Constructing New Representations and the Implications for Decision Making Theory: Learning from Archimedes

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Introduction

In general, whether dealing with a statistical theory of decision making, management science, design science, or economics, especially when complex situations are being analyzed and decision makers are called upon to make optimized choices (e.g. minimal loss, maximum gain, etc.), it is common practice to rely on scientific models, notably those emanating from operations research. What these models have had in common, regardless of whether they are time-dependent, is a theoretical basis that is axiomatized by relying on related concepts of 'the universe' (or 'the world'), 'the state of nature' (or 'state of the world'), or the 'development of a state of nature' (or a true 'state of the world') where knowledge is partitioned as if it were always there, somewhere, in one part or another of the 'universe of possibles' (Savage, 1954; Simon, 1996; Aumann, 1999; Samuelson, 2004). Any alternative models that fail to embrace these basic scientific notions are either ignored or simply deemed irrational.

Furthermore, the ability to construct new representations, to be able to exploit outside information and adapt it for the purposes of other fields of knowledge, is a key to adaptation (Cohen and Levinthal, 1989; Helfat and Winter, 2011). Such construction can be seen, then, as a designing act insofar as, according to Larousse, it entails forming abstract entities in one's mind, arranging the various elements involved and then putting them, or causing them to be put, into practice. It can also be seen as the ability to grasp, assimilate and make a mental image of something in such a manner, and to derive such an idea or interpretation. For Simon (1996: 111), 'Everyone designs who devises courses of action aimed at changing existing situations into preferred ones.' Innovation is a dynamic process leading to the conception of an idea, a behavior or a new object, bringing about its assimilation and widespread application, whereby knowledge accumulates through learning and interactions (Oslo Manual, 2005; de Beaune, 2015). The level of innovative capacity is also linked to an increase in the stock of knowledge through absorption of elements of novelty (Oslo Manual, 2005). Similarly, innovation goes hand and hand with uncertainty, if simply because knowledge, ignorance and uncertainty are intertwined (Nelson and Winter, 1982; Oslo Manual, 2005; Sauce, 2010). In such an ever-changing universe, scientific foundations shift and evolve (Lundvall, 2007); yet, the process for constructing new representations remains the primary missing link in our theories of thinking (Simon,1996). And much is at stake. 'Given its systematic and structured nature, rational decision making can be slow, time-consuming, and effortful, and thus not always appropriate to deal with the time pressure, complexity, and uncertainty of innovation decision making' (Dane and Pratt cited in Calabretta et al., 2017: 367). Toward fostering a greater, generalizing cohesion, where conventional and innovational positions take their respective place according to the reference universe used, we have deliberately chosen to discuss a particular theoretical advance. The article is structured as follows: In the first section of this article we shall present the general decision making theory modeling inferential reasoning, which corresponds to the conventional model. In the second section, we consider an emblematic counter-example of an innovational model, namely Archimedes' thought experiment, which led to his famous 'Eureka' moment[1] (that word being at the root of 'heuristic'), by using a reproducible method described in the Palimpsest.[2] What is revealed is an information asymmetry in the general decision making theory, owing to the fact that the reference universe – a supposedly immutable reality – is not expressly formulated in the operative rules as conditional. Also brought to light is a two-fold rationality, with or without an excluded middle, operating within a reference universe or acting to reframe it. In the third section, we discuss new theoretical foundations, based on a concept of the reference universe, which is to be understood as a universality of reference (and therefore as conditional) relative to a state of knowledge. Any construction of new representations aspiring to be as generalized as possible, that is, so that any prior representation offering less explanatory or predictive strength can be embedded in it, entails a reframing of the reference universe and a redefining of the identity of things or phenomena. A non-reflexive identity relation differing from that operating within the reference universe (especially, when partitioning it conventionally) is therefore at work. This has the effect of adding a new dimension to the knowledge space as well as to the decision making space. In effect, the reframing process fully takes into account innovative situations. Lastly, the link with the paradox theory will also be addressed.

General decision-making theory modeling inferential reasoning.

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[1] An interjection taken from the Greek (ερηκα) translated as 'I found it'.
[2] The equivalent of a publication.

A decision-making model can assume known, invariable probabilities by which one or more individuals weigh their level of confidence in their choices. That is the case, for example, with the Markowitz (1952: 77, 79, 81) portfolio theory: 'The process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage … We assume static probability beliefs. In a general presentation we must recognize that the probability distribution of yields of the various securities is a function of time. The writer intends to present, in the future, the general, mathematical treatment that removes these limitations … This paper does not consider the difficult question of how investors do (or should) form their probability beliefs.' Generally speaking, the scientific models proposed by operations research, in statistical theory of decision making, management science or design science, for analyzing complex situations and intended to enable decision makers to undertake optimized choices (e.g. minimal loss, maximum gain, etc.), use similar assumptions. Optimization of bank portfolios, production networks organization, DNA sequencing, and satellite coverage are only a few of the problems of optimization being tackled through operations research. These scientific models are also used for knowledge discovery in databases, defined as 'the non-trivial process of identifying valid, novel, potentially useful and ultimately understandable patterns in data.' (Fayyad et al., 1996: 40–41). On the whole, this is research using artificial intelligence and adaptive learning, with a specified objective function, taking an empirical approach, with or without a prior knowledge, in a specific pre-defined research space. On a cyclical basis, they collect information, and often stochastically, with a view to enhancing understanding of the problem (based on different phases that can be classified in exploration phases or diversification phases), storing it in a myriad of possible forms, whether collectively (considering the problem as a whole) or inter individually (considering one solution in relation to another), then sorting through it so as to reduce dispersion (in the phase called exploitation or intensification). 'However, there exists a considerable area of design practice where standards of rigor in inference are as high as one could wish. I refer to the domain of so-called 'optimization methods', most highly developed in statistical decision theory and management science but acquiring growing importance also in engineering design theory. (…) The optimization problem is to find an admissible set of values of the command variables, compatible with the constraints, that maximize the (expected value of the) utility function for the given values of the environmental parameters' (Simon, 1996: 116).

A decision-making model can also be time-dependent. Bayesian inference modeling is based on interpreting p-value as a degree of rational belief (in hypothesis θ, etc.) and on conditioning that probability on knowledge of new data E (i.e., evidence, observation, etc.) denoted as $p(\theta|E)$ or expressed as 'the probability p of the hypothesis θ given E (after getting relevant evidence) …'. For the purposes of this theorem, $p(\theta)$ and $p(\theta|E)$ are, respectively, a priori and a posteriori probabilities of θ. They are brought together under the Bayes Theorem: $p(\theta|E).p(E) = p(E|\theta).p(\theta)$. In other words, when considering a hypothesis or a theory θ, comparing the probability that is assigned to it, before and after evidence E is obtained, would be equivalent to comparing in the same relationship, the probability assigned to evidence $p(E)$ and the probability assigned to the likelihood $p(E|\theta)$. More so than other models, the Bayes theorem better captures the way in which an individual updates his beliefs about an event or occurrence in order to make a better judgment, while taking into account what others think, during group decision-making for instance (Seongmin et al., 2017). By reference to probabilities, the individual weighs his trust in his own choice and his perceived credibility in information received from others, before he begins updating his judgment. Also, it reflects 'the phenomenon of a person who tells the truth and is not believed, even though the disbelievers are reasoning consistently. The theory explains why and under what circumstances this will happen (…) New data that we insist on analyzing in terms of old ideas (that is, old models which are not questioned) cannot lead us out of the old ideas (…) Old data, when seen in the light of new ideas, can give us an entirely new insight into a phenomenon' (Jaynes, 2003: xxv–xxvi).[3]

Regardless of whether these models are time-dependent, they operate on the basis of a variety of related concepts implying the existence of 'the universe' (or 'the world'), the 'state of nature' (or 'state of the world') and the reality of a 'state of Nature' (a true 'state of the world') as defined by Savage in his classic text The foundations of statistics: 'The world (is) the object about which the person is concerned. A state (of the world) (is) a description of the world, leaving no relevant aspect undescribed. The true state (of the world) (is) the state that does in fact obtain, i.e., the true description of the world' (Savage, 1954: 9). The same can be said of the general decision-making theory proposed by Jaynes, which posits an objectively derived model of reasoning by inference, including in situations where information is incomplete, requiring reliance on probabilistic inductive reasoning. 'By "inference" we mean simply:

[3] For instance, the visual perception, the discovery of Neptune, etc. (Jaynes, 2003).

deductive reasoning whenever enough information is at hand to permit it; inductive or plausible reasoning when – as is almost invariably the case in real problems – the necessary information is not available. But if a problem can be solved by deductive reasoning, probability theory is not needed for it; thus our topic is the optimal processing of incomplete information' (Jaynes, 2003: xix). The rules to solve the problem of inference are as follows (Jaynes, 2003): (1) enumerate the possible states of nature θ_j, discrete or continuous, as the case might be; (2) assign prior probabilities $p(\theta_j|I)$ which represent whatever prior information I you have about them, before any measurement; (3) assign sampling probabilities $p(E_i|\theta_j)$, which represent the likelihood of the measurements, that is, prior knowledge about the mechanism of measurement process yielding the possible, observable data sets E_i. (4) Digest any additional evidence $E=E_1\,E_2\ldots$ (Sampling data) and, by application of Bayes' Theorem, obtain the posterior probabilities $p(\theta_j|E\ I)$, which means taking into account new data feedback as it comes in. That is the end of the inference problem and probabilities $p(\theta_j|E\ I)$ yield all information regarding the possible states of nature θ_j that can be known a posteriori. In other words, the set of available information can be factored into the calculation of probabilities $p(\theta_j|E\ I)$ pertaining to the states of nature. All of these probabilities are interrelated through Bayes' theorem:

$$p(\theta_j/E\ I).p(E\ I) = p(\theta_j/I).p(E/\theta_j\ I)$$

As for the final steps of the decision-making process, they are: (5) enumerate the possible decisions D_i; (6) express what was sought to be accomplished, as a function of preferences (minimize the expected loss/maximize the expected gains) by associating possible decisions with states of nature $L(D_i|\theta_j)$; and (7) make the decision D_i that leads to the most preferred expected outcome (minimized expected loss/maximum expected gain) for θ_j.

It should be noted, however, that this general theory of decision-making modeling inferential reasoning does not expressly refer to the universe of possible states of nature (or the 'world' as Savage puts it) in its equations. Insofar as the universe has been posited once and for all, there is no need to do so. Moreover, information I on the states of nature, which is assumed to be held a priori, as introduced into the decision process at step (2) through the equations, does not (but should) serve the same function. If that had been the case, information I should have been included either in every step of the process or at least in steps (3) and (4). As we shall see, an information asymmetry (Akerlof, 1970; Akerlof et al., 2001) arises, limiting possibilities for constructing new representations for decision-making. That happens, as is the case here, in all innovative processes involving a reframing of the reference universe. Forgetting the meaning of reference universe previously set forth – as everything that exists and the assumption that it is possible to possess perfect knowledge about it[4] (the 'world' under Savage' definition) – we turn to a consideration of what is held to be universal relative to a given state of knowledge, in the sense of a 'universality of reference.' This takes into account, for instance, the fact that an innovation process is deemed radical or disruptive precisely because it is seen as new to the market or new to the world (Oslo Manual, 2005). For an innovation considered as involving an inventive step, patent law requires that the invention not be obvious to a person skilled in the art, with regard to any matter forming part of the prior art base. Such prior art base is defined as anything already made available to the public anywhere in the world by a written or oral description, use or in any other way (World Intellectual Property Organization, 2004, 2017).[5] As an indication of the importance that such inventions have attained, the value of international property rights traded on world markets has surpassed that of global GDP growth.

In light of the foregoing, we shall consider an emblematic counter-example of such an innovation process, namely: Archimedes' thought experiment (which led to the famous Eureka moment), allowing us to bear out the truth of Cédric Villani' remark, which we might paraphrase by saying that, like mathematics, an innovative idea 'can change the world').[6]

Counter-example: Archimedes' thought experiment.

Before discussing Archimedes' thought experiment and explaining its non-conformity with the general theory of decision-making modeling inferential reasoning, let us first consider how it acquired its status as a counterexample.

Status as a counter-example

[4] 'Semantic formalism consists of a "partition structure": a space Ω of states of the world, together with a partition of Ω for each player, whose atoms represent information sets of that player; Ω is called the universe. Like in probability theory, events are subsets of Ω; intuitively, an event is identified with the set of all those states of the world at which the event obtains. Thus, an event E obtains at a state ω if and only if $\omega \in \Omega$, and a player *i* 'knows' E at Ω if and only if E includes his information set at ω. For *i* to know E is itself an event denoted K_iE: it obtains at some states ω and at others does not' (Aumann, 1999: 264).
[5] Organisation Mondiale de la Propriété Intellectuelle & Centre du Commerce International, 2004, Clés de la propriété intellectuelle, Genève; The Global Innovation Index, 2017, Tenth Edition.
[6] From statements made by 2010 Fields Medal winner at TedxParis conference held in 2012.

According to Larousse (French dictionary), a theory is an organized set of principles, rules, and scientific laws used to describe and explain observed phenomena. In more formal language, a theory is a consistent set of statements containing all of its consequences. In science, a theory cannot demonstrate its own consistency. To be universally valid, it must be provable. Its experimental verification (Popper et al., 1985) would assume an infinite set of favorable outcomes (i.e., outcomes of interest), which is unsustainable. As a next best alternative, a theory is therefore viewed as being constructed from incomplete information and accepted as true until it is contradicted, notably by a counter-example (Tarski, 1969; Taleb, 2010). A counter-example is an indefinitely reproducible experiment that contradicts a theory and may suffice to refute it or refine it, wholly or at least in part, making it more efficient (Séguy-Duclot, 2011). The thought experiment that brought Archimedes to the Eureka moment, occupies the status of counter-example for the general theory of decision-making modeling inferential reasoning. Archimedes constructed a new representation for understanding a class of phenomena (floating bodies) and solved a decision-making situation once thought insoluble. This accomplishment was by no means fortuitous but was, rather, derived from a reproducible method that he described in a 100-page letter written to Eratosthenes[7] (276 BC to 194 BC) (Heath, 2007). The letter, which was lost for nearly 2000 years, reappeared in 1906, only to be lost again until 1998 (the ArchimedesPalimpsest Project). The 'method' consists in having a strategically situated observer who compares an unknown object to a known one (based on a specially designed artificial model whose behavior is known). In sum, he had created a method for modeling and simulation of an unknown object. The thought experiment was ingenious in that he had to find a way to monitor the characteristic properties of an unknown object potentially belonging to a new overarching reference universe, while reasoning within a long-standing reference universe where a known object was taken as the model. Hence, the need for the observer to be appropriately 'situated'. 'Indeed I assume that some one among the investigators of to-day or in the future will discover by the Method here set forth still other propositions which have not yet occurred to us' (Heiberg, 1909 cited by Beauzamy, 2012: 81). Archimedes was so proud of having used this method to determine that the ratio of the volume of a sphere to the volume of the circumscribed cylinder worked out to be two-thirds that he asked for the figure of a sphere and cylinder to be modeled in stone on top of his grave. He is also generally considered to be one of the greatest scientists of all time. 'It is just possible that Archimedes, could he come to life long enough to take a post-graduate course in mathematics and physics, would understand Einstein, Bohr, Heisenberg and Dirac better than they would understand themselves' (Bell, 1986: 19).

The reproducibility of Archimedes' Method stems from its scientific formulation.[8] Like the Law of Floating Bodies, it can be discussed separately from Archimedes. Only a science can build itself on a body of knowledge that can be discussed separately from its formulators and the class of objects and phenomena to which it is applied.

A decision-making situation once thought insoluble

Hiero II, king of Syracuse, had asked a goldsmith to craft a solid-gold crown, in tribute to the immortal gods. He had reason to suspect that some of the gold had been replaced by silver, and asked Archimedes to find out if he had been cheated. No known solution was available to solve the problem as presented. That is, the solution set was empty. At that stage, even operations research would have been unsuccessful. To the great benefit of science, Archimedes, upon entering his bath, noticed the increase in the water level, due to the volume of water his body had displaced, and cried 'I found it! [Eurêka]' (de Chaufepié, 1750). He had two lumps of pure gold and pure silver brought to him, each weighing the exact same as the crown. He immersed the bar of silver into a large vessel, with water filled to the brim, and measured the volume of displaced water. Then he repeated the experiment under exactly the same conditions, using a bar of gold, and observed that a smaller volume of water had been displaced. Next, he conducted the experiment with the solid-gold crown and noted that the volume of displaced water was greater than was the case with the gold bar. He had found out the fraud and his cry of Eureka, upon realizing that he had solved the vexing problem, has since become the emblematic cry of discovery or comprehension. It is of little real importance that the experiment probably did not transpire exactly as reported by Roman architect Marcus Vitruvius Pollio (1st century BC). What is important is the inferential reasoning employed by Archimedes – a line of reasoning that does not conform to the general theory of decision-making modeling inferential reasoning, even during the experimentation stage.

Reframing the reference universe

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[7] Eratosthenes is famous for having calculated the circumference of the Earth with great accuracy (39,375 km), within 10% of the actual figure.

[8] The aim here is not to discuss the reproducibility of Eureka phenomena by intertwining art and science (Toulouse and Danétis, 2008; Koestler, 2011). To be sure, art has its theoretical arguments for putting things aside and focusing on their relationships, so as to achieve a total abstraction, where nothing is recognizable compared to what it was originally. However that may be, they remain attached to an artist, a school of thought or an artistic movement.

It was not simply a matter of comparing the behavior of objects in the context of a given reference universe (air), but also of noting differences in the apparent behavior of a single object while considering the reference universes which, until then, had been deemed to be independent (air and water). Hence, the need to reason independently of the reference universe as it was originally perceived (air) and consider a reference universe broadened to encompass all fluids, in order to conceive a more general relationship linking the apparent weight of objects to the displaced volumes of fluid and, in this way, identify a new property, that is, volumetric weight or density: 'anybody completely or partially submerged in a fluid (gas or liquid) at rest is acted upon by an upward, or buoyant, force the magnitude of which is equal to the weight of the fluid displaced by the body.' The property thought to account for the expected behavior of an object is thus conditioned[9] by the reference universe in which its use is imagined. It would thus appear more logical to talk about embedding or situational properties (Lautman, 2006) rather than intrinsic properties (independent of embedding in a referent) to account for the interpretation of observable phenomena or behaviors: the weight of an object in a fluid, its color in a luminous atmosphere, its price on a market, etc. In economics and social sciences, a methodology based on situational mechanisms has been analyzed and modelized (Caldwell, 1991; Hedström et al., 1998). In the field of design science, the idea of approaching innovation through 'embedding' one design situation into another, was formalized mathematically[10] (Béjean and Ehresmann, 2015). And if certain properties are deemed to be intrinsic, it is because they are dependent on a reference universe[11] (Dehornoy, 2007). In the words of Albert Einstein, it is easier to smash an atom than a prejudice. Over time, a commonsense knowledge base has been built up and ultimately come to prevail, through the force of collective representations. Imagining that things might be otherwise has proved to be a challenging reflective exercise (Carlile, 2002, 2004).

The processual pattern can be applied to immaterial objects (software, commercial offers, etc.) whose properties (weight, volume, value, etc.) and behaviors change in keeping with the reference universe in which they are embedded (situations of use, markets, etc.). Weighing choices, evaluating possibilities, gains, losses or risks, are all part of the underlying process that fuels reasoning during decision-making. The decision maker (whether an individual or an organization) need merely change the name of the reference universes, observables and comparison tools, reconsider the possible states of nature (equilibrium, disequilibrium), to identify with Archimedes' thought experiment.

A new more generalizing representation: reference universes and decidability

Let us consider the following statement – the two scales of a balance have to be in equilibrium to account for two objects of the same weight. Under the initial state of knowledge, a decision maker would be led to accept this proposition as true. And the final state of knowledge renders the statement undecidable (neither provable nor refutable). It is impossible to assign it a truth value (as true or false) as long as the fluid in which the weighing was carried out is not specified. The outcome of the methodology is significant (see Table 1): decidability is linked to a state of knowledge as well as to the meaning ascribed to universal properties. For the same observation, such as that obtained from the weighing of objects where the two scales of a balance are in equilibrium, the explanatory hypothesis in the initial reference universe is necessarily the equality of weight; whereas the explanatory hypothesis in the final reference universe is no longer necessarily equality of weight (if the weighing was performed under water) but, rather, equality of density (a property unknown in the initial reference universe). As a corollary to this, the observation of a balanced weighing in the initial reference universe is consistent with the hypothesis of equality of weight as well as the observation of an unbalanced (immersed) weighing in the final reference universe. An observation such as an unbalanced weighing, considered a priori as unfavorable to an explanatory hypothesis such as the equality of weight in the initial reference universe (air) may be favorable to it in the final reference universe (water). Only the hypothesis of equality of density can dispel the obvious contradiction.

It is not only a matter of reasoning rationally and consistently, by reference to an interpretable universe, while striving to estimate the plausibility of various explanatory hypotheses, based on observations, but also a question of bringing to bear ever more generalizing representations and coherences. In this way, our theoretical constructions are continually put to the test, under experimental conditions that are reproducible, every time that the referential universe changes, which involves a (temporary) absence of decidability (i.e., undecidability). 'Embracing contradictory forces can inspire learning, discovery, and creativity (…) By accepting paradox and applying consistently inconsistent

[9] That depends on the interaction with the other substances comprising the reference universe (molecules of air, water, etc.).
[10] Category theory in mathematics.
[11] Set theory in mathematics (where a provable property, respectively a refutable one, depends on the system of axioms, whereas a property that is neither provable nor refutable--in other words, which is undecidable--is independent of the system of axioms).

strategies of differentiation and integration, systems may thrive, reinforcing paradoxical thinking and management in an upward spiral of peak performance and sustainability' (Lewis and Smith, 2014: 137).

Implications under the general decision-making theory.

First, we will discuss the implications for the general theory of decision-making modeling inferential reasoning. Then we shall examine the foundations of a theory that is intended to be more generalizing.

Reformulating the system of rules

Given that all comparisons between hypotheses or observations must be relativized in relation to the reference universes U_k of possibles (air, water, etc.), rules (3) and (4) of the general theory of decision-making should be reformulated, as follows: (3) assign sampling probabilities $p(E_i|\theta_j\ U_k)$ which represent the likelihood of the measurements, reflecting prior knowledge about the mechanism of measurement process yielding observable data sets E_i in a reference universe U_k of possibles; (4) digest any additional evidence $E = E_1 E_2\ldots$ (Sampling data) and, by application of Bayes' Theorem, obtain the posterior probabilities $p(\theta_j|E\ I\ U_k)$. The probabilities $p(\theta_j|E\ I\ U_k)$ yield all information about possible states of nature θ_j that can be known a posteriori in reference universe U_k. The reformulation of steps (3) and (4) has a notable impact on those that ensue. Assuming that the possible decisions D_i set out in step (5) are not affected (if only two decisions Yes/No or 0/1 were ever proposed), it is obvious that steps (6) and (7) are bound to be affected. Indeed, the expression of what is sought to be accomplished, as a function of preferences (minimize the expected losses/maximize the expected gains) linking possible decisions with states of nature $L(D_i|\theta_j)$ will be transformed into $L(D_i|\theta_j\ U_k)$. A concomitant transformation will take place, in step (7), regarding decision D_i which leads to the most preferred expected outcome for θ_j (minimized expected loss/maximum expected gain) in reference universe U_k, given that the new hypotheses θ_j^* must be formulated pursuant to the reframing of reference universe U^* (encompassing U_k). Otherwise, the problem becomes undecidable (see Table 1). Technically, that would proceed by imposing a forced condition on probabilities $p(\theta_j|E\ I\ U_k)$ within U^* (see Fig. 1).

Table 1 Constructing a more generalizing representation, reference universe and decidability

Observations	Reference Universe	Explanatory Hypotheses
Balanced weighting E_1	Initial U_1 (air)	Θ_0: equality of weights
	New and independent U_2 (water)	Θ_1: not necessarily equality of weights
	Unknown	undecidable (knowing that the existence of U_2 is possible)
	U* broadened to encompass all fluids	Θ_0^*: equality of densities (volumetric weights)
Unbalanced weighing E_2	Initial U_1 (air)	Θ_0: inequality of weights
	New and independent U_2 (water)	Θ_1: not necessarily an inequality of weights
	Unknown	undecidable (knowing that the existence of U_2 is possible)
	U* broadened to encompass all fluids	Θ_1^*: inequality of densities (volumetric weights)

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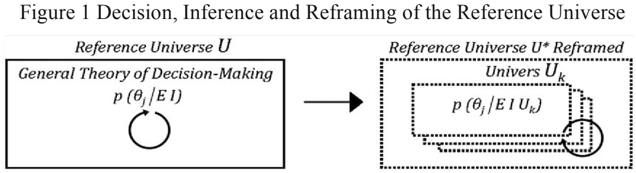
	New and independent U_2 (water)	Θ_1: not necessarily an inequality of weights
	Unknown	undecidable (knowing that the existence of U_2 is possible)
	U* broadened to encompass all fluids	Θ_1^*: inequality of densities (volumetric weights)

Figure 1 Decision, Inference and Reframing of the Reference Universe



The theory of probabilities (Savage, 1972), however, was not intended to apply to several reference universes U_k (insofar as the probability of any reference universe cannot exceed 1). And so the existence a priori of a reference universe U^* has no real possible translation. (A complex translation – partly real and partly imaginary – has been explored and may potentially be the subject of a subsequent article.) It is important to note that it is not a matter of adding any previously unknown or uncertain observations E_i or hypotheses θ_j (Knight, 2006; Shackle, 1961) to a reference universe U_k by conveniently redistributing the probabilities so as to ensure that the total always stands at 1.

Discussion of the theoretical foundations of reframing the reference universe

The representation that we have constructed and the assessment that we make regarding different states of the world and the impact of our decisions (emanating from both our behavior and our actions) thus depend on: (1) our state of knowledge that determines our understanding of the universality of things or phenomena and their decidability; and (2) the potential for reframing the reference universes in order to construct a new, more generalizing representation.

Universality of the reference universe based on the state of knowledge

The belief in an absolute and eternal universality can be posited as an axiom. The implications of this can be seen in the general theory of decision-making modeling inferential reasoning. Theoreticians (in economics, epistemic modal logic, management, etc.) have formulated such a conception, as so-called semantic knowledge, in a unifying axiomatic principle[12](Samuelson, 2004), modeled in terms of partitions on a comprehensive universe of 'possible states of the world' (i.e., knowledge of an observable state of the world already exists somewhere in the partition; Sauce, 2010). But it is possible to posit as an axiom that universality depends on the (time-dependent) state of knowledge. Under this somewhat constructivist concept, every decision maker must know which reference universe serves as the lens through which he can look in order to reframe, decide and act in an informed manner. The acquisition of a new understanding of the world depends on his ability to foresee the broadening of his knowledge beyond the bounds of his rationality and ultimately place into perspective the reasons behind his previous viewpoints and stances. In addition, inferential reasoning must be used to elaborate and nuance the forms of rationality. Indeed, if we consider the initial reference universe, looking at both the void portion and its complement, namely the original reference universe in its entirety, it is clear that, after reframing, the initial reference universe in its entirety is no longer sufficient to fill the gap created by the void. This idea is borne out by a simple example. Etymology helps to infer the meaning that an unknown word takes on in a new context, in reasoning by extension based on the word' original meaning. The negation of the original meaning is not a negation of the new meaning. Reasoning is no longer founded on the law of the excluded middle (where no third possibility is given), which holds that what is not in one part of the reference universe must necessarily be in the complementary part, and vice versa.

Consequently, abandoning the law of the excluded middle appears as a way to avoid the initial reference universes – an idea seized upon by a mathematical theory dubbed intuitionism (Brouwer, 1913). This perspective is in line with the so-called 'paradox theory' (Smith and Lewis, 2011; Lewis and Smith, 2014; MironSpektor and Beenen, 2015; Putnam et al., 2016; Schad et al., 2016). 'Paradoxical tension arises when two practices that seem logical individually are "inconsistent or even absurd when juxtaposed" (Smith and Lewis, 2011: 382) "The intuition–rationality paradoxical tension [that is mental templates in which managers accept and embrace the simultaneous existence of contradictory forces" (Smith and Tushman, 2005)] will be present in any strategic decision-making process, but innovation is by definition a process of change and of uncertain outcomes, which can make the contradiction more salient (Garud, Gehman, and Kumaraswamy, 2011; Jay, 2013; Smith and Lewis, 2011)' (Calabretta et al., 2017: 368, 393, 396).

The decision maker would thus rely on intuition to open up prospects for reframing the reference universe and make anticipatory comparisons as to what they imply, by temporarily abandoning the principle of the excluded middle (paradoxical thinking), even if this means coming off as unreasonable in the eyes of those who strictly adhere only to conventional rationality based on the principle of the excluded middle. 'Intuition not only helps decision makers deal with uncertainty but also stimulates those creative cognitions that are essential to the generation and exploration of novel problem solutions, ideas, and related business opportunities (Claxton, 1998; Hodgkinson et al., 2009; Miller and Ireland, 2005). (…) Understanding better how rationality and intuition interact during decision-making has, however, remained a major challenge (Gray, 2004; Lieberman, 2007). (…) Some researchers suggest that intuition is

[12] The axioms of Conscience, Omniscience, Knowledge, Transparency, and Prudence.

the main mechanism through which choices are made, and the role of rational thinking is to evaluate the product of intuitive processing (Kahneman, Slovic, and Tversky, 1982)' (Calabretta et al., 2017: 366–367).

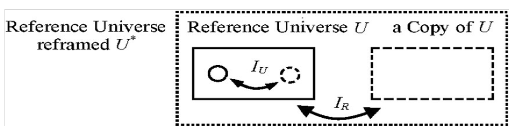
Constructing a more generalizing representation and the restructuration of knowledge

As mentioned earlier, the construction of new representations aims to be as generalizing as possible, so that it would apply to any special framework as well as to any previous representation which has less explanatory or predictive power over a given phenomenon or behavior. We have understood, since Newton, that the falling of an apple and the course followed by the Moon obey one and the same law. Generalizing thus entails redefining the identity of things or phenomena, which are ascribed a new, broader property, reframing the reference universe and restructuring the knowledge space, which, as we shall show, is augmented by a dimension.

The knowledge space can be better understood when considering that its underlying substratum is memory (a base of remembered information). The reframing process involves: (1) a base of remembered information where the initial representation of the known reference universe is stored; and (2) a new base of remembered information where the new representation of the reframed reference universe is recorded, forming a memory that includes the original representation and thus a copy of the entire reference universe that was already kept in memory. Reflexivity (thought taking itself as the object of its own act) can be understood as the memory of memory. A conveniently 'situated' observer, as Archimedes put it. 'Transcendence occurs when actors move 'outside of the paradoxical system to a new level of meaning' (Putnam et al., 2016: 65). In the context of identity, this could be accomplished through the creation of temporary physical and/or cognitive spaces in which new and old identities blur (…) Reflexivity offers another potential strategy' (Besharov and Sharma, 2017: 186). Hence, in seeking to copy[13] a given reference universe in its entirety, the reflexive identity relation I_R that is at work here does not coincide with identity relation I_U used to operate inside this reference universe and, for that reason, they must be distinguished. Namely (see Figure 2):

- Within a reference universe U, an identity is constructed by drawing on a referable knowledge base about which it can be said that a thing, itself or another, share the same identity. An identity relation I_U operating within U is a structural convention for identifying and classifying things in order to facilitate their exchange, memorization or retrieval. Such a partitioning of U as established by the conventional model, requires an identity relation that is simultaneously reflexive, symmetric, and transitive, and thus an equivalence relation. In order to say what makes a thing, in itself or in relation to another, belong to the same identity, certain knowledge is relevant and other knowledge is set aside, insofar as it is taken as independent. At this stage, knowledge space C can be structured into two independent sub-spaces: the knowledge sub-space C_i attached to the identity of things within the reference universe U and the residual knowledge sub-space C_r. These two sub-spaces C_i and C_r can be considered as two independent dimensions which generate the entire comprehensive reference universe U: 'any dimension needs to be studied using the ideas of spaces/sets or sub-spaces/sub-sets or partitions/cuts' (Estrada, 2011: 340).
- Copying the entire reference universe U implies a nonreflexive identity relation I_R given that the resulting reference universe contains a copy of itself. It has therefore been augmented compared to what it had been initially. The reflexivity exercise as defined implies a non-reflexive identity relation I_R, different from I_U. As a result, knowledge C_i attached to I_U is not sufficient. It is therefore necessary to also draw on knowledge sub-space C_r which, consequently, can be configured into two independent dimensions: the dimension attached to knowledge sub-space $C_{r,i}$ that can be used for copying the reference universe U in its entirety, and the dimension attached to residual knowledge $C_{r,r}$.

Figure 2 Reframing of the Reference Universe and non-Reflexive Identity



The reframed comprehensive reference universe U^* is grounded in a knowledge space that has been augmented by a dimension: C_i, $C_{r,i}$ and $C_{r,r}$. Each time, the residual knowledge seems like a hidden flaw in the construction of the identity of things or phenomena, insofar as its informational potential, which is distant and indirect, only becomes visible with the ensuing operation. The benefit that this affords is a reduction in the 'information distance' measuring the identity of phenomena (Delahaye, 2003). These are precisely the kinds of outcomes that have significance in

[13] This operation is common in mathematics (Cartesian Product, Tarski' work on model theory, Gödel' incompleteness theorem, etc.). In each case the existence of such a copy of an entire mathematical universe is presumed possible.

management: 'Building on this work, future research can explore how reflexivity may enable both dynamic process of identity re-construction and the development of stable characteristics describing identity' (Besharov and Sharma, 2017: 186).

Depending on whether an experiment subject is reasoning according to the conventional model or the innovational model, there is always an information asymmetry (given that a subject' understanding of phenomena will hinge on the model chosen, and one model provides better information than the other). Hence, to speak more aptly, we should refer to 'dimensional' information asymmetry. For instance, if we made an analogy to visual perception, the subject would perceive the reference universe and its contents, alternatively, in 2D or in 3D. In the conventional model, the knowledge space does not evolve in terms of dimension and everything transpires within an immutable comprehensive reference universe U. 'Whenever a physical phenomenon is expressed as an equation, there is a causal relation between the quantities appearing in both terms and both have the same dimension.' (Curie, 1894: 399). If we refer back to an equation used to model a phenomenon or a problem, it always involves an identity relation – equality – with the quantities being connected by operators. In the innovational model, simultaneously as the knowledge space assumes an additional dimension, the reframed reference universe U^* is created by conditioning reference universes U_k which, until then, had been held to be independent, as described by Estrada (2011: 340, 357): 'The idea of dimension is complex and rather deep for the human mind to penetrate. (…) the term "dimension" can be defined as the unique mega-space that is built by infinite general-spaces, subspaces and micro-spaces that are systematically interconnected… At the same time, we need to assume that time in each dimension (general dimensions, subdimensions and micro-dimensions) is moving in different levels of speed.' Consequently, a problem without any admissible solutions that arises in a reference universe U, could, in the reframed reference universe U^*, be construed with a non-empty set of admissible solutions, according to the mathematical principles of extension (Bourgne and Azra, 1976; Tappenden, 1995; Chambert-Loir, 2005; Dehornoy, 2007). This new representation cannot be known in advance, as required under the conventional model. To quote Albert Einstein: 'We can't solve problems by using the same kind of thinking we used when we created them.' Sharing the same conceptual basis as the abovementioned definition of a dimension, there is another definition that lends itself well to the disruptive or radical innovation process, adding an element of novelty not included in the body of knowledge made available to the public worldwide (which is what occurs upon reframing the reference universe U): 'between any compactum (compactness makes it possible to proceed from one point to the next) and a point not belonging to it' (Estrada, 2011: 340). The creation of new knowledge augments the knowledge space by adding an extra dimension.

Furthermore, it is clear that substantive or procedural rationality (Simon, 1996) must take into account reference universe, reframing of the reference universe and 'dimensional' information asymmetry. We now have a better understanding of the considerable cognitive work the subjects have to undertake, individually and collectively, in terms of learning, reflexivity and memory, to compensate for such asymmetry. The act of reframing introduces irreversibility in that the erasure of the final body of memory where the new representation of the reframed reference universe is stored (Delahaye, 2003) would, by the same token, lead to the erasure of the copy of the initial base of remembered information included during reframing, thereby making reversibility impossible. For instance, it would be impossible for us to erase a word from our memory and go back to the oldest attestation of the word, because that would involve erasing the known linguistic universe and returning to a language that has long died out. Reframing would also have the effect of increasing the accessible space of understanding and the accessible speed of understanding (to be precise, let us say the space of accessible speeds of understanding). These two transformations can be interpreted as an increase in spatial entropy[14] (Clausius, 1868) and kinetic entropy (Boltzman, 1902).[15] The entropic interpretation of a shift in a body of memory leading to a 'Eureka' moment has made it possible to modelize the phenomena thermodynamically (critical point, entropic diagram, etc.) for the purposes of managerial or pedagogical optimization, especially as the concept of entropy is already widely applied in information science, under Shannon' (1948) theory, which allows quantification of the information content of data sets.

Conclusion

[14] 'en' (for energy) and the Greek
The term dubbed by Clausius from the prefix word tropê (meaning transformation). This notion designates the 'content of transformation' and its basic function is to describe the transformation of a system from a qualitative point of view, irrespective of its energy aspects (from a quantitative point of view).

[15] If it was sufficient, it would be able to copy itself, which would demonstrate its own inconsistency, as illustrated by the famous Catalogue paradox (Gonseth, 1926, 1974).

The general theory of decision-making modeling inferential reasoning is consistent with the conventional model presenting a universe of possibles, together with partitions of this universe in terms of knowledge, as if it were already somewhere in one part or another of the universe. That theory presents an asymmetry of information resulting precisely from the fact that the supposedly immutable reference universe is not explicitly formulated in the operative rules as conditional. The counter-example that has been developed, namely, Archimedes' thought experiment that led to the Eureka moment, is emblematic of an innovational model where the reference universe must be reframed and thus explicitly formulated into operative rules. The concept of the reference universe, which must be understood in terms of a universality of reference regarding a body of knowledge, making it possible to enlarge the concept of the conventional universe and take a step toward a more general coherence where conventional and innovational positions occupy their respective places according to the reference universe. The new theoretical foundations put forward offer a better explanation of the process of constructing new representations: a reframing of the reference universe implies a non-reflexive identity which extends knowledge space – and, consequently, the decision-making space – by an extra dimension. The forms of rationality at work are sometimes based on the principle of the excluded middle (the standard model) when it comes to operating within the reference universe, and sometimes based on rejecting the excluded middle (intuitionistic model) when it comes to reframing the reference universe. However, the theory of probabilities was not intended to apply to several reference universes and cannot be applied when reframing the reference universe. Indeed, only one reference universe may have a 'real' translation. A partly 'real' and partly 'imaginary' translation (so-called 'complex' translation referring to the mathematical formalism of 'complex' numbers) remains to be explored and could open up a new path of investigation. A reframing of the reference universe does not consist in merely redistributing probabilities based on an expansion of the universe of possibles, because the knowledge space takes on an added dimension. All of these considerations are central to the learning systems and the mechanisms required for constructing new representations, exploring and evaluating innovation potential, and for adapting and organizing collective actions within firms. Over time, and under the pressure of progress and change, mankind has adapted its behavior by learning how to put into perspective any biases or preconceived notions that may have served as the basis for its belief in mental representations and by developing the capacity to construct new, more satisfying representations. There are important implications for the general theory of decision-making as well as for artificial intelligence. Often considered as the science of knowledge representation and the science of concept formation and reasoning (Newell and Simon, 1972; Simon, 1996) Artificial intelligence is also the science of the imitation of human abilities. This means that its intersection with the domain of decision-making is necessarily non-empty (Pomerol, 1997).

References

Akerlof, G. A., 1970, "The market for lemons". Quaterly Journal of Economics, 84: 488–500.

Akerlof, G. A., A. M. Spence and S. E. Stiglitz, 2001, "Markets with asymmetric information". In Nobel Prize Committee (ed.), Nobel Prize in Economics documents 2001–2. EconPapers.

Aumann, R., 1999, "Interactive epistemology I: Knowledge". International Journal of Game Theory, 28: 263–300. de Beaune, S. A., 2015, L'homme et l'outil. Broché: CNRS

Editions.

Beauzamy, B., 2012, Archimedes modern works. Paris: SCM.

Béjean, M. and A. Ehresmann, 2015, "DMES: Conceptualizing the working designers". International Journal of Design Boltzman, L., Management and Professional Practice, 9: 1–20.

Bell, E. T., 1986, Men of mathematics. Simon and Schuster (eds.), New York, NY: Touchstone.

Besharov, M. L. and G. Sharma, 2017, "Paradoxes of organizational identity". In Smith, W. K., M. W. Lewis, P. Jarzabkowski and A. Langley (eds.), The Oxford handbook of organizational paradox. Oxford: Oxford University Press, pp. 178–196.1902, Leçons sur la théorie des gaz. Paris:

Gauthier-Villars.

Bourgne, R. and J.-P. Azra, 1976, Evariste Galois: Ecrits et mémoires mathématiques. Paris: Gauthier-Villars.

Brouwer, L. E. J., 1913, "Intuitionism and formalism". Bulletin of the American Mathematical Society, 20: 81–96.

Calabretta, G., G. Gemser and N. M. Wijnberg, 2017, "The interplay between intuition and rationality in strategic decision making: A paradox perspective". Organization Studies, 38: 365–401.

Caldwell, B., 1991, "Clarifying Popper". Journal of Economic Literature, 29: 1–33.

Carlile, P. R., 2002, "A pragmatic view of knowledge and boundaries: Boundary objects in new product development". Organization Science, 13: 442–455.

Carlile, P. R., 2004, "Transferring, translating, and transforming: An integrative framework for managing knowledge across boundaries". Organization Science, 15, n° 5 (Sep.– Oct., 2004): 555–568.
Chambert-Loir, A., 2005, Algèbre corporelle. Paris: Les Editions de l'Ecole polytechnique.
de Chaufepié, J. G., 1750, Nouveau dictionnaire historique et critique. Napoli: Biblioteca Naz. Vittorio Emanuale.
Clausius, R., 1868, Théorie mécanique de la chaleur. Paris: Eugene Lacroix.
Claxton, G., 1998, "Knowing without knowing why". The
Psychologist, 11: 217–220.
Cohen, W. M. and D. A. Levinthal, 1989, "Innovation and learning: The two faces of R&D". The Economic Journal, 99: 569–596.
Curie, P., 1894, "Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique". Journal de Physique Théorique et Appliquée, 3: 393–415.
Dehornoy, P., 2007, "Auidelà du forcing: La notion de vérité essentielle en théorie des ensembles". In Joinet J.-B. (ed.), Logique, dynamique et cognition. Paris: Editions de la Sorbonne, Logique, langage, sciences, philosophie (Coll.), pp. 147–169.
Delahaye, J.-P., 2003, "Ressemblance entre objets". Revue d'Intelligence Artificielle, 17: 885–898.
Estrada, M. A. R., 2011, "Multi-dimensional coordinate spaces". International Journal of the Physical Sciences, 6: 340–357.
Fayyad, U., G. Piatetsky-Shapiro and P. Smyth, 1996, "Advances in knowledge discovery". In Fayyad, U.M., G. Piatetsky-Shapiro and P. Smyth, From Data Mining to Knowledge Discovery: An Overview. Menlo Park, California: AAAI/MIT Press, pp 1–34.
Garud, R., J. Gehman and A. Kumaraswamy, 2011,
"Complexity arrangements for sustained innovation: Lessons from 3M Corporation". Organization Studies, 32: 737–767.
Gray, J. R., 2004, "Integration of emotion and cognitive control". Current Directions in Psychological Science, 13: 46–48.
Gonseth, F., 1926, 1974, Les fondements des mathématiques, de la géométrie d'Euclide à la relativité générale et à l'intuitionnisme. Paris: Librairie scientifique A. Blanchard.
Heath, T. R., 2007, The method of Archimedes. New York: Cosimo Classics.
Hedström, P., R. Swedberg and L. Udèhn, 1998, "Popper' situational analysis and contemporary sociology". Philosophy of the Social Sciences, 28, 339–364.
Heiberg, J. L., 1909, "A newly discovered treatise of
Archimedes: Heiberg' translation from the Greek". The Monist, 19: 202–224.
Helfat, C. E. and S. G. Winter, 2011, "Untangling dynamic and operational capabilities: Strategy for the (n)ever–changing world". Strategic Management Journal, 32: 1243–1250.
Hodgkinson, G. P., E. Sadler-Smith, L. A. Burke, G. Claxton and P. R. Sparrow, 2009, "Intuition in organization: Implications for strategic management". Long Range Planning, 42: 277–297.
Jaynes, E. T., 2003. Probability theory: The logic of science. Cambridge: Cambridge University Press.
Kahneman, D., P. Slovic and A. Tversky, 1982, "Judgment under uncertainty: Heuristics and biases". New York, NY: Cambridge University Press.
Knight, F. H., 2006, Risk, uncertainty, and Profit. New York: Cosimo Classics.
Koestler, A., 2011, Le cri d'Archimède, la découverte de l'Art et l'art de la découverte. Paris: Les Belles Lettres.
Lautman, A., 2006, Les mathématiques, les idées et le réel physique. Paris: Librairie Philosophique Vrin.
Lewis, M. and W. K. Smith, 2014, "Paradox as a metatheoretical perspective: Sharpening the focus and widening the scope". Journal of Applied Behavioral Science, 50: 127–149.
Lieberman, M. D., 2007, "Social cognitive neuroscience: A review of core processes". Annual Review of Psychology, 58: 259–289.
Lundvall, B., 2007, "Innovation system research and policy where it came from and where it might go". Fifth Globelics Conference, Saratov, 19–23 September. Markowitz, H., 1952, "Portfolio Selection". The Journal of Finance, 7: 77–91.
Miller, C. C. and R. D. Ireland, 2005, "Intuition in strategic decision making: Friend or foe in the fast-paced 21st century?". The Academy of Management Executive, 19: 19–30. Miron-Spektor, E. and G. Beenen, 2015, "Motivating creativity: The effects of sequential and simultaneous learning and performance achievement goals on product novelty and usefulness". Organizational Behavior and Human Decision Processes, 127: 53–65.
Nelson, R. R. and S. G. Winter, 1982, An Evolutionary Theory of Economic Change. Cambridge, Mass.: The Belknap Press of Harvard University Press.
Newell, A. and H. A. Simon, 1972, Human problem solving. Englewood Cliffs, NJ: Prentice–Hall.
Oslo Manual, 2005, Guidelines for collecting and interpreting innovation data, 3rd Ed., OCDE Editions.
Organisation Mondiale de la Propriété Intellectuelle & Centre du Commerce International, 2004, Clés de la propriété intellectuelle, Genève.
Pomerol, J.-C., 1997, "Artificial intelligence and human decision making". European Journal of Operational Research, 99: 3–25.
Popper, K. R., M. I. de Launay and M. B. de Launay, 1985, Conjectures et réfutations: La croissance du savoir scientifique. Payot.
Putnam, L. L., G. T. Faihurst and S. G. Banghart, 2016, "Contradictions, dialectics, and paradoxes in organizations: A constitutive approach". Academy of Management Annals, 10: 65–171.
Samuelson, L., 2004, "Modeling knowledge in economic analysis". Journal of Economic Literature, 42, 367–403.

Sauce, L., 2010, “Ordres, structures et coordination: Une contribution à la théorie autrichienne”. Thèse de Doctorat ès Sciences Economiques, Université Nice Sophia-Antipolis. Savage, L., 1954 [1972], The Foundations of Statistics. New York: John Wiley and Sons.

Schad, J., M. Lewis, S. Raisch and W. Smith, 2016, “Paradox research in management science: Looking back to move forward”. Academy of Management Annals, 10: 5–64.

Séguy-Duclot, A., 2011, Recherches sur le langage, Paris: Librairie philosophique. J. Vrin.

Seongmin, A. P., S. Goïame, D. A. O’Connor and J.-C. Dreher, 2017, “Integration of individual and social information for decision making in groups of different size”. PLoS Biology, 28 Jun. https://doi.org/10.1371/journal.pbio.2001958

Shackle, G., 1961, Decision, order and time. London: Cambridge University Press.

Shannon, C. E., 1948, “A mathematical theory of communication”. Bell System Technical Journal, 27: 379– 423: 623–656.

Simon, H. A., 1996, The science of the artificial. Cambridge, MA: MIT Press.

Smith, W. K. and M. W. Lewis, 2011, “Toward a theory of paradox: A dynamic equilibrium model of organizing”. Academy of Management Review, 36: 381–403.

Smith, W. K. and M. L. Tushman, 2005, “Managing strategic contradictions: A top management model for managing innovation streams”. Organization Science, 16: 522–536.

Taleb, N. N., 2010, The black swan: the impact of the highly improbable, 2nd ed. London: Penguin.

Tappenden, J., 1995, “Extending knowledge and ‘fruitful concepts’: Fregean themes in the foundations of mathematics”. Noûs, 29: 427–467.

Tarski, A., 1969, “Truth and proof”. Scientific American, 220: 63–77.

Toulouse, I. and D. Danétis (Eds), 2008, Eurêka, le moment de l’invention. Un dialogue entre art et science. Paris : Éditions. L’Harmattan.