```
> restart;
> interface(showassumed=0) :
      # remove the ~ after the variables when you make assumptions about them
> local D : # use D as a variable
> assume(Rup > Rum, Rup > 0, lambda > 0, Z1 > 0, Z2 > 0, mu > 0, D > 0) :
      # assumptions on the variables
```

Define the equations find their solution

```
> eq1 := D·diff(psi1(x), x, x) + mu·(psi2(x) − psi1(x)) = (k − Rup)·psi1(x); # on (0, 0.5 L)
```

$$eq1 := \mathrm{D}\left(\frac{d^2}{dx^2}\,\psi 1(x)\right) + \mu\left(\psi 2(x) - \psi 1(x)\right) = (k - Rup)\,\psi 1(x) \quad (1)$$

```
> eq2 := D·diff(psi2(x), x, x) + mu·(psi1(x) − psi2(x)) = (k − Rum)·psi2(x); # on (0, 0.5 L)
```

$$eq2 := \mathrm{D}\left(\frac{d^2}{dx^2}\,\psi 2(x)\right) + \mu\left(\psi 1(x) - \psi 2(x)\right) = (k - Rum)\,\psi 2(x) \quad (2)$$

```
> SOL1 := dsolve([eq1, eq2, psi1(0) = 1], {psi1(x), psi2(x)}) :
      # finds the general solution of eq1, eq2
```

> SOL1b := eval(SOL1, {

$$-\frac{1}{2}\,\frac{\sqrt{-2\,\mathrm{D}\left(\sqrt{4\,\mu^2 + Rum^2 - 2\,Rum\,Rup + Rup^2} + Rum + Rup - 2\,k - 2\,\mu\right)}\;x}{\mathrm{D}} = -Z1\cdot x,\quad \frac{1}{2}\,\frac{\sqrt{-2\,\mathrm{D}\left(\sqrt{4\,\mu^2 + Rum^2 - 2\,Rum\,Rup + Rup^2} + Rum + Rup - 2\,k - 2\,\mu\right)}\;x}{\mathrm{D}} = Z1\cdot x,$$

$$-\frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{\mathrm{D}\left(-Rum - Rup + 2\,k + 2\,\mu + \sqrt{4\,\mu^2 + Rum^2 - 2\,Rum\,Rup + Rup^2}\right)}\;x}{\mathrm{D}} = -Z2\cdot x,\quad \frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{\mathrm{D}\left(-Rum - Rup + 2\,k + 2\,\mu + \sqrt{4\,\mu^2 + Rum^2 - 2\,Rum\,Rup + Rup^2}\right)}\;x}{\mathrm{D}} = Z2\cdot x$$

}); # simplifies the expression, using the definition of Z1 and Z2

$$SOL1b := \Big\{\psi 1(x) = (1 - \_C2 - \_C3 - \_C4)\,e^{-Z1\,x} + \_C2\,e^{Z1\,x} + \_C3\,e^{-Z2\,x} + \_C4\,e^{Z2\,x},\ \psi 2(x) \quad (3)$$

$$= -\frac{1}{2}\,\frac{1}{\mu}\Big(\_C4\,e^{Z2\,x}\sqrt{Rum^2 - 2\,Rum\,Rup + Rup^2 + 4\,\mu^2} - \_C4\,e^{Z2\,x}\,Rum$$

$$+ \_C4\,e^{Z2\,x}\,Rup - (1 - \_C2 - \_C3 - \_C4)\,e^{-Z1\,x}\sqrt{Rum^2 - 2\,Rum\,Rup + Rup^2 + 4\,\mu^2}$$

$$- (1 - \_C2 - \_C3 - \_C4)\,e^{-Z1\,x}\,Rum + (1 - \_C2 - \_C3 - \_C4)\,e^{-Z1\,x}\,Rup$$

$$- \_C2\,e^{Z1\,x}\sqrt{Rum^2 - 2\,Rum\,Rup + Rup^2 + 4\,\mu^2} - \_C2\,e^{Z1\,x}\,Rum + \_C2\,e^{Z1\,x}\,Rup$$

$$+\_C3\, e^{-Z2\,x}\sqrt{Rum^2-2\,Rum\,Rup+Rup^2+4\,\mu^2}-\_C3\, e^{-Z2\,x}\,Rum+\_C3\, e^{-Z2\,x}\,Rup\Big)\Big\}$$

> eq3 := D·diff(psi3(x), x, x) + mu·(psi4(x) − psi3(x)) = (k − Rum)·psi3(x); # on (0.5 L,L)

$$eq3 := D\left(\frac{d^2}{dx^2}\,\psi 3(x)\right)+\mu\,(\psi 4(x)-\psi 3(x)) = (k-Rum)\,\psi 3(x) \quad \textbf{(4)}$$

> eq4 := D·diff(psi4(x), x, x) + mu·(psi3(x) − psi4(x)) = (k − Rup)·psi4(x); # on (0.5 L,L)

$$eq4 := D\left(\frac{d^2}{dx^2}\,\psi 4(x)\right)+\mu\,(\psi 3(x)-\psi 4(x)) = (k-Rup)\,\psi 4(x) \quad \textbf{(5)}$$

> SOL2 := dsolve([eq3, eq4], {psi3(x), psi4(x)}) : # finds the general solution of eq3, eq4

> SOL2b := eval(SOL2, {

$$-\frac{1}{2}\,\frac{\sqrt{-2\,D\left(\sqrt{4\,\mu^2+Rum^2-2\,Rum\,Rup+Rup^2}+Rum+Rup-2\,k-2\,\mu\right)}\;x}{D}=-Z1\cdot x,\ \frac{1}{2}\,\frac{\sqrt{-2\,D\left(\sqrt{4\,\mu^2+Rum^2-2\,Rum\,Rup+Rup^2}+Rum+Rup-2\,k-2\,\mu\right)}\;x}{D}=Z1\cdot x,$$

$$-\frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{D\left(-Rum-Rup+2\,k+2\,\mu+\sqrt{4\,\mu^2+Rum^2-2\,Rum\,Rup+Rup^2}\right)}\;x}{D}=-Z2\cdot x,\ \frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{D\left(-Rum-Rup+2\,k+2\,\mu+\sqrt{4\,\mu^2+Rum^2-2\,Rum\,Rup+Rup^2}\right)}\;x}{D}=Z2\cdot x\Big\}\Big)$$

;# simplifies the expression, using the definition of Z1 and Z2

$$SOL2b := \Big\{\psi 3(x)=\_C1\,e^{-Z1\,x}+\_C2\,e^{Z1\,x}+\_C3\,e^{-Z2\,x}+\_C4\,e^{Z2\,x},\ \psi 4(x) = \quad \textbf{(6)}$$

$$-\frac{1}{2}\,\frac{1}{\mu}\Big(\_C4\,e^{Z2\,x}\sqrt{Rum^2-2\,Rum\,Rup+Rup^2+4\,\mu^2}+\_C4\,e^{Z2\,x}\,Rum-\_C4\,e^{Z2\,x}\,Rup$$
$$-\_C1\,e^{-Z1\,x}\sqrt{Rum^2-2\,Rum\,Rup+Rup^2+4\,\mu^2}+\_C1\,e^{-Z1\,x}\,Rum-\_C1\,e^{-Z1\,x}\,Rup$$
$$-\_C2\,e^{Z1\,x}\sqrt{Rum^2-2\,Rum\,Rup+Rup^2+4\,\mu^2}+\_C2\,e^{Z1\,x}\,Rum-\_C2\,e^{Z1\,x}\,Rup$$
$$+\_C3\,e^{-Z2\,x}\sqrt{Rum^2-2\,Rum\,Rup+Rup^2+4\,\mu^2}+\_C3\,e^{-Z2\,x}\,Rum-\_C3\,e^{-Z2\,x}\,Rup\Big)\Big\}$$

> assign(SOL1b) : assign(SOL2b) : # define the functions \psi_i with the above solutions

> P1 := convert(psi1(x), polynom); P2 := convert(psi2(x), polynom) : P3 := convert(psi3(x), polynom) : P4 := convert(psi4(x), polynom) :

$$P1 := (1-\_C2-\_C3-\_C4)\,e^{-Z1\,x}+\_C2\,e^{Z1\,x}+\_C3\,e^{-Z2\,x}+\_C4\,e^{Z2\,x} \quad \textbf{(7)}$$

> P1 := eval(P1, {_C1 = A1, _C2 = A2, _C3 = A3, _C4 = A4}) : P2 := eval(P2, {_C1 = A1, _C2 = A2, _C3 = A3, _C4 = A4}) : P3 := eval(P3, {_C1 = A5, _C2 = A6, _C3 = A7, _C4 = A8}) : P4 := eval(P4, {_C1 = A5, _C2 = A6, _C3 = A7, _C4 = A8}) :
> # replace the constants _Ci by Aj, to take into account that the _Ci are different depending on

the system solved

Define the regularity and periodicity constraints

> $eqc1 := eval\left(P1, x=\frac{L}{2}\right) = eval\left(P3, x=\frac{L}{2}\right)$:# *continuity at* $\frac{L}{2}$:

> $eqc2 := eval\left(diff(P1, x), x=\frac{L}{2}\right) = eval\left(diff(P3, x), x=\frac{L}{2}\right)$: # *derivability at* $\frac{L}{2}$:

> $eqc3 := eval\left(P2, x=\frac{L}{2}\right) = eval\left(P4, x=\frac{L}{2}\right)$: # *continuity at* $\frac{L}{2}$:

> $eqc4 := eval\left(diff(P2, x), x=\frac{L}{2}\right) = eval\left(diff(P4, x), x=\frac{L}{2}\right)$: # *derivability at* $\frac{L}{2}$:

> $eqc5 := eval(P3, x=\mathrm{L}) = \exp(\mathrm{lambda}\cdot L)\cdot eval(P1, x=0)$: # *periodicity at L:*

> $eqc6 := eval(P4, x=\mathrm{L}) = \exp(\mathrm{lambda}\cdot L)\cdot eval(P2, x=0)$: # *periodicity at L:*

> $eqc7 := eval(diff(P3, x), x=L) = eval(\exp(\mathrm{lambda}\cdot L)\cdot diff(P1, x), x=0)$: # *periodicity at L:*

> $eqc8 := eval(diff(P4, x), x=L) = eval(\exp(\mathrm{lambda}\cdot L)\cdot diff(P2, x), x=0)$: # *periodicity at L:*

> *with*(*LinearAlgebra*) : *with*(*linalg*) :

> *eqns* := [*eqc1*, *eqc2*, *eqc3*, *eqc4*, *eqc6*, *eqc7*, *eqc8*] :
> # *Linear system of equations that we solve. We keep eqc5 to compute k*

> (*AA*, *FM*) := *GenerateMatrix*(*eqns*, [*A2*, *A3*, *A4*, *A5*, *A6*, *A7*, *A8*]) : *AI* := *inverse*(*AA*) : *CC* := *multiply*(*AI*, *FM*) : # *solve the linear system*

> *A2* := *CC*[1] : *A3* := *CC*[2] : *A4* := *CC*[3] : *A5* := *CC*[4] : *A6* := *CC*[5] : *A7* := *CC*[6] : *A8* := *CC*[7] : # *define the Aj as the solutions of the system*

At this stage, we have all of the coefficients, but condition eqc5 is not satisfied

> *F1a* := *simplify*(*eval*(*P3*, *x*=L)) : *F1b* := *simplify*(exp(lambda·*L*)·*eval*(*P1*, *x*=0)) : *nF1* := *numer*(*simplify*(*F1a* − *F1b*)) :# *the condition eqc5 can be written like nF1=0*

> *C* := *collect*(*simplify*(*nF1*), exp) : *indets*(*C*); # *finds the exponential terms*

$$\Big\{L, Rum, Rup, Z1, Z2, \lambda, \mu, e^{L(Z1+2\lambda)}, e^{L(2Z1+Z2)}, e^{L(2Z1+\lambda)}, e^{L(2Z1+3\lambda)}, e^{L(3Z1+2\lambda)}, e^{L(Z1+Z2+\lambda)}, e^{L(Z1+Z2+2\lambda)}, e^{L(Z1+Z2+3\lambda)}, e^{L(Z1+2Z2+2\lambda)}, e^{L(2Z1+Z2+2\lambda)}, e^{L(2Z1+Z2+4\lambda)}, e^{L(2Z1+2Z2+\lambda)}, e^{L(2Z1+2Z2+3\lambda)}, e^{L(3Z1+Z2+\lambda)}, e^{L(3Z1+Z2+2\lambda)}, e^{L(3Z1+Z2+3\lambda)}, e^{L(3Z1+2Z2+2\lambda)}, e^{2L(Z1+\lambda)}, e^{2L(Z1+Z2+\lambda)}, e^{\frac{3}{2}L(Z1+Z2+2\lambda)}, e^{\frac{1}{2}L(3Z1+Z2+2\lambda)}, e^{\frac{1}{2}L(3Z1+Z2+4\lambda)}, e^{\frac{1}{2}L(3Z1+3Z2+2\lambda)}, e^{\frac{1}{2}L(3Z1+3Z2+4\lambda)}, e^{\frac{1}{2}L(5Z1+Z2+2\lambda)}, e^{\frac{1}{2}L(5Z1+Z2+4\lambda)}, e^{\frac{1}{2}L(5Z1+3Z2+2\lambda)}, e^{\frac{1}{2}L(5Z1+3Z2+4\lambda)}, e^{\frac{1}{2}L(6\lambda+3Z1+Z2)}, e^{\frac{1}{2}L(6\lambda+5Z1+Z2)}, e^{\frac{1}{2}L(6\lambda+5Z1+3Z2)}\Big\} \quad \textbf{(8)}$$

> # *We manually extract the terms with the highest exponents, using Z2>Z1. These terms are* $-e^{(2Z1+Z2+4\lambda)L}, +e^{(2Z1+2Z2+3\lambda)L}, -e^{(3Z1+2Z2+2\lambda)L}$. *Thus* lambda = *Z1*

> $eqk := eval\left(\text{lambda} = Z1, Z1 = \frac{1}{2}\,\frac{\sqrt{-2\,D\left(\sqrt{4\,\mu^2 + Rum^2 - 2\,Rum\,Rup + Rup^2} + Rum + Rup - 2\,k - 2\,\mu\right)}}{D}\right):$

 # define the equation on k, using lambda=Z1

> $qks := \frac{solve(eqk, k)}{\text{lambda}}$: *# find k and define qks the quotient between k and lambda*

> $\text{lambda2m} := solve(diff(qks, \text{lambda}) = 0, \text{lambda}):$

 # find the value of lambda where the minimum is reached

Warning, solve may be ignoring assumptions on the input variables.

> $c2m := simplify(eval(qks, \text{lambda} = \text{lambda2m}[1]))$ *# this is the limit speed c^{2m} as L converges* **to** infinity

$$c2m := \sqrt{\sqrt{Rum^2 - 2\,Rum\,Rup + Rup^2 + 4\,\mu^2} + Rum + Rup - 2\,\mu}\,\sqrt{D}\,\sqrt{2} \qquad \textbf{(9)}$$