Theory of conjectural variations and its application to natural resource management

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Context: Nash equilibrium and cooperative (Pareto) solution

- The Nash equilibrium is the most common way to define the solution of a non-cooperative game involving two or more players. In a Nash equilibrium, each player is assumed to know the equilibrium strategies of the other players and no player has anything to gain by changing only their own strategy.
- In the cooperative solution players have the opportunity to consult and commit to cooperate before defining the strategy to be adopted (or a regulator takes the best solution following some objective). In this case, it is usual that the players (or the regulator) maximise the sum of the payoffs of all the players.

Context

What are Conjectural Variations ?

A game-theoretical concept in which players have a conjecture about the behaviour of their opponents: they think the others will play in function of their own decision.

What is it useful for?

- As a possible alternative to Nash Equilibria...
  - behavioural model
  - explicative model (suitable for empirical studies)

  specially when information is incomplete
- Or to explain implicit cooperation when agents behave non cooperatively
- A shorthand for dynamic interactions

Plan

1- Static Conjectural Variations Equilibria (CVE)

2- Consistent Conjectures in Dynamic Games

3- Conjectures and Learning

Part I
Static Conjectural Variations Equilibria (CVE)

Conjectural Variations in the literature

The static concept

- Bowley, A.L. (1924), *The Mathematical Groundwork of Economics*, Oxford University Press, Oxford.
- Bresnahan, T.F. (1981), "Duopoly Models with Consistent Conjectures", *American Economic Review*, Vol. 71 (5), pp. 934–945.
- Many "economic" papers in the 80 decade
- Olsder, G.J. (1981), "A Critical Analysis of a New Equilibrium Concept", Memorandum Nr. 329, Dept. Applied Maths., Twente University of Technology, The Netherlands.

Preliminaries

Consider a two-player game with

- $\pi^i(e_1, e_2)$ the payoff of player i,
- e_i the strategy of player i in some set E.

Consider some benchmark strategy profile $e^b = (e_1^b, e_2^b)$.
Assume that player i thinks that if he deviates from e_i^b by some infinitesimal quantity $\mathrm{d}e_i$, then player j will "react" by deviating from e_j^b by the quantity:

$$r_i(e_i^b, e_j^b)\ \mathrm{d}e_i\ ,$$

for some function r_i.
r_i is the conjectural variation assumed by player i.

Preliminaries (continued)

Non infinitesimal deviations?
Player i is logically led to think that if he deviates then player j will play χ_i^c
where χ_i^c is solution of:

$$\frac{\partial \chi_i^c(e_i; e_i^b, e_j^b)}{\partial e_i} = r_i(e_i, \chi_i^c(e_i; e_i^b, e_j^b))$$

with initial condition $\chi_i^c(e_i^b; e_i^b, e_j^b) = e_j^b$.

This function is the conjectured function.

For example: $e_j^c = \chi_i^c(e_i; e_i^b, e_j^b) := e_j^b + r_i(e_i - e_i^b)$.

Definition of CVE

A pair of variational conjectures $r_i(e_j, e_i)$ $i = 1, 2$, together with a pair of strategies $(e_1^c, e_2^c) \in \mathrm{E} \times \mathrm{E}$ is a General Conjectural Variations Equilibrium (GCVE) if (e_1^c, e_2^c) is solution of the optimization problem:

$$\max_{e_i}\{\pi^i(e_i, e_j) \mid (e_1, e_2) \in \mathrm{E} \times \mathrm{E} \text{ and } \mathrm{e_j} \;=\; \chi_{\mathrm{i}}^{\mathrm{c}}(\mathrm{e_i}; \mathrm{e_i^b}, \mathrm{e_j^b})\} \;,$$

simultaneously for $i = 1, 2$.

A very positive point

It is not necessary to know the payoff of the other player in order to compute an equilibrium

Empirical findings show that agents do not play Nash equilibrium.

Objections

- Credible behaviour.
  Both players are assumed to act "à la Stackelberg".
  However there is not sequentiality. No player observes the opponent's play.
- Static framework.
- Refutability.
  By selecting carefully the conjectures, any outcome may be a CVE. The theory accounts for all observable results!

Trying to “save” the concept of CVE in the static framework. Steady state of a dynamic game

The stationary state of a dynamic game may be “summarized” by a static conjectural variations equilibrium. The conjecture captures the inter-temporal forces of the game.

Some papers with this idea....

Public goods dynamic game

Feshtman and Nitzan (1991), Itaya and Shimomura (2001) have considered the game:

$$\max_{e_i(\cdot)} \int_0^\infty e^{-\theta t} \pi^i(e_i(t), x(t)) \, \mathrm{d}t \ ,$$

$$\dot{x} \ = \ \sum_{i=1}^{n} e_i(t) \ - \ \delta x \ .$$

The associated static game is:

$$\max_{e_i} \pi_i(e_i, x) \qquad \text{with} \quad \delta x \ = \ \sum_{i=1}^{n} e_i \ .$$

Public goods, dynamic game (continued)

The first order equations write as:

open loop Nash equilibrium $$\pi_e^i \ + \ \frac{1}{\theta + \delta} \, \pi_x^i \ = \ 0$$

feedback Nash equilibrium

with $e_i = \phi_1 + \phi_2 x$ $$\pi_e^i \ + \ \frac{1}{\delta - (n-1)\phi_2} \, \pi_x^i \ = \ 0$$

CVE with variation r $$\pi_e^i \ + \ r \, \pi_x^i \ = \ 0$$

Nice economic interpretation!

Trying to "save" the concept of CVE in the static framework. Consistent CVE

- Conjectures are endogenized.
- Consistency is the requirement that conjectural best responses and conjectured reactions coincide.

In general, computing CCVE requires solving systems of difference-differential equations.
Few solutions are known, all with constant conjectures.

In particular the nice idea: "it is not necessary to know the payoff of the other player" is lost with the concept of consistency.

Part II
Consistent Conjectures in Dynamic Games

A Dynamic Game

- n players, time horizon T
- $x(t) = (x_1(t), ...x_n(t)) \in R^m$ state variable
- $e_i(t)$ control variable of i in $[t, t+1]$, $e(t)$

Dynamics

$$x(t+1) = f(x(t), e(t)), \quad x(0) = x_0$$

Payoff

$$\sum_{t=0}^{T} \rho^t \pi^i(x(t), e(t))$$

Conjectures

Conjecture of i

$$e_j^c(t) = \phi_t^{ij}(x(t)) \quad \rightarrow \quad x(t+1) = \tilde{f}_i(x(t), e_i(t)) \ .$$

optimal control problem

optimal policy $e_i^{i*}(t)$ that we suppose unique. Player i can compute $e_j^{i*}(t)$ and $x^{i*}(t+1)$ via ϕ_t^{ij}.

A Dynamic Game with consistent conjectures

- $\phi_t^1, \ldots \phi_t^n$ is a (weak) control-consistent conjectural equilibrium $\iff$

$$e_j^{i*}(t) = e^{j*}(t), \quad \forall i \neq j,\ t,\ x(0) = x_0 \ \ (\textit{with } x(0) \textit{ given})$$

- Optimization problem: $\rightarrow e_i^{i*}(t) = \psi_t^i(x(t))$
  $\phi_t^1, \ldots \phi_t^n$ is a feedback-consistent conjectural equilibrium $\iff$

$$\psi_t^i = \phi_t^{ji}, \quad \forall i \neq j,\ t,\ x(0) = x_0$$

Consistency in dynamic games

Fershtman and Kamien (1985) in differential games.
Jean-Marie and Tidball (2005) in discrete-time games, prove:

- Open-loop Nash equilibria coincides with weak control-consistent conjectural equilibria
- Feedback Nash equilibria coincides with feedback-consistent conjectural equilibria

Fershtman and Kamien (1985) propose to consider other kinds of conjectures (state and strategy based conjectures) of the form

$$e_j^c = \phi^{ij}(x(t), e_i(t)).$$

What about state and strategy based conjectures. Quérou and Tidball (2014)

x is a non renewable resource and c_i is consumption of player i. The problem of player i is

$$\max_{\{c_t^i\}} \sum_{t=0}^{\infty} \rho^t log c_{i,t}, \quad x_{t+1} = x_t - c_{i,t} - c_{j,t}.$$

Player i conjectures that player j's consumption decision at period t is given by:

$$c_{j,t}^c = a_i x_t + b_i c_{i,t-1},$$

where a_i and b_i model the player's beliefs. In other words, agent i assumes that the consumption policy of agent j at period t is a function of the state of the resource at period t and his own consumption strategy at period $t-1$.

What about state and strategy based conjectures.

Consistent Conjectural problem

$$\max_{\{c_t^i\}} \sum_{t=0}^{\infty} \rho^t logc_{i,t}, \quad x_{t+1} = x_t - a_i x_t - b_i y_t - c_{i,t} \quad y_{t+1} = c_{i,t}, c_0, x_0, \text{ given}$$

Benchmarks

- Non cooperative problem (Nash feedback)

$$\max_{\{c_t^i\}} \sum_{t=0}^{\infty} \rho^t logc_{i,t}, \quad x_{t+1} = x_t - c_{i,t} - c_{j,t}, \; x_0, \text{ given.}$$

- Cooperative solution (Pareto solution)

$$\max_{\{c_t^i, c_t^j\}} \sum_{t=0}^{\infty} \rho^t \sum_{i=1}^{2} logc_{i,t}, \quad x_{t+1} = x_t - c_{i,t} - c_{j,t}, \; x_0, \text{ given.}$$

What about state and strategy based conjectures. Results

- In the conjectured model the optimal consumption policies depend on the initial consumption level. This implies that one might influence the consumption path that will be chosen by focusing on the initial level of consumption.
- The feedback consistent conjectural equilibrium coincides with the cooperative solution under complete information provided that the initial level of consumption is the same in both cases. If agents cooperate initially then asking for consistency ensures that cooperation will be sustained in the long run.

What about state and strategy based conjectures. Results

- If the initial consumption is too high, then the present procedure leads to a more aggressive pattern than even in the non-cooperative case under full information. In such a case the effects of strategic behaviours are reinforced by incomplete information.
- When initial consumption is sufficiently low, we obtain the surprising conclusion that the procedure leads to an under-exploitation of the resource compared to the full information cooperative benchmark.

Part III
Conjectures and Learning

The learning model

Each player (two players) maximises at each period t (myopic behaviour)

$$\max_{w_i^t} \pi_i(w_i^t, w_j^t, H^t), \quad H^{t+1} = f(w_i^t, w_j^t, H^t).$$

Each player at each period makes a conjecture about the behaviour of the other player

$$w_j^{t,c} = \chi_i(w_i^t, H^t).$$

At each period player i solves the following optimisation problem

$$\max_{w_i^t} \pi_i(w_i^t, \chi_i(w_i^t, H^t), H^t).$$

The general learning model

Call the optimal solution for player i $(i = 1, 2)$, w_i^* and the corresponding conjectured solution of the other player $w_j^{c*} = \chi_i(w_i^*, H)$.

This conjecture is in general different of w_j^* (at time t). Then player i updates his conjecture

Adapting conjectures

This update process can take one of the general forms:

$$\chi_i^{t+1} = \mathcal{U}_i(\chi_i^t, w_i^{t*}, w_j^{t*}, H^t) \quad (1)$$

$$\chi_i^{t+1} = \mathcal{S}_i(w_j^{0*}, H^0, \ldots, w_j^{t*}, H^t) \ . \quad (2)$$

In the first form (1), the "functions" are updated (hence "$\mathcal{U}$") based on the most recent observation of the opponent. In the second form (2), some "statistic" (hence "$\mathcal{S}$") is performed on the whole history of observations.

Note that χ_i^t is a sequence of functions. In practical situations we are going to consider particular functional forms for the conjecture. This functional form will depend on a certain numbers of parameters. These parameters are going to be learned in the learning procedure.

$\mathcal{U}$ conjectures at the steady state

Assuming limits exist ... the procedure gives

$$w_i^\infty,$$

$$\chi_i^\infty(w_i^\infty, H^\infty) = w_j^\infty,$$

$$H^\infty = f(w_i^\infty, w_j^\infty, H^\infty).$$

Learning model. Some general results

Assuming convergence of the processes

- Conjectures in H converge to Nash myopic equilibrium with complete information.
- Conjectures in w + "some" hypothesis on the learning parameters and symmetry, converge to myopic cooperative solution with complete information.

But the idea is to compare with farsighted Nash equilibrium and farsighted cooperative solution (when maximizing in infinite horizon).

A groundwater exploitation problem

- Water extraction is the only input in the production process of the farmers and gives a profit $P_i(w_i^t)$,
- Unitary cost increases when the level of the water table is low,
- Players can also take into account the state of the resource and have an extra (subjective) profit of maintaining the resource, in this case ρ is the discount factor and γ_i is his resource preference,
- The dynamics is given by the evolution of the level of the water table.

$$\pi_i(w_i^t, w_j^t, H^t, H^{t+1}) = P_i(w_i^t) - w_i^t C(H^t, H^{t+1}) + \rho\gamma_i H^{t+1},$$

such that

$$H^{t+1} = f(w_i^t, w_j^t, H^t) = H^t + R - \alpha(w_i^t + w_j^t).$$

Near-sighted or short-sighted procedures.

Three kinds of conjectures

- Affine conjecture in w

$$w_j^c = \bar{w}_j + \beta_i(w_i - \bar{w}_i) := \chi_i(w_i), \quad i \neq j.$$

- Copy w (imitation)

$$w_j^c = \beta_i + w_i, \quad i \neq j.$$

- Affine conjecture in H

$$w_j^c = \beta_i(H_t + R), \quad i \neq j.$$

Simulations

The "performance" of a learning scheme can be assessed using many criteria. Taking into account environmental and economic concerns, we select for our comparisons:

a) the limiting/steady state stock of water H^∞, representing the environment, and
b) the total discounted sum of profits for both players, representing the welfare of the society, V^∞.

Experiments:

- The graphs represent the result of simulations in the plane (H^∞, V^∞).
- Near-sighted solution (H^∞, V^∞) are functions of γ.

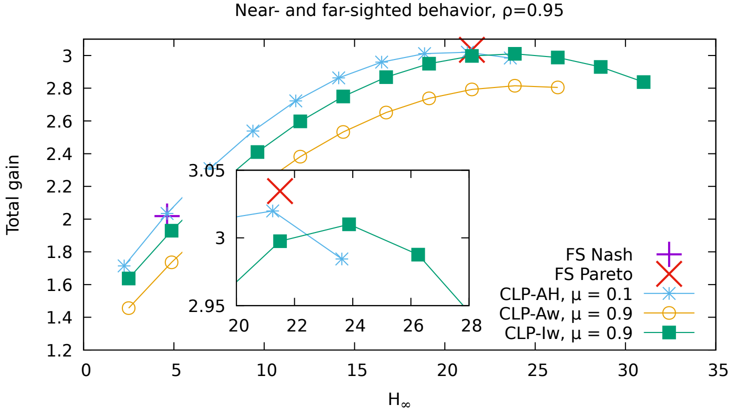

Figure: $(H^\infty(\gamma), V(\gamma))$. All behaviors, $\rho = 0.95$

The maximum of $V(\gamma)$ for the CLPs affine in w and "copy w" is in $\gamma = 0.125$. For the CLP affine in H, $\gamma = 0.225$.

Simulations

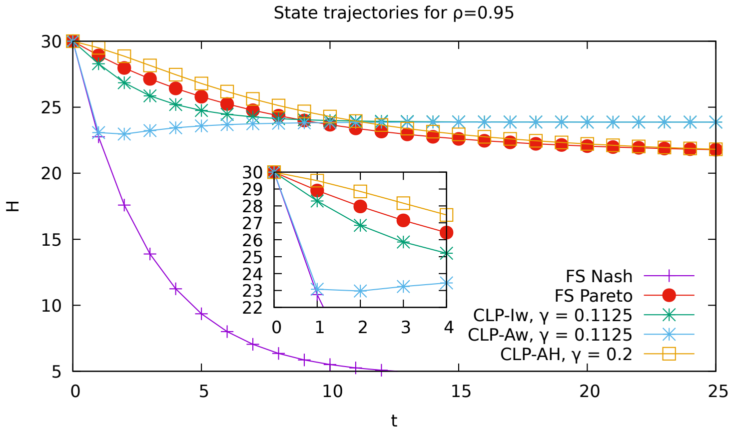

Figure: Evolution of the resource

With learning, when players take into account the near future (γ) we can approach the long term cooperative behaviour.

Conclusion - Extensions

Conjectures replace the need of information about other players payoffs. Conjectures as a way for implicit cooperation. Conjectures as a meta model that involves different kinds of behaviours.

Things to do (to do = TODO)

- Several players and state variables, constraints
- More complex learning process
- Different economic applications
- Models that explain behaviours in experimental economics
- Specially for modelling behavioural economics.
  A conformist agent: agent gets joy from doing like others. A positional agent gets joy from contributing above others. A prosocial agent gets joy when contribution approaches an ideal level
- ...

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