About optimal control problem under action duration constraint and infimum-gap

Dan Goreac, Alain Rapaport

About optimal control problem under action duration constraint and infimum-gap

Dan Goreac[1,2] and Alain Rapaport[3]

[1] School of Mathematics and Statistics, Shandong University, Weihai, China
[2] LAMA, Univ. Eiffel, UPEM, Univ. Paris Est Creteil, CNRS, Marne-la-Vallée, France
[3] MISTEA, Univ. Montpellier, INRAE, Institut Agro, Montpellier, France
E-mails: dan.goreac@u-pem.fr, alain.rapaport@inrae.fr

December 15, 2023

Abstract

We consider optimal control problems with scalar control in $[0, 1]$ under the constraint that the length of time during which the control is non-null is bounded by a prescribed value. We show that such problems present an infimum-gap when the solution of the relaxed problem is not bang-bang. Then, we show that the gap can be closed if one considers the infimum over all survival probability functions dominated by the survival probability function of the uniform law as a constraint on the control function.

1 Introduction

The Covid-19 pandemic has led to a surge in activity among researchers in applied mathematics about epidemiological modeling, and in particular for optimization and decision support issues, which are found to be of general interest in epidemiology beyond the Sars diseases.

Thus, the problems of minimizing the epidemic peak or maximizing the final size of the susceptible sub-population under duration constraints on the interventions have recently been tackled in the literature [13, 12, 2, 3] for the well-known SIR model [9]. Optimal solutions have been mathematically demonstrated for this model. The contributions of these theoretical analyses versus purely numerical solutions are to provide explicit structures of the optimal solution in terms of feedback strategies. Let us stress that these criteria are not standard, i.e. not in the usual Mayer, Lagrange or Bolza form, or over infinite horizon with moreover unconventional constraints on the control variable. This deserves special interests because one cannot apply straightforwardly usual tools such as the Maximum Principle of Pontryagin to prove the structure of the optimal solution. In the three contributions [13, 12, 2], the controlled SIR model that is considered is as follows

$$\begin{aligned}\dot{S} &= -\beta(1-u)SI\\ \dot{I} &= \beta(1-u)SI - \gamma I\\ \dot{R} &= \gamma I\end{aligned}$$

where the control $u \in [0, u_{max}]$ with $u_{max} \leq 1$ represents actions or interventions (such as lock-downs and curfew) which reduce contacts between infected and susceptible individuals. In [12] the problem consists in minimizing the L^∞ norm of the variable I (the so-called "epidemic peak")

$$\inf_{u(\cdot)} \max_{t\geq 0} I(t), \tag{1}$$

under a L_1 budget constraint on the action

$$\int_0^{+\infty} u(t)dt \leq Q. \tag{2}$$

Alternatively, the authors in [11, 1, 5] have considered the "dual" problem which consists in minimizing the L^1 norm of the control

$$\inf_{u(\cdot)} \int_0^{+\infty} u(t)dt,$$

under the state constraints

$$I(t) \leq \bar{I}, \quad t \in [0, +\infty).$$

The optimal strategies of these two problems turn out to be identical (bang-singular-bang), as this has indeed been proved to be true in a more general framework [7]. In [13], the same criterion (1) has been considered but for the class of controls $u(\cdot)$ that verify a duration constraint

$$u(t) = 0,\ t \notin [t_i, t_i + \tau], \quad u(t) > 0,\ t \in [t_i, t_i + \tau], \tag{3}$$

where τ is fixed and t_i has to be chosen. The structure of the optimal solution (band-singular-bang-bang) is different.

In [2, 3] , the problem of maximizing the final size

$$\sup_{u(\cdot)} \lim_{t \to +\infty} S(t) \tag{4}$$

has been investigated for the same class of controls (3), and the optimal solutions has been proved to be bang-bang i.e. $u(t) = u_{max}$, for $t \in [t_i, t_i + \tau]$, where t_i has to be optimized. In [4] the authors have considered the same criterion (4) but for the class of controls in L_1 with a budget constraint (2), and have shown that the same bang-bang control on a single time interval of interventions was optimal, when $u_{max}\tau = Q$. Here, this means that the optimal solution for the L_1 constraint is not only optimal for the class of controls (3), but also for the more general class of controls such that

$$\int_0^{+\infty} I_{\mathbb{R}_\star^+}(u(t)dt \leq \tau, \tag{5}$$

where $I_{\mathbb{R}_\star^+}$ denotes the indicator function of positive numbers, that is measurable controls for which the occupation measure of $\mathbb{R}_\star^+$ is bounded by τ.

More generally, when the optimal solution under a L^1 constraint of the control is not bang-bang, as for problem (1) for instance, the problems with duration constraints are no longer equivalent. Motivated by these observations, the objective of the present note is to investigate problems for scalar controls under the "action duration constraint" (5). The use of occupation measures in control theory has already been considered to deal with state or mixed constraints to reformulate nonlinear optimal control problems as infinite dimensional linear programming problems on spaces of occupational measures generated by control-state trajectories [10, 6]. Here, we consider the occupation measure to define the constraint on the control function, which has not been yet considered in this way up to our knowledge.

The following is organized as follows. In the next section, we propose an equivalent reformulation of optimal control problems with action duration constraint ad show that a infimum-gap occurs when bang-bang controls are not optimal. Then, in Section we generalize the constraint (5) for different measures on the control and show under which conditions an infimum-gap is avoided.

2 Reformulation with extended velocity set and relaxation

We consider a control system in $\mathbb{R}^n$

$$\begin{cases} \dot{x} = f(x) + g(x)u, \\ x(0) = x_0 \end{cases} \qquad u \in U := [0,1], \tag{6}$$

where f, g are C^1 maps with linear growth, and the set of control functions

$$\mathcal{U} := \{u : [0,T] \to U \text{ Borel mesurable}\}.$$

Define the Mayer problem for $\Phi \in C^1$

$$(\mathcal{P}_1): \quad \inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)),$$

under a constraint on the action duration, that is for

$$\mathcal{U}_\tau := \{u(\cdot) \in \mathcal{U} : \text{ meas}\, E(u) \leq \tau\} \text{ where } E(u) := \{t \in [0,T];\ u(t) > 0\}.$$

As recalled in the introduction, such a constraint is not classical in optimal control theory.

Alternatively, we consider the extended dynamics

$$\begin{cases} \dot{x} = f(x) + g(x)u,\ x(0) = x_0, \\ \dot{z} = v,\ z(0) = 0, \end{cases} \tag{7}$$

where

$$w := (u,v) \in W := \{(u,v) \in [0,1]^2,\ uv = 0\},$$

and the (more classical) optimal control problem with a target condition

$$(\mathcal{P}_2): \quad \inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau,$$

where $\mathcal{W} := \{w : [0,T] \to W \text{ Borel mesurable}\}$.

Lemma 2.1. *Problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are equivalent.*

Proof. Take $\epsilon > 0$ and let $u_\epsilon(\cdot) \in \mathcal{U}_T$ be such that the corresponding solution $x_\epsilon(\cdot)$ satisfies

$$\Phi(x_\epsilon(T)) < \inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)) + \epsilon.$$

Let

$$v_\epsilon(t) = \begin{cases} 0 & t \in E(u_\epsilon), \\ 1 & t \notin E(u_\epsilon). \end{cases}$$

Clearly (u_ϵ, v_ϵ) belongs to $\mathcal{W}$ and the corresponding solution $z_\epsilon(\cdot)$ verifies

$$z_\epsilon(T) = \int_0^T v_\epsilon(t)dt = T - E(u_\epsilon) \geq T - \tau.$$

Then one gets

$$\Phi(x_\epsilon(T)) \geq \inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau,$$

and, thus,

$$\inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)) > \left(\inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau\right) - \epsilon.$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)) \geq \inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau. \tag{8}$$

Conversely, let $(u_\epsilon, v_\epsilon) \in \mathcal{W}$ such that the corresponding solution $(x_\epsilon(\cdot), z_\epsilon(\cdot)$ verifies $z_\epsilon(T) \geq T - \tau$ and

$$\Phi(x_\epsilon(T)) < \left(\inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau \right) + \epsilon.$$

One has

$$T - \tau \leq z_\epsilon(T) = \int_0^T v_\epsilon(t)dt = \int_{t\notin E(u_\epsilon)} v_\epsilon(t)dt \leq T - \text{meas}\, E(u_\epsilon),$$

and, thus, $u_\epsilon(\cdot) \in \mathcal{U}_\tau$. This allows to write

$$\Phi(x_\epsilon(T)) \geq \inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)),$$

from which we get

$$\left(\inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau \right) > \inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)) - \epsilon.$$

Letting ϵ be arbitrary small, one obtains

$$\left(\inf_{w(\cdot)\in\mathcal{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau \right) \geq \inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)). \tag{9}$$

Inequalities (8) and (9) give the equivalence of problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. □

Note that the control set W is not convex and thus existence of optimal solution is not guaranteed. However, one has

$$\overline{\text{co}}\, W = \{(u, v) \in [0, 1]^2,\ u + v \leq 1\},$$

(where $\overline{co}$ denotes the closed convex hull, see Figure 1) and one can consider the "convexified" or relaxed

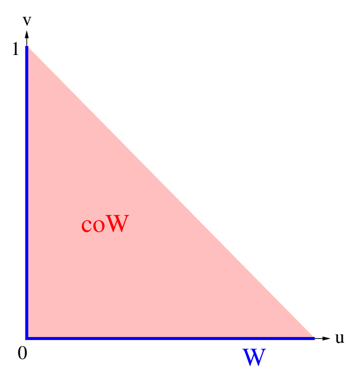

Figure 1: The set W and its convexification $\text{co}\, W$

problem

$$(\overline{\mathcal{P}_2}) : \quad \inf_{w(\cdot)\in\overline{\mathcal{W}}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau,$$

where $\overline{\mathcal{W}} := \left\{ w : [0, T] \to \overline{\text{co}}\, W \text{ mesurable} \right\}$. Let us also consider the problem with L^1 constraint on the control, that is

$$(\mathcal{P}_3) : \quad \inf_{u(\cdot)\in\mathcal{U}^1_\tau} \Phi(x(T)),$$

where

$$\mathcal{U}_\tau^1 := \{u(\cdot) \in \mathcal{U} \text{ s.t. } ||u||_1 \leq \tau\}, \text{ where } ||u||_1 := \int_0^T u(t)dt.$$

Note that problems $(\overline{\mathcal{P}_2})$ and $(\mathcal{P}_3)$ fulfill the usual convexity assumption that guarantees the existence of optimal solutions.

Lemma 2.2. *Problems* $(\overline{\mathcal{P}_2})$ *and* $(\mathcal{P}_3)$ *are equivalent.*

Proof. Take $(u(\cdot), v(\cdot)$ in $\overline{\mathcal{W}}$ such that the corresponding solution satisfies $z(T) \geq T - \tau$. Then, one can write

$$\int_0^T u(t)dt \leq \int_0^T 1 - v(t)dt = T - z(T) \leq \tau,$$

that is $u(\cdot)$ belongs to $\mathcal{U}_\tau^1$ and one has

$$\inf_{u(\cdot)\in\mathcal{U}_\tau^1} \Phi(x(T)) \leq \inf_{w(\cdot)\in\overline{\mathcal{W}}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau. \tag{10}$$

Conversely, let $u(\cdot)$ belong to $\mathcal{U}_\tau^1$ and posit $v(t) = 1 - u(t)$ for any $t \in [0, T]$. Clearly, $(u(\cdot), v(\cdot)$ belongs to $\overline{\mathcal{W}}$ and one has

$$z(T) = \int_0^T v(t)dt = \int_0^T 1 - u(t)dt = T - \int_0^T u(t)dt \geq T - \tau.$$

Therefore, one has

$$\inf_{u(\cdot)\in\mathcal{U}_\tau^1} \Phi(x(T)) \geq \inf_{w(\cdot)\in\overline{\mathcal{W}}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau. \tag{11}$$

Finally, inequalities (10), (11) show the equivalence of problems $(\overline{\mathcal{P}_2})$ and $(\mathcal{P}_3)$.

□

From Lemmas 2.1 and 2.2, we immediately deduce the following property.

Proposition 2.1. *Problem* $(\mathcal{P}_3)$ *is equivalent to problem* $(\mathcal{P}_1)$ *with relaxed controls.*

Remark. *If problem* $(\mathcal{P}_3)$ *admits an optimal solution with a control* $u(\cdot)$ *that takes values* 0 *or* 1 *only, then it is optimal for problem* $(\mathcal{P}_1)$*, as* $||u||_1 =$ *meas* $E(u)$ *in this case.*

Finally, we obtain the following result about existence of infimum-gap in the original problem.

Proposition 2.2. *If any optimal solution of problem* $(\mathcal{P}_3)$ *saturates the* L^1 *constraint and possesses a singular arc, then there is an infimum-gap between between problem* $(\mathcal{P}_1)$ *and problem* $(\mathcal{P}_3)$*, that is*

$$\inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)) > \min_{u(\cdot)\in\mathcal{U}_\tau^1} \Phi(x(T)).$$

Proof. Assume by contradiction that

$$\inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)) = J^\star := \min_{u(\cdot)\in\mathcal{U}_\tau^1} \Phi(x(T)).$$

Then, for any $n \in \mathbb{N}$, there exists a control $u_n(\cdot) \in \mathcal{U}_\tau$ such that

$$\Phi(x_n(T)) < J^\star + \frac{1}{n},$$

where $x_n(\cdot)$ denotes the solution of (6) associated to $u_n(\cdot)$. Thanks to the convexity of the velocity set of the dynamics (6) and the usual regularity assumptions, the theorem of compactness of solutions of (6) applies and there exists a sub-sequence, also denoted x_n, such that $x_n(\cdot)$ converges pointwise to a certain

$x^\star(\cdot)$ solution of (6) a certain $u^\star \in \mathcal{U}$, and $\dot{x}_n(\cdot)$ converges weakly to $\dot{x}^\star(\cdot)$, that is $u_n(\cdot)$ converges weakly to $u^\star(\cdot)$. Moreover, we obtain passing at the limit $\Phi(x^\star(T)) = J^\star$.

On another hand, $\mathbb{P}(u) := \frac{1}{T}\text{meas } E(u)$ is the occupation probability for the open set $(0, +\infty)$ for any measurable function $u(\cdot)$. By the Portmanteau Theorem (see for instance [8]), we get

$$\liminf_n \mathbb{P}(u_n) \geq \mathbb{P}(u^\star).$$

and as meas $E(u_n) \leq \tau$ for any n, we deduce that

$$\tau \geq \text{meas } E(u^\star) \geq ||u^\star||_1.$$

Therefore $u^\star$ belongs to $\mathcal{U}_\tau^1$ with $\Phi(x^\star(T)) = J^\star$. The control $u^\star$ is thus optimal for problem $(\mathcal{P}_1)$ with meas $E(u^\star) = \tau$. Then, having $||u^\star||_1 = \text{meas } E(u^\star)$ implies that $u^\star(t)$ is equal to 0 or 1 for a.e. $t \in [0, T]$, which contradicts the presence of a singular arc. □

3 Generalization of the action duration constraint

The action duration constraint defined by the set $\mathcal{U}_\tau$ in Section 2 can be seen as a particular measure or "energy" imposed on the control functions $u(\cdot)$, where energy and time spend by an action are the same. If one draws analogy with mechanics, this means that any action count the same one unit of energy. From a Lagrangian view point, this amounts to measure the energy to go from one point to another simply by the euclidean distance. One may consider other ways to define energy of actions $u(\cdot)$ with a non euclidean geometry. For pure bang-bang controls, there is no need to distinguish how counts each possible action but this is no longer the case with singular arcs, as revealed by Proposition 2.2. For such arcs, the energy needs to be computed taking into account the geometry.

This intuition roughly corresponds to considering some measure on the control set $[0, 1]$, or, alternatively, some affine transformation of a cumulative distribution function (c.d.f.) $u \mapsto \gamma(u)$ and compute the action energy (instead of the simple duration) $\int_0^T \gamma(u(t))dt$, or, even more generally $\int_0^T \gamma(t, u(t))dF(t)$ with F being a c.d.f. of a $[0, T]$-supported random variable.

3.1 A control point of view for measuring

Having a look at the constraint defining the set W in Section 2, i.e. $uv = 0$ and the ensuing formulation for $\overline{co}\ W$, one may consider a similar approach for any constraint $\phi(u, v) = 0$, provided that the kernel of ϕ is included in $\overline{co}\ W$ and $\phi(0, v) = \phi(u, 0) = 0$ for any $(u, v) \in [0, 1]^2$. In particular, constraints that are expressed as $v \in \Gamma(u)$ possess these properties, provided that the set-valued map Γ verifies

$$0 \in \Gamma(u) \subset [0, 1 - u], \text{ for all } u \in [0, 1] \text{ and } \Gamma(0) = [0, 1]\,. \tag{12}$$

For ensuring the existence of optimal solutions, we shall require the set-valued map Γ to take convex compact values and to be upper hemicontinuous, which basically amounts to asking $\Gamma(u) = [0, \theta(u)]$, where θ is a $[0, 1]$-valued upper semicontinuous function such that $\theta(0) = 1$ and $\theta(u) \leq 1 - u$, for every $u \in [0, 1]$. Indeed, we are going to offer a slight generalization by considering a function

$$h : [0, 1] \mapsto [0, 1] \text{ convex continuous s.t. } h(u) = 0 \Leftrightarrow u = 0. \tag{13}$$

Having fixed such a function h, we then define the set of functions

$$\begin{aligned}\Theta(h) := \Big\{ &\theta : [0, 1] \longrightarrow [0, 1]\,;\ \theta \text{ is } u.s.c., \\ &\theta(u) \leq 1 - h(u),\ \forall u \in [0, 1] \text{ and } \theta(0) = 1 \Big\}.\end{aligned} \tag{14}$$

Let $\theta \in \Theta(h)$ and consider the control set

$$W_\theta := \left\{(u,v) \in [0,1]^2 : \ v \le \theta(u)\right\}.$$

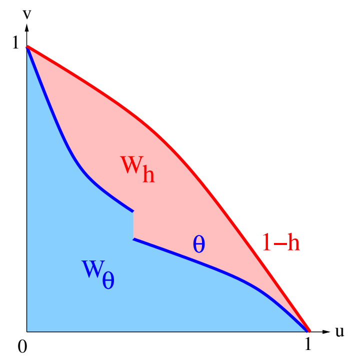

Figure 2: Example of sets W_θ and $\overline{W}_h$

For F being the cummulative distribution function (c.d.f.) of a $[0,T]$-supported random variable, we define the following optimal control problem

$$(\mathcal{P}_{1,F,\theta}) \ : \inf_{u \in \mathcal{U}_{F,\tau}(\theta)} \Phi(x(T)),$$

for the dynamics (6), where

$$\mathcal{U}_{F,\tau}(\theta) := \left\{ u(\cdot) \in \mathcal{U} : \ \mathbb{E}_F\left[1 - \theta(u(\cdot))\right] := \int_0^T \left(1 - \theta(u(t))\right) dF(t) \le \frac{\tau}{T} \right\}.$$

As in Section 2, we consider for the extended dynamics

$$\left\{ \begin{array}{l} \dot{x} = f(x) + g(x)u, \ x(0) = x_0, \\ dz_F = v \ dF, \ z_F(0) = 0, \end{array} \right. \tag{15}$$

with control $w = (u,v) \in W_\theta$ the optimization problem

$$(\mathcal{P}_{2,F,\theta}) : \quad \inf_{w(\cdot) \in \mathcal{W}_\theta} \phi(x(T)) \text{ s.t. } z_F(T) \ge 1 - \frac{\tau}{T},$$

for the family of Borel measurable functions $\mathcal{W}_\theta := \mathbb{L}^0\left([0,T]\,; W_\theta\right)$.

Remark. *The initial problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ in Section 2 are obtained for F corresponding to the uniformly distributed r.v. on $[0,T]$ (i.e. $F(u) = \frac{u}{T}$, on $[0,T]$) and the indicator function $\theta = \mathbf{1}_0$.*

Lemma 3.1. *Problems $(\mathcal{P}_{1,F,\theta})$ and $(\mathcal{P}_{2,F,\theta})$ are equivalent.*

Proof. The proof is quite similar to the one provided in Section 2. For convenience, let us denote $V_{j,F,\theta}(x_0)$ the value functions for the problems $\mathcal{P}_{j,F,\theta}$, with $j \in \{1,2\}$ and $x_0 \in \mathbb{R}^n$.
For $\varepsilon > 0$ and u_ε being an ε-optimal control for the problem $(\mathcal{P}_{1,F,\theta})$, we set $v_\varepsilon := \theta\left(u_\varepsilon\right)$. It is clear that the associated solution $z_F^{v_\varepsilon}$ satisfies

$$z_F^{v_\varepsilon}(T) = \int_0^T v_\varepsilon(t) dF(t) = 1 - \int_0^T \left(1 - \theta(u_\varepsilon(t)\right) dF(t) \ge 1 - \frac{\tau}{T},$$

which, by invoking the arbitrariness of $\varepsilon > 0$, leads to

$$V_{1,F,\theta}(x_0) \geq V_{2,F,\theta}(x_0).$$

For the converse, again with $\varepsilon > 0$ and $w_\varepsilon = (u_\varepsilon, v_\varepsilon)$ being an ε-optimal control for problem $\mathcal{P}_{2,F,\theta}$, i.e., $\Phi\left(x_\varepsilon(T)\right) \leq V_{2,F,\theta} + \varepsilon$, one has

$$1 - \frac{\tau}{T} \leq z_F^{v_\varepsilon}(T) = \int_0^T v_\varepsilon(t) dF(t) \leq \int_0^T \theta(u_\varepsilon(t)) dF(t),$$

leading to u_ε being admissible for problem $\mathcal{P}_{1,F,\theta}$, i.e., $u_\varepsilon \in \mathcal{U}_{F,\tau}(\theta)$.As a consequence, one has

$$V_{2,F,\theta}(x_0) + \varepsilon \geq \Phi\left(x_\varepsilon(T)\right) \geq V_{1,F,\theta}(x_0).$$

Again, we invoke the arbitrariness of $\varepsilon > 0$ to complete the proof of our assertions. □

A simple glance at the argument developed at the beginning shows that, having fixed h and $\theta \in \Theta(h)$, one has

$$\overline{co}\, W_\theta \subset \overline{W}_h := \left\{ (u,v) \in [0,1]^2 \ : \ h(u) + v \leq 1 \right\} \tag{16}$$

(see Figure 2 as an illustration). Note that we have equality in the particular case of the identity function $h = Id$.

We now consider the relaxed control problems that are defined independently of $\theta \in \Theta(h)$ (but may still depend on h)

$$\left(\overline{\mathcal{P}}_{2,F,h}\right): \quad \inf_{w(\cdot) \in \overline{\mathcal{W}}_h} \phi(x(T)) \text{ s.t. } z_F(T) \geq 1 - \frac{\tau}{T},$$

where $\overline{\mathcal{W}}_h := \mathbb{L}^0\left([0,T]; \overline{W}_h\right)$, and

$$\left(\mathcal{P}_{3,F,h}\right): \quad \inf_{u \in \mathcal{U}^1_{F,\tau}(h)} \Phi(x(T)),$$

where

$$\mathcal{U}^1_{F,\tau}(h) := \left\{ u(\cdot) \in \mathcal{U} \text{ s.t. } \|h(u(\cdot))\|_{\mathbb{L}^1([0,T], dF; \mathbb{R}_+)} := \int_0^T h(u(t)) dF(t) \leq \frac{\tau}{T} \right\}.$$

With an argument identical (up to replacing in the inequalities u by $h(u)$ and dt by dF and recalling that the total mass of dF is 1) to the one employed in Lemma 2, one establishes the following equivalence result.

Lemma 3.2. *The problems* $\left(\mathcal{P}_{3,F,h}\right)$ *and* $\left(\overline{\mathcal{P}}_{2,F,h}\right)$ *are equivalent.*

We emphasize that this equivalence is provided for every cummulative distribution function F associated to some $[0,T]$-supported real-valued random variables.

3.2 A non infimum-gap result

We are now ready to state and prove the following result stating a no-gap property between problems $\left(\mathcal{P}_{3,F,h}\right)$ and a minimization of $\left(\mathcal{P}_{1,F,\theta}\right)$ over $\theta \in \Theta(h)$.

Proposition 3.1. *Let h be a function satisfying the property* (13) *and F be a cumulative distribution function (c.d.f.) of a real-valued random variable taking its values in $[0,T]$. Then, the following equality holds true.*

$$\inf_{\theta \in \Theta(h)} \inf_{u(\cdot) \in \mathcal{U}_{F,\tau}(\theta)} \Phi(x(T)) = \inf_{u(\cdot) \in \mathcal{U}^1_{F,\tau}(h)} \Phi(x(T)).$$

Proof. By Lemmas 3.1 and 3.2, it follows that, for every $\theta \in \Theta$, the optimal value of problem $(\mathcal{P}_{1,F,\theta})$ is no lower than the one of problem $(\mathcal{P}_{3,F,h})$. By taking the infimum over $\theta \in \Theta(h)$, it follows

$$\inf_{\theta \in \Theta(h)} \inf_{u(\cdot) \in \mathcal{U}_{F,\tau}(\theta)} \Phi(x(T)) \geq \min_{u(\cdot) \in \mathcal{U}^1_{F,\tau}(h)} \Phi(x(T)).$$

Let $u_\varepsilon \in \mathcal{U}^1_{F,\tau}(h)$ be an optimal control for $(\mathcal{P}_{3,F,h})$. By definition, one has

$$\int_0^T h\left(u_\varepsilon(t)\right) dF(t) \leq \frac{\tau}{T}.$$

Then, by simply taking $\theta^*(u) := 1 - h(u)$, we complete the proof since $\theta^* \in \Theta(h)$ and

$$\mathbb{E}_F\left[1 - \theta^*(u_\varepsilon(\cdot))\right] = \int_0^T h\left(u_\varepsilon(t)\right) dF(t),$$

thus showing that $u_\varepsilon \in \mathcal{U}_{F,\tau}(\theta^*)$. □

Remark. *For $F(u) = \min\left(\max\left(0, \frac{u}{T}\right), 1\right)$ (i.e., F corresponding to the uniform law on $[0,T]$), and $h(u) = u$), let us we drop for the dependence on F and h and simply write $(\mathcal{P}_{1,\theta})$, $(\mathcal{P}_{2,\theta})$, $\left(\overline{\mathcal{P}}_2\right)$, $(\mathcal{P}_3)$ for simplicity. The result states that the control problem $(\mathcal{P}_3)$ of minimizing the final cost under the integral constraint on the control is equivalent to minimizing $(\mathcal{P}_{1,\theta})$ over all survival probability functions θ of r.v. on $[0,1]$ and dominated by the survival probability function of the uniform law. This closes the gap and refers to the mechanical heuristic depicted at the beginning of this section: it is not only the action/non action that need to be taken into account (as for the action duration constraint of Section 2) but one has to minimize over all "energy" curves with $\theta(0) = 1$ and $\theta(1) = 0$.*

4 Conclusion

In the present work, we show the benefit of extending the dynamics with an additional control to reformulate the optimal control problem with action duration constraint. This allows us to show that the relaxed problem is indeed the problem with the simple constraint on the L_1 norm on the control. An infimum-gap appears then when the optimal control under the L_1 constraint presents a singular arc. We generalize this approach to "energy" constraints and show that there is no infimum-gap when taking the infimum over the family of optimal control problems with control subject to a probability measure dominated by the one defining the energy constraint.

References

[1] F. Avram, L. Freddi, and D. Goreac. Optimal control of a SIR epidemic with ICU constraints and target objectives. *Applied Mathematics and Computation*, 418:126816, 2022.

[2] P.-A. Bliman and M. Duprez. How best can finite-time social distancing reduce epidemic final size? *Journal of Theoretical Biology*, 511:110557, 2021.

[3] P.-A. Bliman, M. Duprez, Y. Privat, and N. Vauchelet. Optimal immunity control and final size minimization by social distancing for the SIR epidemic model. *Journal of Optimization Theory and Applications*, 189(2):408–436, 2021.

[4] P.-A. Bliman and A. Rapaport. On the problem of minimizing the epidemic final size for SIR model by social distancing. In *Proceedings of the 22nd IFAC World Congress*, Yokohama, Japan, July 2023.

[5] L. Freddi, D. Goreac, J. Li, and B. Xu. SIR Epidemics with State-Dependent Costs and ICU Constraints: A Hamilton–Jacobi Verification Argument and Dual LP Algorithms. *Applied Mathematics & Optimization*, 86(2):23, Jul 2022.

[6] V. Gaitsgory and M. Quincampoix. On sets of occupational measures generated by a deterministic control system on an infinite time horizon. *Nonlinear Analysis: Theory, Methods & Applications*, 88:27–41, 2013.

[7] D. Goreac and A. Rapaport. L^{∞}/L^{1} duality results in optimal control problems. Preprint arXiv:2305.02585, under revision for *IEEE Trans. on Aut. Cont.*, 2023.

[8] O. Kallenberg. *Foundations of Modern Probability.* Springer, series Probability and Its Applications (2nd Edition), 2001.

[9] W. O. Kermack and A. G. McKendrick. Contributions to the mathematical theory of epidemics—I. *Proceedings of the Royal Society*, 115A:700–721, 1927.

[10] J. B. Lasserre, D. Henrion, C. Prieur, and E. Trélat. Nonlinear optimal control via occupation measures and lmi-relaxations. *SIAM Journal on Control and Optimization*, 47(4):1643–1666, 2008.

[11] L. Miclo, D. Spiro, and J. Weibull. Optimal epidemic suppression under an ICU constraint: An analytical solution. *Journal of Mathematical Economics*, page 102669, 2022.

[12] E. Molina and A. Rapaport. An optimal feedback control that minimizes the epidemic peak in the SIR model under a budget constraint. *Automatica*, 146(110596), 2022.

[13] D. Morris, F. Rossine, J. Plotkin, and S. Levin. Optimal, near-optimal, and robust epidemic control. *Communications Physics*, 4(1):1–8, 2021.