\alpha^{\texttt{en}}$	$\beta^{\texttt{ec}}$	$\beta^{\texttt{en}}$	
a.	$[D^{\texttt{ec}}_{\texttt{t}}(\cdot)]^{-1}$	0	0	1	0	Färe et *al.* (2004)
b.	$D^{\texttt{em}}_{\texttt{t}}(\cdot)$	0	0	0	-1	
c.	$H^{o}_{\texttt{t}}(\cdot)$	0	0	1	-1	Färe et *al.* (1989)
d.	$\Psi_{\texttt{t}}(\cdot)$	1	1	1	-1	Abad and Ravelojaona (2022)
	$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\texttt{t}}(\cdot) \equiv$	$\gamma^{\texttt{ec}}$	$\gamma^{\texttt{en}}$	$\sigma^{\texttt{ec}}$	$\sigma^{\texttt{en}}$	
e.	$\overrightarrow{D}^{\texttt{ec}}_{\texttt{t}}(\cdot)$	0	0	1	0	Picazo-Tadeo et *al.* (2014)
f.	$\overrightarrow{D}^{\texttt{em}}_{\texttt{t}}(\cdot)$	0	0	0	-1	
g.	$\overrightarrow{D}_{\texttt{t}}(\cdot)$	0	0	1	-1	Chung et *al.* (1997)
h.	$\Xi_{\texttt{t}}(\cdot)$	1	1	1	-1	Abad and Ravelojaona (2022)

Table 4: Environmentally-adjusted and classical distance functions equivalence

The upcoming statements display equivalence conditions for the proposed environmentally-adjusted distance functions (2.2)-(2.3) and the aforementioned additive- and multiplicative-based distance functions.

Proposition 5.2 *For any* $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$ *and any logarithmically transformed production set* $\ln(T^{++}_{\mathtt{t}})$*, the environmentally-adjusted additive and multiplicative distance functions are equivalent to classical additive- and multiplicative-based efficiency measures, as follows:*

	Efficiency measures	**Parameters**				**Reference**
	$\ln(\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(\cdot)) \equiv$	$\alpha^{\mathtt{ec}}$	$\alpha^{\mathtt{en}}$	$\beta^{\mathtt{ec}}$	$\beta^{\mathtt{en}}$	
i.	$\overrightarrow{D}^{\mathtt{ec}}_{\mathtt{t}}(\ln(\cdot))$	0	0	$\ln(y^{\mathtt{ec}}_{\mathtt{t}})$	0	Picazo-Tadeo et *al.* (2014)
ii.	$\overrightarrow{D}^{\mathtt{em}}_{\mathtt{t}}(\ln(\cdot))$	0	0	0	$-\ln(y^{\mathtt{em}}_{\mathtt{t}})$	
iii.	$\overrightarrow{D}_{\mathtt{t}}(\ln(\cdot))$	0	0	$\ln(y^{\mathtt{ec}}_{\mathtt{t}})$	$-\ln(y^{\mathtt{em}}_{\mathtt{t}})$	Chung et *al.* (1997)
iv.	$\Xi_{\mathtt{t}}(\ln(\cdot))$	$\ln(x^{\mathtt{ec}}_{\mathtt{t}})$	$\ln(x^{\mathtt{em}}_{\mathtt{t}})$	$\ln(y^{\mathtt{ec}}_{\mathtt{t}})$	$-\ln(y^{\mathtt{em}}_{\mathtt{t}})$	Abad and Ravelojaona (2022)
	$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}(\ln(\cdot)) \equiv$	$\gamma^{\mathtt{ec}}$	$\gamma^{\mathtt{en}}$	$\sigma^{\mathtt{ec}}$	$\sigma^{\mathtt{en}}$	
v.	$-\ln(D^{\mathtt{ec}}_{\mathtt{t}}(\cdot))$	0	0	$\frac{1}{\ln(y^{\mathtt{ec}}_{\mathtt{t}})}$	0	Färe et *al.* (2004)
vi.	$\ln(D^{\mathtt{em}}_{\mathtt{t}}(\cdot))$	0	0	0	$-\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}$	
vii.	$\ln(H^{o}_{\mathtt{t}}(\cdot))$	0	0	$\frac{1}{\ln(y^{\mathtt{ec}}_{\mathtt{t}})}$	$-\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}$	Färe et *al.* (1989)
viii.	$\ln(\Psi_{\mathtt{t}}(\cdot))$	$\frac{1}{\ln(x^{\mathtt{ec}}_{\mathtt{t}})}$	$\frac{1}{\ln(x^{\mathtt{ec}}_{\mathtt{t}})}$	$\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}$	$-\frac{1}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}$	Abad and Ravelojaona (2022)

Table 5: Equivalence conditions: environmentally-adjusted and classical efficiency measures.

Remark that the hyperbolic-based distance function (**c.**) can be approximated through a Taylor's expansion as proposed by Färe et *al.* (2016), with respect to some constraints. However, this paper assesses the hyperbolic efficiency measure by using the equivalence (**vi.**) in Proposition 5.2.

Appendix II

Proof of Proposition 2.2: Assume that $T_{\mathtt{t}}$ is a production technology that satisfies properties *T1-T4*. For any $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$, the environmentally-adjusted additive distance function (2.3) is defined as follows,

$$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) := \sup_{\delta} \left\{ \delta \geq 0 : \Big((I_n - \delta\gamma)\mathbf{x'}_{\mathtt{t}}, (I_m + \delta\sigma)\mathbf{y'}_{\mathtt{t}} \Big) \in T_{\mathtt{t}} \right\} \tag{5.1}$$

Therefore,

$$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}} (\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}})) := \sup_{\delta} \left\{ \delta \geq 0 : \Big((I_n - \delta\gamma)\ln(\mathbf{x}_{\mathtt{t}})', (I_m + \delta\sigma)\ln(\mathbf{y}_{\mathtt{t}})' \Big) \in T^{\ln}_{\mathtt{t}} \right\}, \tag{5.2}$$

such that $\ln(\mathbf{x}_{\mathtt{t}}) = (\ln(x^{\mathtt{ec}}_{\mathtt{t}}), \ln(x^{\mathtt{em}}_{\mathtt{t}}))$, $\ln(\mathbf{y}_{\mathtt{t}}) = (\ln(y^{\mathtt{ec}}_{\mathtt{t}}), \ln(y^{\mathtt{em}}_{\mathtt{t}}))$ and $T^{\ln}_{\mathtt{t}} := \{(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}})) : (\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in T^{++}_{\mathtt{t}}\} =: \ln(T^{++}_{\mathtt{t}})$.

Moreover, according to the definition of the environmentally-adjusted multiplicative distance function (2.2),

$$\ln\Big(\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\Big) := \sup_{\lambda} \left\{ \lambda \geq 1 : (-\alpha\ln(\lambda) + \ln(\mathbf{x}_{\mathtt{t}})', \beta\ln(\lambda) + \ln(\mathbf{y}_{\mathtt{t}})') \in \ln\left(T^{++}_{\mathtt{t}}\right) \right\}. \tag{5.3}$$

As a result, since $T^{\ln}_{\mathtt{t}} \equiv \ln(T^{++}_{\mathtt{t}})$ then, $\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}} (\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}})) \equiv \ln\Big(\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}})\Big)$ with $\gamma = \left(\frac{\alpha^{\mathtt{ec}}}{\ln(x^{\mathtt{ec}}_{\mathtt{t}})}, \frac{\alpha^{\mathtt{em}}}{\ln(x^{\mathtt{em}}_{\mathtt{t}})}\right)$ and $\sigma = \left(\frac{\beta^{\mathtt{ec}}}{\ln(y^{\mathtt{ec}}_{\mathtt{t}})}, \frac{\beta^{\mathtt{em}}}{\ln(y^{\mathtt{em}}_{\mathtt{t}})}\right)$. $\square$

Appendix III

Non parametric specification of the distance functions.

Multiplicative environmentally-adjusted distance function

The multiplicative environmentally-adjusted distance function is defined within the multiplicative technology $T^{\mathtt{M}}_{\mathtt{t}}$. Let $\alpha = (\alpha^{\mathtt{ec}}, \alpha^{\mathtt{em}}) \in \mathbb{R}^n_+$ and $\beta = (\beta^{\mathtt{ec}}, \beta^{\mathtt{em}}) \in \mathbb{R}^{m_{\mathtt{ec}}}_+ \times \mathbb{R}^{m_{\mathtt{em}}}_-$, for any $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$:

$$
\begin{aligned}
\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \quad &= \min \quad \lambda \\
\text{s.t.} \qquad & \lambda^{\alpha}\mathbf{x}_{\mathtt{t}} \geq \prod_{j\in\mathcal{J}} \left(\mathbf{x}^{j}_{\mathtt{t}}\right)^{\lambda_j} \\
& \lambda^{\beta}\mathbf{y}_{\mathtt{t}} \leq \prod_{j\in\mathcal{J}} \left(\mathbf{y}^{j}_{\mathtt{t}}\right)^{\lambda_j} \\
& \lambda^{\alpha^{\mathtt{em}}} x^{\mathtt{em}}_t \leq \prod_{j\in\mathcal{J}} \left(x^{\mathtt{em},j}_t\right)^{\mu_j} \\
& \lambda^{\beta^{\mathtt{em}}} y^{\mathtt{em}}_t \geq \prod_{j\in\mathcal{J}} \left(y^{\mathtt{em},j}_t\right)^{\mu_j} \\
& \sum_{j\in\mathcal{J}} \lambda_j = 1 \\
& \sum_{j\in\mathcal{J}} \mu_j = 1 \\
& \lambda, \mu \geq 0.
\end{aligned}
$$

Additive environmentally-adjusted distance function

The additive environmentally-adjusted distance function is defined within the natural logarithmic transformed multiplicative production technology $\ln(T^{\mathtt{M}}_{\mathtt{t}})$. For any $(\mathbf{x}_{\mathtt{t}}, \mathbf{y}_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$ and any $\gamma = (\gamma^{\mathtt{ec}}, \gamma^{\mathtt{em}})$ and $\sigma = (\sigma^{\mathtt{ec}}, \sigma^{\mathtt{em}})$ where $\gamma^{\mathtt{ec}}, \gamma^{\mathtt{em}} \in \mathbb{R}^n_+$, $\sigma^{\mathtt{ec}} \in \mathbb{R}^{m_{\mathtt{ec}}}_+$ and $\sigma^{\mathtt{em}} \in \mathbb{R}^{m_{\mathtt{em}}}_-$,

$$
\begin{aligned}
\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}\left(\ln(\mathbf{x}_{\mathtt{t}}), \ln(\mathbf{y}_{\mathtt{t}})\right) \quad = \max \quad & \delta \\
\text{s.t.} \qquad\qquad & (1-\delta\gamma)\ln(\mathbf{x}_{\mathtt{t}}) \geq \sum_{j\in\mathcal{J}} \lambda_j \ln\left(\mathbf{x}^{j}_{\mathtt{t}}\right) \\
& (1+\delta\sigma)\ln(\mathbf{y}_{\mathtt{t}}) \leq \sum_{j\in\mathcal{J}} \lambda_j \ln\left(\mathbf{y}^{j}_{\mathtt{t}}\right) \\
& (1-\delta\gamma^{\mathtt{em}})\ln(x^{\mathtt{em}}_{t}) \leq \sum_{j\in\mathcal{J}} \mu_j \ln\left(x^{\mathtt{em},j}_{t}\right) \\
& (1+\delta\sigma^{\mathtt{em}})\ln(y^{\mathtt{em}}_{t}) \geq \sum_{j\in\mathcal{J}} \mu_j \ln\left(y^{\mathtt{em},j}_{t}\right) \\
& \sum_{j\in\mathcal{J}} \lambda_j = 1 \\
& \sum_{j\in\mathcal{J}} \mu_j = 1 \\
& \lambda, \mu \geq 0.
\end{aligned}
$$

Appendix IV

Observations	M^{ec}	$\ln(M^{ec})$	L^{ec}	HM^{ec}	$\ln(HM^{ec})$	LHM^{ec}
Canada	1.0053	0.0053	0.0053	0.974	-0.0263	-0.0263
U. S. A	1.0245	0.0242	0.0242	0.9811	-0.0191	-0.0191
Japan	1.0131	0.0130	0.013	1.0116	0.0115	0.0115
Austria	1.0002	0.0002	0.0002	1.0256	0.0253	0.0252
Belgium	1.0224	0.0222	0.0221	1.0205	0.0203	0.0203
Denmark	1.0049	0.0049	0.0049	1.0063	0.0063	0.0063
Finland	1	0	0	0.9852	-0.0149	-0.015
France	1.0001	0.0001	0.0001	1.0159	0.0158	0.0158
Germany	1.0003	0.0003	0.0003	1.0351	0.0345	0.0345
Greece	1.0179	0.0177	0.0177	0.9982	-0.0018	-0.0018
Ireland	1.048	0.0469	0.0469	1.0759	0.0732	0.0731
Italy	1.0098	0.0098	0.0098	1.0002	0.0002	0.0002
Norway	1.0016	0.0016	0.0016	1.0122	0.0121	0.0122
Spain	1.0018	0.0018	0.0018	0.9878	-0.0123	-0.0122
Sweden	0.9991	-0.0009	-0.0009	1.0006	0.0006	0.0006
U. K.	1.0303	0.0299	0.0299	0.9698	-0.0307	-0.0307
Australia	1.0183	0.0181	0.0181	0.9537	-0.0474	-0.0474

Table 6: Economic-based productivity scores over the period 1989-1990.

Observations	M^{em}	$\ln(\mathrm{M}^{em})$	L^{em}	HM^{em}	$\ln(\mathrm{HM}^{em})$	LHM^{em}
Canada	∞	∞	∞	1.0754	0.0727	0.0727
U. S. A	∞	∞	∞	0.9832	-0.0169	-0.017
Japan	1.002	0.0020	0.002	0.9382	-0.0638	-0.0638
Austria	1	0	0	0.9077	-0.0968	-0.0968
Belgium	∞	∞	∞	0.928	-0.0747	-0.0747
Denmark	∞	∞	∞	1.0738	0.0712	0.0712
Finland	∞	∞	∞	0.9614	-0.0394	-0.0394
France	1	0	0	0.9634	-0.0373	-0.0373
Germany	0.9986	-0.0014	-0.0014	0.9484	-0.0530	-0.053
Greece	∞	∞	∞	0.8943	-0.1117	-0.1117
Ireland	∞	∞	∞	0.882	-0.1256	-0.1256
Italy	1.0015	0.0015	0.0015	1.0018	0.0018	0.0018
Norway	∞	∞	∞	1.0197	0.0195	0.0195
Spain	1	0	0	1.1122	0.1063	0.1064
Sweden	1	0	0	1.0589	0.0572	0.0572
U. K.	∞	∞	∞	0.979	-0.0212	-0.0212
Australia	∞	∞	∞	0.9116	-0.0926	-0.0925

Table 7: Polluting-based productivity scores over the period 1989-1990.