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Abstract

We propose a simplified model of fertirrigation for which the control variable is the irrigation flow rate. We first characterize conditions fo which the system is viable in the sense that it allows the maximal production of biomass. Then, we consider the problem of minimizing the quantity of water delivered during a season under the viability constraint. We demonstrate that depending on the initial nitrogen content in soil, the optimal strategies can be radically different. Moreover, we show the possibility of having an infinity of optimal singular trajectories.
Key-words. Crop models, water irrigation, viability theory, optimal control, singular arcs.

1 Introduction

We consider a simplified crop model describing hydric and nitrogen stresses. We assume here that the other resources necessary for crop growth, such as carbon, phosphorus... are not limiting. The originality compared to existing models of the literature is to consider explicitly in the model that nitrogen can be brought with irrigation, modeling the so-called 'fertirrigation'. Depending on the concentration of nitrogen in the irrigation water, a dilution of existing nitrogen available for crop could occur during irrigation, increasing then the nitrogen stress while hydric stress could be avoided. The objective of the present work is to grasp with the help of a model the right balance between water and nitrogen inputs to be reached for good growth performances. The nitrogen input can be made either by classical fertilization, for instance at seed time, but also with fertirrigation. Fertirrigation is typically used when soil nitrogen content is not sufficient to guarantee the maximal growth when supplying water avoiding hydric stress, but can be compensated by the nitrogen present in the irrigation water. However, when the nitrogen concentration in supply water is too low, an initial input has to be considered. Therefore, we expect also the model to characterize situations for which initial nitrogen inputs are necessary or not. The viability domain that we consider are the set of initial conditions, moisture and nitrogen contents at seeding time, for which it is possible with fertirrigation to obtain the maximal production at harvesting time. The viability analysis that we have conducted in this model allows precisely to characterize the operating parameters, nitrogen concentration in irrigation water and maximal irrigation rate, for which the viability property is satisfied. The boundaries of the viability conditions that we have identified with the model provides then the minimal inputs to obtain the maximal production at harvesting time, avoiding then to unnecessarily overload soil with nitrogen content. Then, we look for the optimal control strategies which consist in having the maximal production (that amounts to stay in the viability domain) while minimizing the total water consumption.

2 The model

In the spirit of existing models [5, 4] and former works [3, 1, 2], we consider the following simplified crop model, where S denotes the level of water in the soil (between 0 and 1), N and B the nitrogen and biomass per unit of surface

$$\dot{S} = k_1\Big(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u\Big) \tag{1}$$

$$\dot{N} = -k_3\varphi(t)K_S(S)f\Big(\frac{N}{S}\Big) + k_4 C_N^{in} u \tag{2}$$

$$\dot{B} = \varphi(t)K_S(S)f\Big(\frac{N}{S}\Big)g(B) \tag{3}$$

where

$$K_S(S) = \begin{cases} 0, & S \in [0, S_w] \\ \dfrac{S-S_w}{S^\star - S_w}, & S \in [S_w, S^\star] \\ 1, & S \in [S^\star, 1] \end{cases} \qquad K_R(S) = \begin{cases} 0, & S \in [0, S_h] \\ \dfrac{S-S_h}{1-S_h}, & S \in [S_h, 1] \end{cases}$$

with threshold values $0 < S_h < S_w < S^\star < 1$,

$$f(C_N) = \begin{cases} \dfrac{C_N}{\eta_C}, & C_N \in [0, \eta_C] \\ 1, & C_N > \eta_C \end{cases}$$

and φ is a smooth increasing function from $[0,T]$ to $[0,1]$ that represents the crop radiation interception efficiency. The growth function g is strictly positive for $B > 0$. We shall say that a control $u(\cdot)$ (the irrigation flow rate) is admissible if it takes values in $[0, u_{max}]$ and the solution $S(\cdot)$ remains on the domain $[0,1]$, which amounts to impose the condition

$$\{S = 1\} \Rightarrow u \leq \frac{1}{k_2}\ .$$

We shall say that crop suffers from *hydric stress* when the value of the function K_S is not maximal (that is when $S < S^\star$), and *nitrogen stress* when the value of the function f is not maximal (that is when $N - \eta_C S < 0$). Parameters C_N^{in}, u_{max} are the operating parameters of the fertirrigation.

3 The viability analysis

From equation (3), one can see that the biomass growth is maximal when

$$K_S(S(t))f\Big(\frac{N(t)}{S(t)}\Big) = 1, \quad t \in [0,T]$$

which amounts to claim that the state (S,N) belongs to the set

$$E := \{(S,N) \in [0,1] \times \mathbb{R}_+ \ ;\ S \geq S^\star,\ N \geq \eta_c S\} \tag{4}$$

(see Figure 1) for any $t \in [0,T]$.

Being in E amounts to state that both hydric and nitrogen stresses are avoided. Therefore, we shall consider the following viability property.

Definition 1. *The domain E is viable if for any initial condition $(t_0, S_0, N_0) \in [0,T] \times E$, there exists an admissible control $u(\cdot)$ such that the solution of (1)-(2) with $S(t_0) = S_0$, $N(t_0) = N_0$ verifies $(S(t), N(t)) \in E$ for any $t \in [t_0, T]$.*

For convenience, we define the numbers

$$C_1 = \eta_C k_1 - k_3, \quad C_2 = k_4 C_N^{in} - \eta_C k_1 k_2.$$

Proposition 1. *The domain E is viable in the sense of Definition 1 exactly when the condition*

$$C_N^{in} \geq \underline{C_N^{in}} := \frac{k_2}{k_4} \max\Big(\eta_C k_1\big(1 - K_R(S^\star)\big), k_3\Big) \tag{5}$$

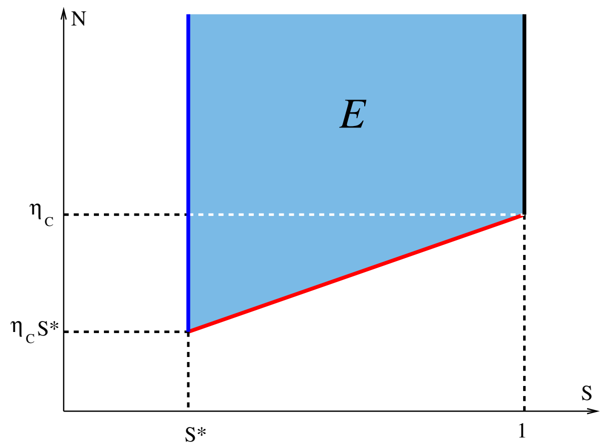

Figure 1: Illustration of the set E. The boundaries of the hydric stress is depicted in blue and of the nitrogen one in red

is fulfilled and u_{max} *is such that*

$$u_{max} \geq \underline{u_{max}} := \begin{cases} \max\left(\dfrac{1}{k_2}, \dfrac{-C_1}{C_2}\right) & \text{if } C_1 < 0 \text{ and } C_2 > 0, \\ \dfrac{1}{k_2} & \text{otherwise} \end{cases} \tag{6}$$

Proof. At $S = S^\star$, one has from equation (1)

$$\dot{S} = k_1\big(-\varphi(t) - (1-\varphi(t))K_R(S^\star) + k_2 u\big).$$

A necessary condition to have $\dot{S} \geq 0$ is that $u \in [0, u_{max}]$ satisfies

$$u \geq \frac{\varphi(t) + (1-\varphi(t))K_R(S^\star)}{k_2}.$$

This has to be fulfilled for any possible $t \in [0, T]$, which implies the condition

$$u_{max} \geq \max_{t\in[0,T]} \frac{\varphi(t) + (1-\varphi(t))K_R(S^\star)}{k_2} = \frac{1}{k_2}. \tag{7}$$

When $N = \eta_C S$, one has from equations (1)-(2)

$$\dot{N} - \eta_C\dot{S} = k_1\eta_C\big(\varphi(t) + (1-\varphi(t))K_R(S)\big) - k_3\varphi(t) + \big(k_4 C_N^{in} - \eta_C k_1 k_2\big)u.$$

Then, a necessary condition to have $\dot{N} - \eta_C\dot{S} \geq 0$ is to have for any $(t, S) \in [0, T] \times [S^\star, 1]$ the existence of $u \in [0, u_{max}]$ such that

$$(k_1\eta_C - k_3)\varphi(t) + k_1\eta_C(1-\varphi(t))K_R(S) + \big(k_4 C_N^{in} - \eta_C k_1 k_2\big)u \geq 0$$

which implies the existence of $u \in [0, u_{max}]$ such that

$$\min\big(k_1\eta_C - k_3, k_1\eta_C K_R(S^\star)\big) + \big(k_4 C_N^{in} - \eta_C k_1 k_2\big)u \geq 0$$

that is

$$\min\big(C_1, k_1\eta_C K_R(S^\star)\big) + C_2 u \geq 0. \tag{8}$$

At $S = 1$ one should have also $\dot{S} \leq 0$, that is $u \leq \frac{1}{k_2}$, along with

$$\min\big(C_1, k_1\eta_C\big) + C_2 u \geq 0. \tag{9}$$

Let us distinguish cases depending on the signs of C_1 and C_2.

i. If $C_1 < 0$ and $C_2 \leq 0$, clearly condition (8) cannot be fulfilled for a non-negative u.

ii. If $C_1 < 0$ and $C_2 > 0$, one has to have

$$k_2 C_1 + C_2 \geq 0 \tag{10}$$

for condition (9) to be fulfilled with $u \leq \frac{1}{k_2}$. Condition (8) implies that u_{max} satisfies

$$u_{max} \geq \frac{-C_1}{C_2} > 0$$

and combining with condition (7)

$$u_{max} \geq \max\left(\frac{1}{k_2}, \frac{-C_1}{C_2}\right).$$

iii. If $C_1 \geq 0$ with $C_2 \geq 0$, conditions (8) and (9) are fulfilled for any $u \geq 0$. Therefore, only condition (7) needs to be satisfied.

iv. If $C_1 \geq 0$ with $C_2 < 0$, u has to satisfy

$$u \leq \frac{\min\left(C_1, k_1 \eta_C K_R(S^\star)\right)}{-C_2}.$$

This condition is compatible with condition (7) if

$$k_2 \min\left(C_1, k_1 \eta_C K_R(S^\star)\right) + C_2 \geq 0 \tag{11}$$

and then condition (9) with $u \leq \frac{1}{k_2}$ is necessarily satisfied. Finally, u_{max} has simply to satisfy condition (7).

Note that conditions for the set E to be viable in case ii with (10), or case iii, or case iv with (11), are satisfied when the single condition (11) is satisfied. If this later one is not verified, we are exactly in cases i, ii or iv when the set E is not viable. Therefore, this condition is necessary and sufficient for the set E to be viable. Equivalently, this condition is fulfilled if and only if (5) is verified. Finally, u_{max} has to fulfill condition (6) depending on the case ii, iii or iv. $\square$

4 The optimal control problem

When the set E defined in (4) is viable, that is according to Proposition 1 for the conditions

$$C_N^{in} \geq \underline{C_N^{in}}, \quad u_{max} \geq \underline{u_{max}}$$

where $\underline{C_N^{in}}$ and $\underline{u_{max}}$ are given in (5)-(6), one can consider the optimization problem which consists in minimizing the total amount of water delivered on $[0, T]$ while remaining in the set E.

Problem $\mathcal{P}$: For $(S_0, N_0) \in E$, we look for

$$\inf_{u(\cdot)\in\mathcal{U}} \int_0^T u(\tau)d\tau \quad \text{s.t.} \quad (S^u_{0,S_0}(t), N^u_{0,S_0,N_0}(t)) \in E, \, t \in [t_0, T]$$

where $(S^u_{t_0,S_0}(\cdot), N^u_{t_0,S_0,N_0}(\cdot))$ denotes the solution of (1)-(2) with $S(t_0) = S_0$, $N(t_0) = N_0$ and control $u(\cdot)$. $\mathcal{U}$ denotes the set of admissible controls $u(\cdot)$ on $[0, T]$.

In the following, we shall consider initial conditions with $S_0 = 1$ (and thus $N_0 \geq \eta_c$) only which are often met in practice (assuming the soil humidity S to be maximal at the beginning of the agronomic season).

Remark 1. *From equation* (2), *solutions in the set E satisfy*

$$N(t) = N_0 - k_3 \int_0^t \varphi(\tau)d\tau + k_4 C_N^{in} \int_0^t u(\tau)d\tau, \quad t \in [0, T].$$

Therefore, minimizing $\int_0^T u(\tau)d\tau$ amounts to minimizing $N(T)$ among all admissible solutions in E. In particular, if there exists an admissible control such that the solution remains in E and verifies $N(T) = \eta_C S^\star$, which is the smallest value of N in the set E, then it is necessarily optimal.

Problem $\mathcal{P}$ takes the form of a standard optimal control problem with state constraints. A usual approach to study optimal solutions is to write necessary optimality conditions from the Maximum Principle with state constraints. In general, these conditions do not provide straightforwardly the optimal solution, because one has to analyze the solutions of the adjoint system with the maximization condition of the Hamiltonian and terminal conditions. We failed to characterize the optimal solutions of our problem in this way, due the non-autonomous behavior of the dynamics (the Hamiltonian is no constant along optimal solutions) and the measure-multipliers of the state constraints (or jumps in the adjoint variables) to be determined, which make together the analysis quite intricate. We have chosen here another route exploiting the particular structure of the model, where the S-dynamics does not depend on the other variable N (but the optimal control has to depend on both variables because of the constraints), and using comparison arguments with the introduction of intermediate variables. Guided by the intuition that for high values of the initial nitrogen content N_0 one expects the nitrogen stress to be never met, we characterize below values of N_0 for which the optimal solution either saturates the hydric constraint $S \geq S^\star$ only (section 4.1), or the nitrogen constraint $N \geq \eta S$ only (section 4.2), or both (section 4.3). As the dynamics is linear with respect to the control, we have looked for candidate optimal solutions composed of extreme arcs i.e. with extreme values of the control or on the boundary of the constraints set. However, we found a gap among all these values of N_0, but showed that it corresponds to singular trajectories that can be interpreted as convex combinations of two particular extreme solutions (section 4.4).

4.1 Saturation of the hydric stress only

We define a control strategy the ensures the hydric stress to be avoided, as follows.

Definition 2. *The S-strategy is given by the time-varying feedback*

$$u_S(t,S) := \begin{cases} 0, & S > S^\star, \\ u_S^{sing}(t), & S = S^\star \end{cases} \tag{12}$$

where

$$u_S^{sing}(t) := \frac{1}{k_2}\big(\varphi(t) + (1-\varphi(t))K_R(S^\star)\big) \tag{13}$$

Note that under the condition (6), the control (13) does take values in $[0, u_{max}]$ and is thus admissible. This strategy consists in no irrigation until the humidity level reaches the threshold $S^\star$ if it can, and in this latter case maintaining then the humidity S constant at the value $S^\star$. We characterize now the optimality of this strategy. Let us define the hitting time related to the hydric constraint with the null control

$$t_S := \sup\{t \in [0,T];\ S^0_{0,1}(t) > S^\star\}, \quad S_S^\star := S^0_{0,1}(t_S). \tag{14}$$

When the time horizon T is large enough, one has $S_S^\star = S^\star$.

Proposition 2. *The S-strategy is optimal for problem $\mathcal{P}$ when condition*

$$C_S := C_1\varphi(t_S) + k_1\eta_C K_R(S_S^\star)(1-\varphi(t_S)) \geq 0 \tag{15}$$

or

$$N_0 \geq N_0^\flat := \max\Big(\eta_C, \eta_C S_S^\star + k_3\int_0^{t_S}\varphi(t)dt\Big) \tag{16}$$

is fulfilled. Then, the optimal value of the criterion is

$$V^\star = \begin{cases} \dfrac{1}{k_2}\displaystyle\int_{t_S}^{T}\big(\varphi(t) + (1-\varphi(t))K_R(S^\star)\big)dt, & t_S < T \\ 0, & t_S = T \end{cases} \tag{17}$$

Proof. From equations (1)-(2), one gets

$$\begin{aligned} \dot N - \eta_C \dot S &= k_1\eta_C\big(\varphi(t) + (1-\varphi(t))K_R(S)\big) - k_3\varphi(t) + \big(k_4 C_N^{in} - \eta_C k_1 k_2\big)u \\ &= C_1\varphi(t) + (1-\varphi(t))k_1\eta_C K_R(S) + C_2 u \end{aligned} \tag{18}$$

For $S > S^\star$ and $u = 0$, one obtains from equation (18) the following properties

1. when $C_1 \geq 0$, one has necessarily with the control $u = 0$ the property
$$N(t) - \eta_C S(t) \geq 0, \quad t \in [0, t_S] \tag{19}$$

2. when $C_1 < 0$, one has
$$\ddot{N}(t) - \eta_C \ddot{S}(t) = C_1 \varphi'(t) - \varphi'(t) K_R(S(t)) + (1 - \varphi(t)) k_1 \eta_C K_R'(S(t)) \dot{S}(t) \leq 0. \tag{20}$$
and thus
$$\dot{N}(t) - \eta_C \dot{S}(t) \geq \dot{N}(t_S) - \eta_C \dot{S}(t_S) = C_1 \varphi(t_S) + (1 - \varphi(t_S)) k_1 \eta_C K_R(S_S^\star), \quad t \in [0, t_S]$$
Therefore, when condition (15) is fulfilled, property (19) is also verified. Otherwise, as (20) shows that the map $t \mapsto N(t) - \eta_C S(t)$ is concave on $[0, t_S]$, property (19) is verified exactly when
$$N(0) - \eta_C S(0) \geq 0 \text{ and } N(t_S) - \eta_C S(t_S) \geq 0. \tag{21}$$
From equation (2) with $u = 0$, one gets
$$N(t_S) = N(t_0) - k_3 \int_0^{t_S} \varphi(t) dt$$
Then (21) is fulfilled exactly when condition (16) is verified.

At $S = S^\star$, which is attainable before T when $t_S < T$, one obtains with the control (13)
$$\dot{N} - \eta_C \dot{S} = \Big(C_1 + \frac{C_2}{k_2} \Big) \big(\varphi(t) + (1 - \varphi(t)) K_R(S^\star) \big) + k_3 (1 - \varphi(t)) K_R(S^\star)$$
where
$$C_1 + \frac{C_2}{k_2} = \frac{k_4}{k_2} C_N^{in} - k_3$$
which is non-negative under the condition $C_N^{in} \geq \underline{C_N^{in}}$. Therefore, one has the property
$$\dot{N}(t) - \eta_C \dot{S}(t) \geq 0, \quad t \in [t_S, T] \tag{22}$$
with the S-strategy. Finally, property (19) along with property (22) shows that the solution $S_{0,1}^{u_S}(\cdot)$ generated by the S-strategy remains in the set E on $[0, T]$ when condition (15) or (16) is fulfilled. Let u show now that it is necessarily optimal.

Let us denote by $S(\cdot)$ the solution generated by the S-strategy, and $u(\cdot)$ the corresponding control function. For any other admissible solution $\tilde{S}(\cdot)$ with a control $\tilde{u}(\cdot)$, one has
$$\frac{d}{dt}(\tilde{S}(t) - S(t)) = -k_1 (1 - \varphi(t)) \frac{\tilde{S}(t) - S(t)}{1 - S_h} + k_1 k_2 \tilde{u}(t) := F(t, \tilde{S}(t) - S(t)), \quad t \in [0, t_S]$$
As the function F satisfies $F(t, 0) \geq 0$ for any $t \in [0, t_S]$, we deduce that the solution $\delta(\cdot) = \tilde{S}(\cdot) - S(\cdot)$ of $\dot{\delta} = F(t, \delta)$, $\delta(0) = 0$ stays non-negative on this time interval. When $T > t_S$, one has on $[t_S, T]$ $S(\cdot) = S^\star$ which is the smallest admissible value to stay in E. Therefore, any admissible solution $\tilde{S}(\cdot)$ that stays in E verifies $\tilde{S}(t) \geq S(t)$ for any $t \in [0, T]$. Then one can write
$$\begin{aligned} & S(T) - S(0) = -k_1 \int_0^T \varphi(t) + (1 - \varphi(t)) K_R(S(t)) dt + k_1 k_2 \int_0^T u(t) dt \\ \leq \quad & \tilde{S}(T) - \tilde{S}(0) = -k_1 \int_0^T \varphi(t) + (1 - \varphi(t)) K_R(\tilde{S}(t)) dt + k_1 k_2 \int_0^T \tilde{u}(t) dt \end{aligned}$$
from which we obtain
$$\int_0^T u(t) dt \leq \frac{1}{k_2} \int_0^T (1 - \varphi(t)) (K_R(S(t)) - K_R(\tilde{S}(t))) dt + \int_0^T \tilde{u}(t) dt$$
and as the function K_R is increasing, we deduce
$$\int_0^T u(t) dt \leq \int_0^T \tilde{u}(t) dt$$
that is the optimality of the S-strategy. Finally, expression (17) follows straightforwardly from (12). □

Let us stress that conditions (15), (16) do not depend on operating parameters, and are thus intrinsic to the crop-soil system. We focus now on cases for which these two conditions are both invalidated. Note that having condition (16) not fulfilled is possible (i.e. the existence of initial conditions $(0, 1, N_0)$ in E) when $N_0^\flat > \eta_c$. Therfore, we shall consider now the following conditions

$$C_S = C_1\varphi(t_S) + k_1\eta_C K_R(S_S^\star)(1 - \varphi(t_S)) < 0 \tag{23}$$

and

$$N_0^\flat = \eta_C S_S^\star + k_3 \int_0^{t_S} \varphi(t)dt > \eta_C. \tag{24}$$

4.2 Saturation of the nitrogen stress only

We consider here cases for which its is possible with a "bang-bang" control the saturate the constraint $N - \eta_C S \geq 0$ exactly at time T, keeping the constraint $S \geq S^\star$ unsaturated. Indeed, we found that saturating the constraint $N - \eta_C S \geq 0$ on a time interval of positive length is not optimal if the trajectory does not touch $S = S^\star$.

Definition 3. *The N-strategy is given by the bang-bang control with commutation time $t_c \in [0, T]$, as follows*

$$u_N(t, t_c) := \begin{cases} \bar{u}, & t < t_c \\ 0 & t \geq t_c \end{cases} \qquad \text{where } \bar{u} := \frac{1}{k_2} \tag{25}$$

Clearly, this control is admissible with condition (6). It will be relevant to define the time

$$\underline{t}_c = \inf\{t \in [0, T];\ S^0_{T,S_S^\star}(t) < 1\} \tag{26}$$

which consists in integrating backward the S dynamics with the control $u = 0$ from the terminal state $S(T) = S_S^\star$ up to the target $\{S = 1\}$. This time is well defined as the solution $S(\cdot)$ with the control $u = 0$ is decreasing in forward time and $S_{0,1}(T) \leq S_{0,1}(t_S) = S_S^\star$. If $t_S = T$, then one has $\underline{t}_c = 0$, and if $t_S < T$ then $\underline{t}_c > 0$. The value

$$N_0^\sharp := \eta_C S_S^\star + k_3 \int_0^T \varphi(t)dt - k_4 C_N^{in} \bar{u}\underline{t}_c \tag{27}$$

is such that the solution for $N_0 = N_0^\sharp$ and the control u_N with commutation time $\underline{t}_c$ reaches exactly the point $(S_S^\star, \eta_C S_S^\star)$ in E at the final time T. Note that one has necessarily $N_0^\sharp \leq N_0^\flat$ because any solution with $N_0 > N_0^\flat$ cannot reach the set $\{N - \eta_C S = 0\}$ at time T (see Section 4.1). One has the following result about the optimality of the control u_N.

Proposition 3. *Assume that conditions (23)-(24) are fulfilled. If $N_0^\sharp \geq \eta_C$, then for any $N_0 \in [\eta_C, N_0^\sharp]$, there exists a unique $t_c^\star \in [\underline{t}_c, T]$ such that the solution with control $u_N(\cdot, t_c^\star)$ verifies $N(T) = \eta_C S(T) \geq \eta_C S^\star$. Then, the N-strategy with $t_c = t_c^\star$ is optimal for the problem $\mathcal{P}$, and the optimal cost is $V^\star = \bar{u}t_c^\star$.*

Proof. Note first that condition (23) implies $C_1 < 0$ and then condition (5) gives $C_2 \geq -k_2 C_1 > 0$.

One can check that the solution $S^{\bar{u}}_{0,1}(\cdot)$ is constant equal to 1. This implies that one has $S^{\bar{u}}_{0,1}(t_c) = 1$. Moreover, $N_c(t_c) := N^{\bar{u}}_{0,1,N_0}(t_c)$ is continuous with respect to t_c. Then, the composed function

$$\xi(t_c) := N^0_{t_c,1,N_c(t_c)}(T) - \eta_C S^0_{t_c,1}(T)$$

is continuous with respect to t_c (from the continuity of solutions of ordinary differential equations w.r.t. the initial condition). We look now for the existence of a zero of this function. For $t_c \in (0, T]$, one has $S = 1$ on the time interval $[0, t_c]$ and from equation (2) one gets

$$\dot{N} = -k_3\varphi(t) + \frac{k_4 C_N^{in}}{k_2} > -k_3 + \frac{k_4 C_N^{in}}{k_2} > -k_3 + \frac{k_4 \underline{C_N^{in}}}{k_2} \geq 0,$$

the last inequality being provided by condition (5). Therefore, the function

$$\gamma(t) := N(t) - \eta_C S(t)$$

is increasing on $[0, t_c]$. We immediately deduce the inequality $\xi(T) = \gamma(T) > \gamma(0) \geq 0$.

Let us show that ξ is increasing. Take $t_c' > t_c$ in $[0,T]$ and denote $S'(\cdot)$, $\gamma'(\cdot)$ the solutions with control u_N and commutation time t_c'. One has $S'(t_c) = S(t_c)$ and $\gamma'(t_c) = \gamma(t_c)$. As $u_N(\cdot, t_c') = \bar{u} > 0 = u_N(\cdot, t_c)$ on (t_c, t_c'), one gets from equation (1) the inequality $S'(t) > S(t)$ for $t \in (t_c, t_c']$. Then, by uniqueness of the solutions of the scalar dynamics (1) with control $u = 0$, we get $S'(t) > S(t)$ for any $t \in (t_c', T]$. From equation (18), one can write

$$\xi(t_c') = \gamma'(T) = \gamma'(t_c) + \int_{t_c}^{T} \left(C_1\varphi(t) + (1-\varphi(t))k_1\eta_C K_R(S'(t))\right)dt + C_2\bar{u}(t_c' - t_c)$$

As K_R is increasing and $C_2 > 0$, we obtain

$$\xi(t_c') > \gamma(t_c) + \int_{t_c}^{T} \Big(C_1\varphi(t) + (1-\varphi(t))k_1\eta_C K_R(S(t))\Big)dt = \gamma(T) = \xi(t_c)$$

which proves that ξ is increasing.

For $t_c = 0$, the control u is equal to 0 on the whole interval $[0,T]$, and one has for $N_0 < N_0^\flat$

$$\gamma(t_S) = N(t_S) - \eta_C S(t_S) = N_0 - k_3\int_0^{t_S}\varphi(t)dt - \eta_C S_S^\star < N_0^\flat - k_3\int_0^{t_S}\varphi(t)dt - \eta_C S_S^\star = 0.$$

As in the proof of Proposition 2 with $C_1 < 0$, we obtain that the function γ is concave (with control $u = 0$). If $t_S < T$, one has with condition (23)

$$\dot{\gamma}(t) < \dot{\gamma}(t_S) = C_1\varphi(t_S) + (1 - \varphi(t_S)k_1\eta_C K_R(S_S^\star) < 0, \quad t > t_S.$$

Therefore, one has $\gamma(T) \leq \gamma(t_S) < 0$ when $t_c = 0$, that is $f(0) < 0$. The function ξ being continuous increasing with $\xi(T) > 0$, we deduce that the exists an unique $t_c^\star$ such that $\xi(t_c^\star) = 0$, which belongs to $(0,T)$. However, we have to check that the corresponding solution remains in E. We have already shown that γ is increasing on $[0, t_c^\star]$, which implies $\gamma(t_c^\star) \geq 0$, and that γ is concave on $(t_c^\star, T)$ with $\gamma(T) = 0$. This implies that one has necessarily $\gamma(t) > 0$ for any $t \in [0,T)$. The constraint $N \geq \eta_C S$ is thus satisfied at any time. The solution S remains equal to 1 on $[0, t_c^\star]$, and is decreasing on $[t_c^\star, T]$ (as the control u is equal to 0 for $t > t_c^\star$). Therefore, the solution remains in E exactly when $S(T) \geq S^\star$. If $t_S = T$ then one has necessarily $S(T) \geq S^\star$. Otherwise, note that $T - \underline{t}_c$ is the largest time for which it is possible to reach $S^\star$ at T and thus one has $S(T) \geq S^\star$ exactly when $t_c^\star \geq \underline{t}_c$. From equation (2) one gets

$$N(T) = N_0 - k_3\int_0^T \varphi(t)dt + k_4 C_N^{in}\bar{u}t_c$$

and for $t_c = \underline{t}_c$, one has $S(T) = S^\star$ with the property

$$N_0 \leq N_0^\sharp \iff N(T) \leq N_0^\sharp - k_3\int_0^T\varphi(t)dt + k_4C_N^{in}\bar{u}\underline{t}_c = \eta_c S^\star \iff \xi(\underline{t}_c) \leq 0.$$

As the function ξ is increasing, we deduce the equivalence

$$N_0 \leq N_0^\sharp \iff t_c^\star \geq \underline{t}_c.$$

We conclude that the solution with the control u_N such that $N(T) - \eta_C S(T) = 0$ satisfies also the constraint $S \geq S^\star$ when $N_0 \leq N_0^\sharp$. When $t_S = T$, note that one has necessarily $N_0 \leq N_0^\sharp$ as $\underline{t}_c = 0$.

Let us show now that the solution $(S(\cdot), N(\cdot))$ with the control $u(\cdot) = u_N(\cdot, t_c^\star)$ is optimal. If not, there should exist another solution in E, denoted $(\tilde{S}(\cdot), \tilde{N}(\cdot))$ with an admissible control $\tilde{u}(\cdot)$ such that

$$\int_0^T \tilde{u}(t)dt < \int_0^T u(t)dt.$$

Note from equation (2) that this implies the inequality

$$\tilde{N}(T) < N(T). \tag{28}$$

On the another hand, one has from equation (18)

$$N(T) - \eta_C S(T) = N_0 - \eta_C + C_1\int_0^T\varphi(t)dt + k_1\eta_c\int_0^T(1-\varphi(t))K_R(S(t))dt + C_2\int_0^T u(t)dt = 0$$

and

$$\tilde{N}(T) - \eta_C \tilde{S}(T) = N_0 - \eta_C + C_1 \int_0^T \varphi(t)dt + k_1 \eta_c \int_0^T (1-\varphi(t)) K_R(\tilde{S}(t))dt + C_2 \int_0^T \tilde{u}(t)dt \geq 0$$

that imply

$$k_1 \eta_c \int_0^T (1-\varphi(t))\big(K_R(S(t)) - K_R(\tilde{S}(t))\big)dt \leq C_2 \int_0^T \tilde{u}(t) - u(t)dt < 0.$$

For $t \in [0, t_c^\star]$, one has $K_R(S(t)) - K_R(\tilde{S}(t)) = 1 - K_R(\tilde{S}(t)) \geq 0$, which implies that there exists necessarily an non-empty interval $(t_1, t_2) \subset (t_c^\star, T)$ such that $K_R(S(t)) - K_R(\tilde{S}(t)) < 0$ for $t \in (t_1, t_2)$. As the function K_R, we deduce that one has $\tilde{S}(t) > S(t)$ for $t \in (t_1, t_2)$. But then, as $\tilde{u}(t) \geq 0 = u(t)$ for any $t \in [t_1, T]$, the solutions $S(\cdot)$, $\tilde{S}(\cdot)$ cannot cross i.e. are such that $\tilde{S}(t) > S(t)$ for any $t > t_1$. In particular, at $t = T$, one has

$$\tilde{N}(T) \geq \eta_C \tilde{S}(T) > \eta_C S(t) = N(T)$$

which contradicts the inequality (28). □

4.3 Saturation of both stress

From the analysis of Section 4.1, we know that when the hydric constraint is hit first, then the nitrogen constraint remains non saturated up to the terminal time. We thus consider here a NS-strategy which consists in saturating the nitrogen constraint first. Remark that the case for which $S_S^\star > S^\star$ (that is when $S = S^\star$ cannot be reached) is already treated with Proposition 3. So, we consider here cases for which $S_S^\star = S^\star$.

Definition 4. *The NS-strategy is given by the time-varying feedback*

$$u_{NS}(t, S, N) := \begin{cases} 0, & S > \dfrac{N}{\eta_C}, \\ \max(0, u_N^{sing}(t, S)), & S = \dfrac{N}{\eta_C} > S^\star, \\ u_S^{sing}(t), & S = S^\star \end{cases} \tag{29}$$

where

$$u_N^{sing}(t, S) := \frac{C_1 \varphi(t) + k_1 \eta_C K_R(S)(1-\varphi(t))}{-C_2} \tag{30}$$

and $u_S^{sing}(\cdot)$ is defined in (13).

Note that condition (23) implies $C_1 < 0$ and condition (5) gives $C_2 \geq -k_2 C_1 > 0$. Then for any $(t, S) \in [0, T] \times [0, 1]$ one has $u_N^{sing}(t, S) \leq \frac{-C_1}{C_2}$, which is upper bounded by u_{max} with condition (6). The control $u_{NS}(\cdot)$ is thus admissible.

From the definitions (14) of t_S and (24) of $N_0^\flat$, one has $N^0_{0,1,N_0}(t_S) = N^{u_{NS}}_{0,1,N_0}(t_S) = \eta_C S^\star$ for any $N_0 \leq N_0^\flat$, and the following number is thus well defined.

$$N_0^\dagger := \inf\{N_0 \in [\eta_C, N_0^\flat];\ \exists t \in [0, T] \text{ s.t. } N^{u_{NS}}_{0,1,N_0}(t) = \eta_c S^\star\} \tag{31}$$

One has the following result about the optimality of the NS-strategy.

Proposition 4. *Assume that conditions* (23)-(24) *are satisfied with $S_S^\star = S^\star$. For any $N_0 \in [N_0^\dagger, N_0^\flat]$, the NS-strategy is optimal for problem $\mathcal{P}$. The optimal cost is*

$$V^\star = \frac{1}{k_4 C_N^{in}} \left(\eta_C S^\star - N_0 + k_3 \int_0^{t^\star} \varphi(t)dt \right) + \frac{1}{k_2} \int_{t^\star}^T \varphi(t) + (1-\varphi(t)) K_R(S^\star)dt$$

where

$$t^\star := \inf\{t \in (t_S, T];\ S^{u_{NS}}_{0,1,N_0}(t) = S^\star\}.$$

If $N_0^\dagger > \eta_C$, then the optimal solution for $N_0 = N_0^\dagger$ verifies $N(T) = \eta_C S^\star$.

Proof. For $N_0 = N_0^\flat$, the control generated by NS-strategy coincides with the one generated by the S-strategy, the case $S = N/\eta_C = S^\star$ being not met by the feedback (29), and we already know from Proposition 2 that it is optimal.

For $N_0 < N_0^\flat$ (when $N_0^\dagger < N_0^\flat$), the solution with the control $u = 0$, verifies from the definition of t_S,

$$S(t_S) = S^\star, \quad N(t_S) = N_0 + \int_0^{t_S} \varphi(t)dt < \eta_C S^\star$$

and by continuity the number

$$t_N := \inf\{t \in [0,T];\ N(t) - \eta_C S(t) = 0\}$$

is well defined, with $t_N < t_S$ and $S(t_N) > S^\star$. Moreover t_N is increasing w.r.t. $N_0 < N_0^\flat$.

For $t > t_N$, one has with the control (29)

$$\dot{S} \leq \frac{k_1}{-C_2}\Big((k_2C_1 + C_2)\varphi(t) + K_R(S)(k_1k_2\eta_C + C_2)(1-\varphi(t))\Big) < 0$$

as long as $S(t) > S^\star$. Then, one obtains

$$\frac{d}{dt}\Big(C_1\varphi(t) + k_1\eta_C K_R(S)(1-\varphi(t))\Big) = C_1\varphi'(t) - k_1\eta_C K_R(S)\varphi'(t) + k_1\eta_C K_R(S)(1-\varphi(t)\dot{S} < 0$$

which implies

$$C_1\varphi(t) + k_1\eta_C K_R(S(t))(1-\varphi(t)) < C_1\varphi(t_N) + k_1\eta_C K_R(S(t_N))(1-\varphi(t_N))$$

On the other hand, at time t_N one should have $\dot{N}(t_N^-) - \eta_C\dot{S}(t_N^-) \leq 0$ (if not, one obtains a contradiction of t_N being an infimum), that is

$$\dot{N}(t_N^-) - \eta_C\dot{S}(t_N^-) = C_1\varphi(t_N) + k_1\eta_C K_R(S(t_N))(1-\varphi(t_N)) \leq 0$$

We deduce that the control $u_n^{sing}(t, S(t))$ defined in (30) is positive for any $t > t_N$ such that $S(t) > S^\star$ (and thus $u_{NS} = u_N^{sing}$). With this control, one can also check that one has

$$\dot{N}(t) - \eta_C S(t) = 0 \Rightarrow N(t) - \eta_C S(t) = 0 \tag{32}$$

for such times. Moreover, one has $S_{0,1}^{u_{NS}}(t) > S_{0,1}^0(t) > S^\star$ for $t \in (t_N, t_S)$, as the control $u_{NS} = u_N^{sing}$ takes positive values. For $N_0 \geq N_0^\dagger$, we know that there exists a time

$$t^\star = \inf\{t \in (t_S, T];\ S(t) = S^\star\}$$

for the solution $(S(\cdot), N(\cdot))$ with the control u_{NS}, and one has $N(t^\star) = \eta_C S^\star$ from (32). Let us show that $S(t_N)$ is decreasing with respect to N_0. Take $N_0 < N_0' < N_0^\flat$ and denote $S'(\cdot)$ the solution generated by the control u_{NS} for the initial condition $N(0) = N_0'$. As t_N is increasing w.r.t. N_0, one has $\dot{S}'(t) = \dot{S}(t)$ for $t \in [0, t_N]$ and $\dot{S}'(t) < \dot{S}(t)$ for $t \in (t_N, t_N')$ (where $t_N' > t_N$ is the time to reach $N' - \eta_C S' = 0$), which gives $S'(t_N') < S(t_N)$. By uniqueness of the solutions with the control u_n^{sing}, we obtain $S'(t) < S(t)$ for $t > t_N'$ as long as $S'(t) > S^\star$. We deduce that $t^\star$ is decreasing w.r.t. $N_0 \in (N_0^\dagger, N_0^\flat)$ and consequently the final state $N(T)$ is increasing w.r.t. N_0. This implies that one has $N(T) > \eta_C S^\star$ for any $N_0 \in (N_0^\dagger, N_0^\flat)$. For $N_0 = N_0^\dagger$, if one has $N_{0,1,N_0^\dagger}(T) = \eta_C S^\star$, then u_{NS} is necessarily optimal, following Remark 1. Note that when $N_0^\dagger > \eta_C$, one has necessarily $N(T) = \eta_C S^\star$. If not, one should have $t^\star < T$ and by continuity there exists $N_0' \in (\eta_C, N_0^\dagger)$ such that the solution with the NS-strategy reaches $S^\star$ at a time $t^{\star\prime} \in (t^\star, T)$, in contradiction with the definition (31).

Let us show now that a solution $(S\cdot), N(\cdot)))$ with $N_0 \in [N_0^\dagger, N_0^\flat)$ and control u_{NS} such that $N(T) > \eta_C S^\star$ is optimal. It will be convenient to consider the function

$$\beta(t) := k_1k_2N(t) - k_4C_N^{in}S(t).$$

From equations (1)-(2), one obtains for any admissible solution in E

$$\dot{\beta}(t) = k_1(C_2 + k_2C_1)\varphi(t) + (1-\varphi(t))k_2k_4C_N^{in}K_R(S(t)), \quad t \in [0,T]. \tag{33}$$

Assume there exists an optimal solution $(\tilde{S}(\cdot), \tilde{N}(\cdot))$ in E with an admissible control $\tilde{u}(\cdot))$ such that

$$\int_0^T \tilde{u}(t)dt < \int_0^T u(t)dt.$$

From equation (2), one gets $\tilde{N}(T) < N(T)$ and as $\tilde{S}(T) \geq S^\star = S(T)$, one should have

$$\tilde{\beta}(T) < \beta(T) \tag{34}$$

where $\tilde{\beta}$ denotes the function $\tilde{\beta}(t) := k_1 k_2 \tilde{N}(t) - k_4 C_N^{in} \tilde{S}(t)$. From the integration of (33) between 0 and T, one gets

$$\int_0^T (1 - \varphi(t)) K_R(\tilde{S}(t)) dt < \int_0^T (1 - \varphi(t)) K_R(S(t)) dt.$$

However, one has $\tilde{S} \geq S$ on the intervals $[0, t_N]$ and $[t^\star, T]$, and K_R is increasing. We deduce that $\tilde{S} - S$ has to be negative on some non-empty interval of $[t_N, t^\star]$. Therefore, by continuity, there exists $t_e \in [t_N, t^\star]$ such that $\tilde{S}(t_e) = S(t_e)$ and $\tilde{S}(t) > S(t)$ for $t \in (t_e, t^\star)$. If $\tilde{S}(t^\star) = S^\star$, then one can reproduce the argumentation of Proposition 2 to show that one should have $\tilde{S}(t) = S^\star$, $t \in [t^\star, T]$ for $\tilde{S}(\cdot)$ to be optimal, but then one has $\tilde{N}(T) = N(T) + (\tilde{N}(t^\star) - N(t^\star)) \geq N(T)$, as $N(t^\star) = \eta_C S^\star$ is the smallest value of N in E. This contradicts $\tilde{N}(T) < N(T)$. Therefore, one has necessarily $t_e < t^\star$ and can write from the integration of (33) between t_e and $t^\star$

$$\tilde{\beta}(t^\star) - \beta(t^\star) = \tilde{\beta}(t_e) - \beta(t_e) + k_2 k_4 C_N^{in} \int_{t_e}^{t^\star} (1 - \varphi(t))(K_R(\tilde{S}(t)) - K_R(S(t))) dt > \tilde{\beta}(t_e) - \beta(t_e)$$

where

$$\tilde{\beta}(t_e) = k_1 k_2 \tilde{N}(t_e) - k_4 C_N^{in} \tilde{S}(t_e) \geq k_1 k_2 \eta_C \tilde{S}(t_e) - k_4 C_N^{in} \tilde{S}(t_e) = k_1 k_2 \eta_C S(t_e) - k_4 C_N^{in} S(t_e) = \beta(t_e)$$

from which one deduces the inequality $\tilde{\beta}(t^\star) > \beta(t^\star)$. From the integration of (33) between $t^\star$ and T, one gets

$$\tilde{\beta}(T) - \beta(T) = \tilde{\beta}(t^\star) - \beta(t^\star) + \int_{t^\star}^T (1 - \varphi(t))(K_R(\tilde{S}(t)) - K_R(S^\star) dt \geq \tilde{\beta}(t^\star) - \beta(t^\star) > 0$$

which contradicts (34). Finally, the expression of the optimal cost comes from a straightforward calculation with the integration of (2) between 0 and T and (33) between $t^\star$ and T. □

Remark 2. *When $N_0^\dagger = \eta_C$ and initial condition is $N_0 = \eta_C$, one has $S = N/\eta_C > S^\star$ at initial time. The control u_N^{sing} defined in* (30) *is equal to $u_N^{sing}(0, 1) = -k_1 \eta_C / C_2 < 0$. The optimal control is thus $u = 0$ at initial time, which takes the trajectory out of the boundary $N - \eta_C S = 0$ until it later joins this edge again. This is why there is* $\max(0, \cdot)$ *in the expression* (29) *for $S = N/\eta_C > S^\star$.*

4.4 The singular case

Note from Propositions 2, 3 and 4, that when condition (15) is not fulfilled and $N_0^\sharp < N_0^\dagger$, the case $N_0 \in (N_0^\sharp, N_0^\dagger)$ is not covered by these Propositions. This is the matter of this section. We call this case "singular" because, as we shall see, the optimal control does not take extreme values nor saturates the constraints, alike singular controls in the theory of optimal control with linear dependency in control, differently to the cases covered by Propositions 2, 3, 4.

Proposition 5. *Assume that conditions* (23)-(24) *are satisfied with $S_S^\star = S^\star$. If $N_0^\sharp < N_0^\dagger$, where $N_0^\sharp$, $N_0^\dagger$ are defined in* (27), (31), *then for any $N_0 \in (N_0^\sharp, N_0^\dagger)$, the control*

$$u(t) = \lambda u^\sharp(t) + (1 - \lambda) u^\dagger(t), \quad \lambda = \frac{N_0^\dagger - N_0}{N_0^\dagger - N_0^\sharp}, \quad t \in [0, T] \tag{35}$$

is optimal, where $u^\sharp(\cdot)$, $u^\dagger(\cdot)$ are the optimal open loop controls given by Propositions 3, 4 for the initial condition $N_0 = N_0^\sharp, N_0^\dagger$ respectively. The optimal cost is

$$V^\star = \frac{\lambda t_c^\star}{k_2} + \frac{1 - \lambda}{k_4 C_N^{in}} \left(\eta_C S^\star - N_0 + k_3 \int_0^T \varphi(t) dt \right). \tag{36}$$

Proof. Let $(S^\sharp(\cdot), N^\sharp(\cdot))$, $(S^\dagger(\cdot), N^\dagger(\cdot))$ be the optimal solutions given by Propositions 3, 4 for the initial condition $N_0 = N_0^\sharp, N_0^\dagger$ respectively. From equations (1), (2) one has

$$\begin{aligned}
\lambda\dot{S}^\sharp + (1-\lambda)\dot{S}^\dagger &= -k_1\varphi(t) - k_1\big(\lambda K_R(S^\sharp) + (1-\lambda)K_R(S^\dagger)\big) + k_1k_2\big(\lambda u^\sharp(t) + (1-\lambda)u^\dagger(t)\big),\\
\lambda\dot{N}^\sharp + (1-\lambda)\dot{N}^\dagger &= -k_3\varphi(t) + k_4C_N^{in}\big(\lambda u^\sharp(t) + (1-\lambda)u^\dagger(t)\big)
\end{aligned}$$

Note that one has $\lambda K_R(S^\sharp) + (1-\lambda)K_R(S^\dagger) = K_R(\lambda S^\sharp + (1-\lambda)S^\dagger))$ as K_R is an affine function in the domain E. Therefore

$$S(t) = \lambda S^\sharp(t) + (1-\lambda)S^\dagger(t),\ N(t) = \lambda N^\sharp(t) + (1-\lambda)N^\dagger(t), \quad t \in [0,T]$$

is the solution of (1), (2) for the control $u(\cdot)$ and the initial condition $(1, N_0)$. As the solutions $(S^\sharp(\cdot), N^\sharp(\cdot))$, $(S^\dagger(\cdot), N^\dagger(\cdot))$ remain in the convex set E, we deduce that the solution $(S(\cdot), N(\cdot))$ does also, and the control $u(\cdot)$ is admissible. From $N^\sharp(T) = N^\dagger(T) = \eta_C S^\star$ (with $t^\star = T$), we get $N(T) = \eta_c S^\star$ and according to Remark 1, we conclude that $u(\cdot)$ is optimal. The expression of the optimal cost is obtained straightforwardly. □

Remark 3. *There is no uniqueness of the optimal control in the case of Proposition 5. Let $S(\cdot)$, $N(\cdot)$ be the solution with the control $u(\cdot)$ defined on (35) for an initial condition $N_0 \in (N_0^\sharp, N_0^\dagger)$. Take for instance an absolutely function $\delta(\cdot)$ such that*

$$\int_0^T (1-\varphi(t))\delta(t) = 0.$$

with $\delta(t) = 0$ for $t \in [0,\tau] \cup [T-\tau, T]$ for some positive $\tau < T/2$, and for any t with $u(t) = 0$ Then, the control

$$\tilde{u}(t) = u(t) + \varepsilon\left(\frac{1-\varphi(t)}{k_2(1-S_h)}\delta(t) + \frac{\dot{\delta}(t)}{k_1k_2}\right), \quad a.e.\ t \in [0,T]$$

takes values in $[0, u_{max}]$ for a.e. $t \in [0,T]$, provided the number $\varepsilon > 0$ to be small enough. The solution $\tilde{S}(\cdot)$, $\tilde{N}(\cdot)$ with control $\tilde{u}$ satisfies

$$\begin{aligned}
\dot{\tilde{S}} &= -k_1\varphi(t) - k_1(1-\varphi(t))K_R(\tilde{S}) + k_1k_2\tilde{u}(t)\\
&= -k_1\varphi(t) - k_1(1-\varphi(t))K_R(\tilde{S} - \varepsilon\delta(t)) + k_1k_2u(t) + \varepsilon\dot{\delta}(t)
\end{aligned}$$

from which we deduce that it satisfies $\tilde{S}(t) = S(t) + \varepsilon\delta(t)$ for any $t \in [0,T]$. One has also

$$\tilde{N}(t) - \eta_C\tilde{S}(t) = N(t) - \eta_C S(t) + \varepsilon\left(\frac{k_4C_N^{in}}{k_2(1-S_h)}\int_0^t (1-\varphi(s))\delta(s)ds + \left(\frac{k_4C_N^{in}}{k_1k_2} - \eta_C\right)\delta(t)\right)$$

For $t \in [0,\tau] \cup [T-\tau, T]$, one has $\tilde{S}(t) = S(t)$, $\tilde{N}(t) - \eta_C\tilde{S}(t) = N(t) - \eta_C S(t)$, and as the inequalities $S^\star < S(t) < 1$, $N(T) - \eta_C S(t) > 0$ are satisfied for any $t \in (0,T)$, we deduce that for ε small enough, one has also $S^\star < \tilde{S}(t) < 1$, $\tilde{N}(T) - \eta_C\tilde{S}(t) > 0$ for $t \in (0,T)$. Thus, one has $(\tilde{S}(t), \tilde{N}(t)) \in E$ for any $t \in [0,T]$, and as $\tilde{N}(T) = N(T) = \eta_c S^\star$ we conclude that the control $\tilde{u}(\cdot)$ is also optimal.

5 Numerical illustrations and discussion

We have considered a class of concave functions φ, given by the following expression

$$\varphi(t) = \frac{t(1+\alpha)}{t+\alpha}, \quad \alpha > 0$$

parameterized by α, which in some sense measures how quickly the vegetation cover progresses over time, impacting the LAI (Leaf Aarea Index). We have chosen a plausible set of values of model parameters (see Table 1), inspired by the literature.

For these parameters, we have computed the lower bounds (5), (6) on the operating parameters C_N^{in}, u_{max} for the set E to be viable (see Table 2). The values in Table 1 of these parameters being above these bounds, we deduce that the set E is viable fo these operating parameters.

Then, we have determined by solving numerically the differential equation (1) the various quantities considered in Propositions 2, 3, 4, that are reported them in Table 3.

k_1	k_2	k_3	k_4	$S^\star$	S_w	S_h	T	α	η_C	C_N^{in}	u_{max}
1	1	1.4	1	0.7	0.4	0.2	1	0.07	0.5	1.45	2

Table 1: Parameters values of the model

C_N^{in}	u_{max}
1.4	1

Table 2: Lower bounds on the operating parameters for the viability of the set E

t_S	C_S	$N_0^\flat$	$\underline{t}_c$	$N_0^\sharp$	$N_0^\dagger$
0.313413	-0.748012	0.641166	0.699568	0.547681	0.579387

Table 3: Values of the various quantities considered in Propositions 2, 3 and 4

From Table 3, one has $C_S < 0$ and from Proposition 2 we get that the S-strategy is optimal for $N_0 \geq N_0^\flat$. As $t_S < T$, we have $S_S^\star = S^\star$ and from Proposition 4 we know that the NS-strategy is optimal for $N_0 \in [N_0^\dagger, N_0^\flat]$. In Table 3, we see also that one has $N_0^\sharp > \eta_C$, and then the N-strategy is optimal for $N_0 \in [\eta_C, N_0^\sharp]$ according to Proposition 3. Finally, as $N_0^\dagger > N_0^\sharp$, the singular control defined in Proposition 5 is optimal for $N_0 \in (N_0^\sharp, N_0^\dagger)$. For this set of parameters, all the four possible strategies are met, as summarized in Table 4.

N_0	$[\eta_C, N_0^\sharp]$	$(N_0^\sharp, N_0^\dagger)$	$[N_0^\dagger, N_0^\flat]$	$[N_0^\flat, \rightarrow)$
optimal control	N-strategy	singular	NS-strategy	S-strategy
	act on stress		*anticipate stress*	

Table 4: Optimality of the various strategies depending on N_0

For different values of N_0, we have determined numerically the optimal solutions with the help of the Bocop software [6], which is based on a direct method. On can see on Figures 2 to 9 that the solutions are in perfect accordance with the theoretical results, as predicted by Table 4. For $N_0 \in (N_0^\sharp, N_0^\dagger)$, we have plotted on Figure 5 the optimal solution as the convex combination of trajectories and control of the particular cases given in Figures 3 and 6. On can compare on Figure 4 with the solution given by Bocop. As underlined in Remark 3, the optimal solution is not unique in this case. The numerical software has provided a kind of regularization of the solution given by Proposition 5, which ends also at the corner point $(S^\star, \eta_C S^\star)$ at time T and is thus optimal (cf Remark 1).

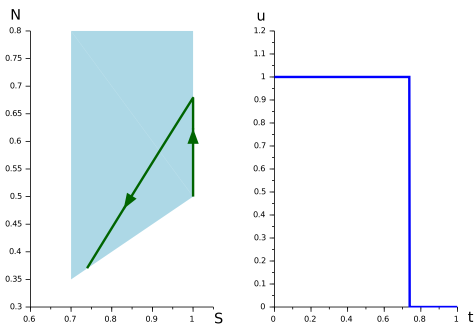

Figure 2: Optimal solution for $N_0 = \eta_C$

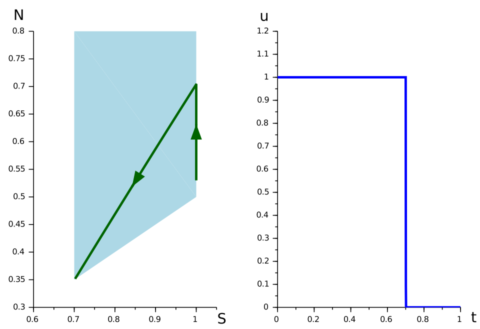

Figure 3: Optimal solution for $N_0 = N_0^\sharp$

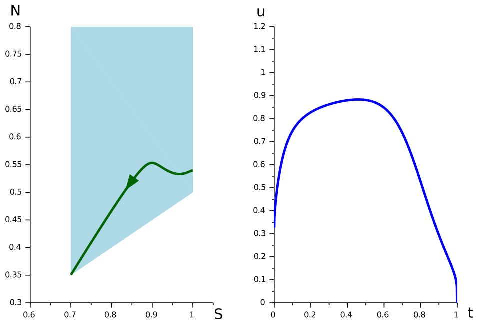

Figure 4: Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$

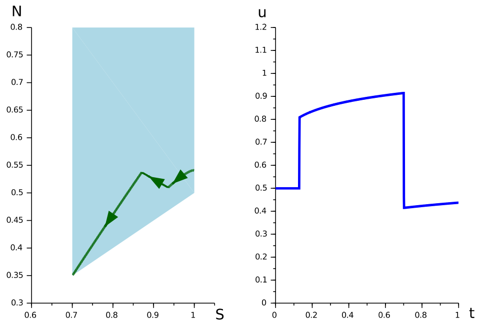

Figure 5: Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$ with the control (35)

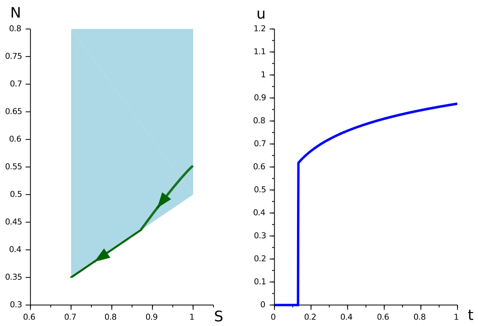

Figure 6: Optimal solution for $N_0 = N_0^\dagger$

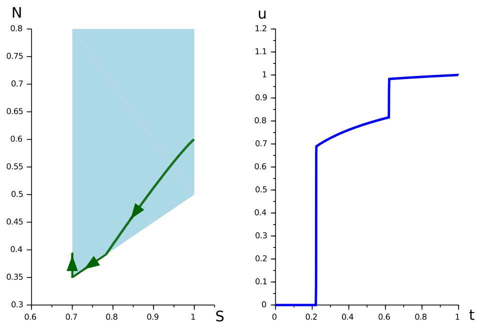

Figure 7: Optimal solution for $N_0 \in (N_0^\dagger, N_0^\flat)$

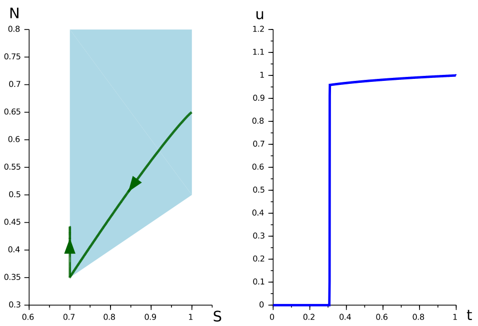

Figure 8: Optimal solution for $N_0 = N_0^\flat$

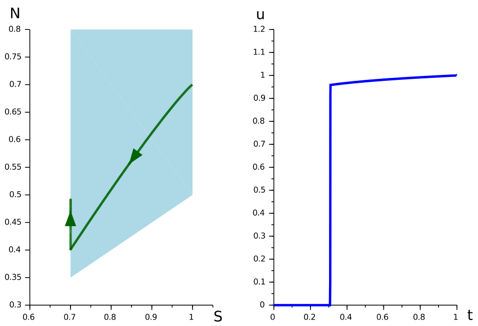

Figure 9: Optimal solution for $N_0 > N_0^\flat$

Let us comment about these optimal solutions.

- When the initial quantity of nitrogen is high ($N_0 > N_0^\flat$), it is not surprising that the nitrogen constraint is not saturated. The S-strategy coincides with the optimal one already found in the literature for the simplified model with no nitrogen stress. It consists in no irrigation up to the time the humidity threshold $S^\star$ is met and then keeping the humidity level exactly equal to this threshold up to the final time. An sensor measuring the humidity level S is required for the application of this control strategy.

- For lower value of initial nitrogen $N_0 \in (N_0^\dagger, N_0^\flat)$, the lack of irrigation conducts the system to face the nitrogen stress before the hydric one. Then, the optimal solution consists in keeping the system at the edge of this constraint until it reaches the humidity threshold and then keeping the humidity level exactly equal to this threshold (as for the S-strategy). In this way, the NS-strategy generalizes the S-strategy in a context of nitrogen stress. Here, the measurement of the nitrogen content N (or its concentration N/S) is required, in addition to the humidity level S, as online information.

- A surprising feature occurs when keeping the system at the edge of the nitrogen stress does not allow to reach the humidity threshold (because the time horizon as been reached before), for small initial content of nitrogen $N_0 < N_0^\sharp$. Then the NS-strategy is no longer optimal and the best one is fundamentally different. The N-strategy consists in irrigating since the beginning to maintain the humidity level at his maximal level up to a precise time $t_c^\star$, from which stopping the irrigation conducts the system to touch the N-stress exactly at the terminal time. Differently to the S-strategy, the N-strategy is not a particular instance of the NS-strategy. On the opposite, the N-strategy requires an anticipation of the future needs providing water since the beginning. From a practical viewpoint, this "open-loop" strategy requires the precise determination of the optimal commutation time $t_c^\star$ from the data of the model an initial N_0, without the need of future on-line measurement.

- Another non-intuitive feature is the possibility for a subset of initial conditions $N_0 \in (N_0^\sharp, N_0^\dagger)$ to reach the "corner" point $(S^\star, \eta_C S^\star)$, defined as the intersection of the two constraints, exactly at the final time. This requires particular irrigation strategies which no longer consist in keeping the system at boundaries of the set E or absence of irrigation. The optimal control has to irrigate all the time with particular profiles (that are non-unique), avoiding the boundaries of E, differently to the other strategies. This strategy is the most demanding to implement because it requires the prior determination of the whole time profile depending on the initial N_0. Let us stress that for these cases the final nitrogen content is the smallest among all possible ones.

On Figure 10, we have plotted the optimal value of the water consumption V and the corresponding final nitrogen $N(t)$, as function of the initial nitrogen content. This shows that although the optimal

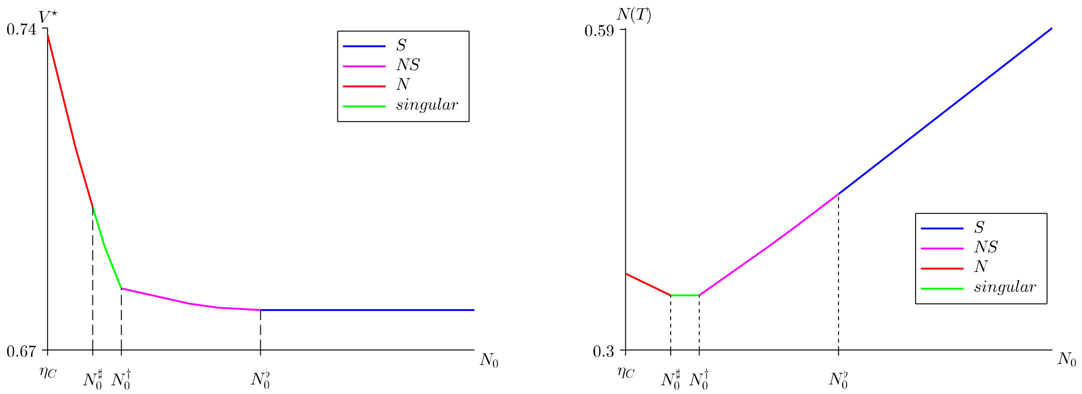

Figure 10: Minimal water consumption (left) and final nitrogen content (right) as a function of N_0 (the colors indicate which strategy is optimal)

irrigation strategies can be quite different for $N_0 < N_0^\sharp$ or $N_0 > N_0^\dagger$, the minimal water consumption $V^\star$ is always decreasing with initial N_0, down to the constant level for which the system receives enough nitrogen to never face a nitrogen stress, that is when $N_0 \geq N_0^\flat$. Note that the part of the curve for $N_0 \in (N_0^\sharp, N_0^\dagger)$ is linear, justified by the expression (36) given in Proposition 5. Due to the particular structure of the singular cases with $N_0 \in (N_0^\sharp, N_0^\dagger)$ for which the final nitrogen content $N(T)$ is always the minimal value $\eta_C S^\star$ of N in E, the map $N_0 \mapsto N(T)$ is non monotonic and flat for $(N_0^\sharp, N_0^\dagger)$. If one intends to minimize both the water consumption and the residual nitrogen content (while ensuring the maximal biomass production), a Pareto diagram might be useful. On Figure 11, one can see that the Pareto front is for $N_0 \in [N_0^\dagger, N_0^\flat]$, for which a compromise between water consumption and final nitrogen content has to be chosen. Note that this locus corresponds to the optimality of the NS-strategy.

The conditions to obtain the various cases depicted above depend on several parameters of the model. We have studied numerically the impact of two of them: the function φ related to the LAI coefficient, and the input concentration C_N^{in}.

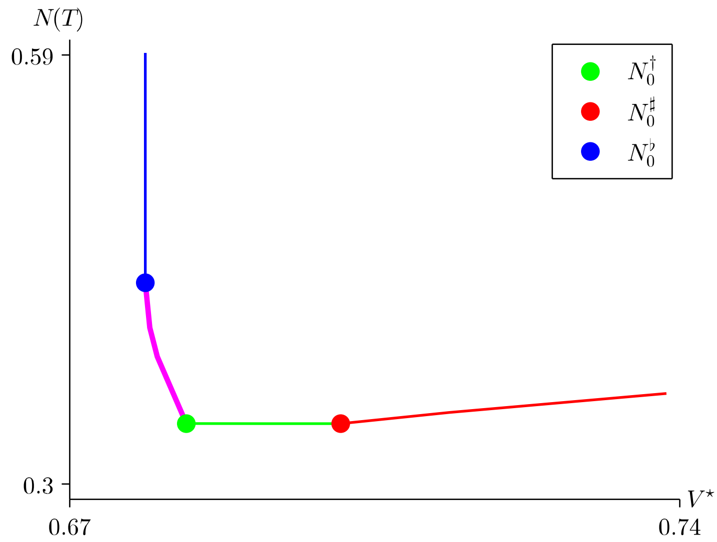

Figure 11: Set of optimums $(N(T), V^\star)$. The Pareto front is depicted in magenta

5.1 Increasing parameter α

To grasp the impact of the shape of the function φ, we have increased α to $\alpha = 0.17$ in Table 1 to reduce the concavity of φ (see Figure 15). This gives the Table 5 for which one has $N_0^\sharp < \eta_C$.

t_S	C_S	$N_0^\flat$	$\underline{t}_c$	$N_0^\sharp$	$N_0^\dagger$
0.322878	-0.616823	0.582464	0.699052	0.437236	η_C

Table 5: Values of the various quantities considered in Propositions 2, 3 and 4 for $\alpha = 0.4$

Consequently, the N-strategy is never optimal and there is no singular case. As underlined in Remark 2, for $N_0 = \eta_C$, the optimal trajectory does not consist in staying first on the boundary $N - \eta_C S = 0$. With the control $u = 0$, it leaves this edge until it later reaches it again, as depicted on Figure 12. The other kinds of optimal trajectories are illustrated on Figures 13 and 14.

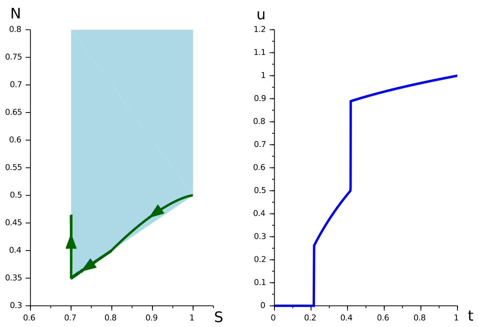

Figure 12: Optimal solution for $N_0 = \eta_C$

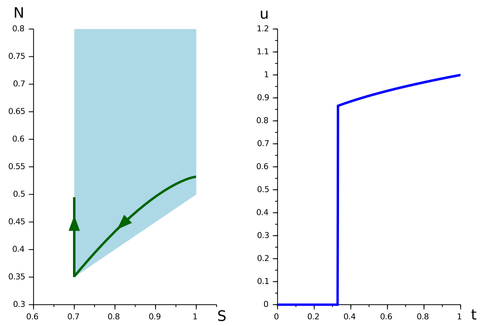

Figure 13: Optimal solution for $N_0 = N_0^\flat$

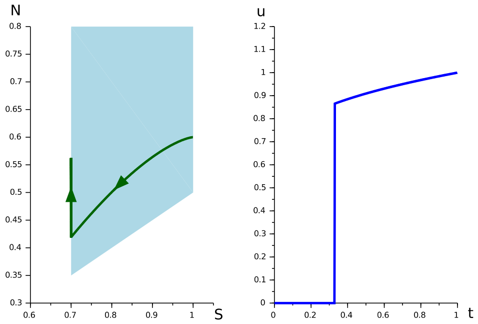

Figure 14: Optimal solution for $N_0 > N_0^\flat$

If one increases again the parameter α with $\alpha = 0.8$ for the function φ to become close from a linear one, one obtains the figures given in Table 6. This time, one has $N_0^\flat = \eta_C$ and from Proposition 2 we

t_S	C_S	$N_0^\flat$
0.340431	-0.339000	η_C

Table 6: Values of the various quantities considered in Proposition 2 for $\alpha = 0.8$

conclude that the S-strategy is always optimal i.e. the nitrogen stress is never met.

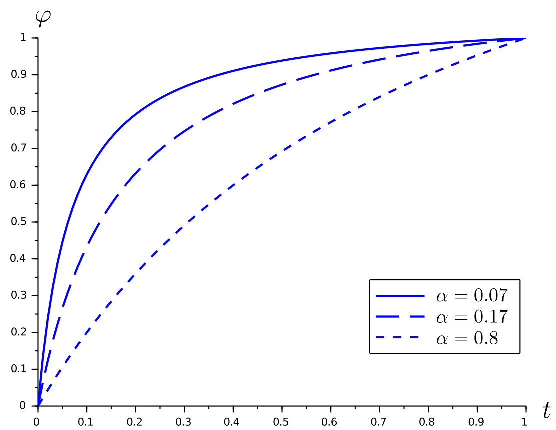

Figure 15: Increasing parameter α

5.2 Increasing parameter C_N^{in}

Note that when changing in Table 1 the parameter C_N^{in} only, all the quantities related to the solutions with the control $u = 0$ (t_S, C_S, $N_0^\flat$, $\underline{t}_c$) do not change. The values of $N_0^\sharp$ and $N_0^\dagger$ are changing only.

t_S	C_S	$N_0^\flat$	$\underline{t}_c$	$N_0^\sharp$	$N_0^\dagger$
0.313413	-0.748012	0.641166	0.699568	0.302832	η_C

Table 7: Values of the various quantities considered in Propositions 2, 3 and 4 for $C_N^{in} = 1.8$

Table 7 gives these new numbers when $C_N^{in} = 1.5$. One can observe the inequality $N_0^\sharp < \eta_C$, which implies that the N-strategy nor the singular one can be optimal. However, one can see on Figure 16 that the optimal trajectory from $N_0 = \eta_C$ seems to remain on the edge $N - \eta_C S = 0$ since the initial time, differently to Figure 12 for instance, which is in contradiction with Remark 2. Indeed, the initial duration for which $u = 0$ is optimal is very small so that the optimal trajectory almost immediately returns to the edge $N - \eta_C S = 0$ (this is due to the relatively small value of α). The other configurations are depicted on Figures 17, 18, 19.

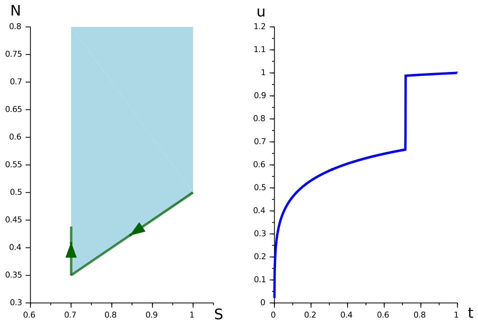

Figure 16: Optimal solution for $N_0 = \eta_C$

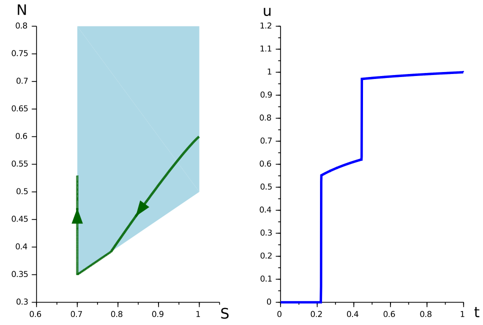

Figure 17: Optimal solution for $N_0 \in (\eta_c,)N_0^\flat)$

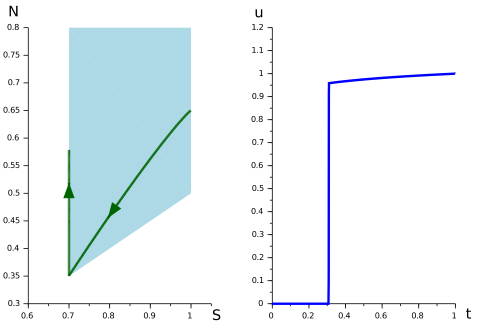

Figure 18: Optimal solution for $N_0 = N_0^\flat$

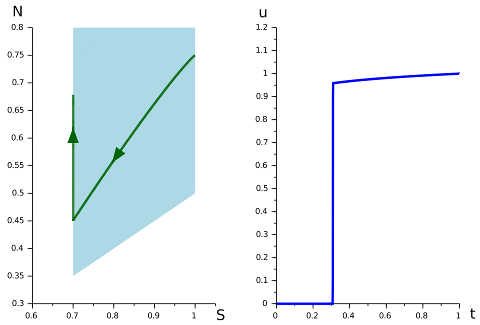

Figure 19: Optimal solution for $N_0 > N_0^\flat$

If one increases again C_N^{in}, the picture does not change because the value of $N_0^\flat$ is not impacted while $N_0^\dagger$ remains equal to η_C. The NS-strategy is thus always optimal for $N_0 < N_0^\flat$.

6 Conclusion

This study reveals that when the crop radiation efficiency increases rapidly with time and the nitrogen concentration in the irrigation water is low, then the minimal residual nitrogen content in soil is obtained when the initial nitrogen content belongs to a particular interval of values, and not for the the smallest initial nitrogen content. Therefore, having a low initial content of nitrogen in soil could lead paradoxically to larger residual contents. This is indeed explained by the larger water consumption needed to maintain the maximal production. This non intuitive feature is due to the interplay between nitrogen dilution and water supply in fertirrigation.

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