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Abstract

We give conditions for a non-monotone system to preserve the usual vector order of solutions for a subset of initial conditions. Our approach consists of separating terms that meet Kamke's sign conditions from other ones in the dynamics, and considering Picard iterations. These conditions amount for the dynamics to preserve a partial order, which is not necessarily induced by a cone. Examples illustrate the results.

Key-words. ordinary differential equations, solutions comparison, monotone systems, partial order.

1 Introduction

Monotone dynamical systems have received a great attention in the literature (see for instance the monograph [9], the review [6] and the references herein). Let us recall that the semi-flows of monotone systems preserve a vector order, and that their asymptotic behaviors present some strong properties (see [4, 5]). In particular, systems $\dot{x} = f(t, x)$ in $\mathbb{R}^n$ that are cooperative preserve the partial order relatively to the positive orthant in $\mathbb{R}^n_+$:

$$y_0 \geq x_0 \Rightarrow y(t) \geq x(t),\ t \geq t_0 \tag{1}$$

(where $y(\cdot)$, $x(\cdot)$ are solutions of the initial value problems $y(t_0) = y_0$, $x(t_0) = x_0$ and $\geq$ is considered component-wise). The Kamke's condition

$$\frac{\partial f_i}{\partial x_j}(t, x) \geq 0,\ i \neq j \tag{2}$$

characterizes such systems from the single knowledge of the Jacobian matrix of f. This condition can been extended to partial orders relatively to the other orthants of $\mathbb{R}^n$, that are $\{x \in \mathbb{R}^n;\ (-1)^{m_i} x_i \geq 0;\ i = 1 \cdots n\}$ where $m_i \in \{0, 1\}$

$$(-1)^{m_i + m_j} \frac{\partial f_i}{\partial x_j}(t, x) \geq 0,\ i \neq j$$

(see [9]), or for even more general positive cones P of $\mathbb{R}^n$

$$\lambda \left(\frac{\partial f}{\partial x}(t, x).y \right) \geq 0,\ y \in \partial P,\ \lambda \in \Lambda(P) \tag{3}$$

where $\Lambda(P)$ is the set of supporting linear forms of P (see [10]). More recently, the cooperativity property with respect to cones has been characterized for non-smooth dynamics [1].

The preservation of vector order for solutions of dynamical systems has important implications in several applications. In particular, this property is at the core of the interval observers techniques (see for instance [2, 8]). However, in some practical problems, one may observe an order preservation of trajectories (relatively to the positive orthant $\mathbb{R}^n_+$) for some subsets of initial conditions, while the system

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is not cooperative. For instance, the anaerobic digestion model studied in [7] exhibits the monotone property (1) of solutions for some realistic operating conditions and initial conditions when augmenting the initial density of organic matter, while the dynamics is nowhere cooperative in the domain. This property has a practical impact on the performance of the biogaz production of the system, when it is increasing with respect to the initial organic matter (we refer the reader to [7] for further details). The aim of the present work is to characterize theoretically such situations for a class of systems. For this purpose, we shall consider a decomposition of the map f by isolating terms into a partial map h that prevent the Kamke's condition to be fulfilled, and give conditions on the maps f and h for the ordering property (1) to hold for a subset of initial conditions. Note that in practice it is not always easy to find a cone P that verify condition (3) (if it exists). This is also a motivation of our work to propose a methodology that could facilitate this search.

For simplicity of the presentation, we shall consider autonomous dynamics only, but extension of the results to non-autonomous ones does not present any particular difficulty and is left to the reader. The paper is organized as follows. In Section 2, we define a partial order in the positive quadrant induced by the decomposition of the dynamics, under some hypotheses, and give preliminaries results that will be used in the following. Section 3 gives our main results about properties of the maps f and h that ensure the preservation of the partial order. Finally, Section 4 illustrates the results and the methodology on examples.

2 Hypotheses and preliminaries

Consider a dynamical system on a domain $\Omega \subset \mathbb{R}^n$

$$\dot{x}(t) = f(x(t)),\ t \geq 0. \tag{4}$$

We assume that f can be written as

$$f(x) = g(x, h(x)),\ x \in \Omega \tag{5}$$

where h a map from Ω to $\mathbb{R}^m$ such that the following hypotheses are satisfied.

H0. The maps g, h belong to $\mathcal{C}^1(\Omega \times h(\Omega), \mathbb{R}^n)$, $\mathcal{C}^1(\Omega, \mathbb{R}^m)$ respectively.

H1. For any continuous function $\phi : \mathbb{R}_+ \mapsto h(\Omega)$, Ω is positively invariant by the non-autonomous system

$$\dot{x}(t) = g(x(t), \phi(t)),\ t \geq 0 \tag{6}$$

which is moreover forward complete.

H2. The dynamics $\dot{x} = g(x, z)$ is cooperative on Ω, for any fixed $z \in h(\Omega)$ i.e. the Kamke's condition (2) is fulfilled

$$\frac{\partial g_j}{\partial x_k}(x, z) \geq 0,\ j \neq k,\ x \in \Omega.$$

H3. The map $z \mapsto g(x, z)$ is monotone (i.e. component-wise non-decreasing) on $h(\Omega)$, for any $x \in \Omega$ i.e.

$$z, \bar{z} \in h(\Omega),\ \bar{z} \geq z\ \Rightarrow\ g(x, \bar{z}) \geq g(x, z).$$

H4. The components of h have no critical point in Ω i.e.

$$\nabla h_j(x) \neq 0,\ j = 1 \cdots m,\ \ x \in \Omega.$$

Remark 1. *When the dynamics* (4) *is not cooperative but satisfy H2 and H3, the map h is necessarily non monotone.*

We then consider a partial order $\succeq$ on Ω, defined as follows

Definition 1. *For x, y in Ω,*

$$y \succeq x \iff y \geq x \text{ and } h(y) \geq h(x) \text{ (component-wise).}$$

Let us underline that this partial order is not necessarily induced by a cone when the function h is not homogeneous (see for instance examples in Section 4.1 and 4.2). Otherwise, when h is for instance linear, $h(x) = Ax$ where A is a $m \times n$ matrix, the partial order $\succeq$ is induced by the cone $\mathcal{C} := \{x \in \mathbb{R}^n_+;\ Ax \geq 0\}$ and $y \succeq x$ amounts exactly to write $y - x \in \mathcal{C}$.

As the partial order $\succeq$ is not necessarily induced by a cone, we shall use a different technique than the usual one based on the Jacobian matrix of f [9, 10]. For a given time interval $[0, T]$, we associate to x_0, $\bar{x}_0$ in Ω the sequence of functions ϕ_i, $i = 0, 1, \cdots$ defined as follows

1. the function ϕ_0 is given by
$$\phi_0(t) = h(x(t)),\ t \in [0, T]$$
where $x(\cdot)$ is the solution of (4) for the initial condition $x(0) = x_0$,

2. the functions ϕ_i for $i = 1, \cdots$ are given recursively by
$$\phi_i = \mathcal{O}[\phi_{i-1}],\ i = 1, \cdots$$
where $\mathcal{O}$ is the operator defined on the set Φ of continuous functions $\phi : [0, T] \mapsto h(\Omega)$ as
$$\mathcal{O}[\phi](t) := h(x_\phi(t)),\ t \in [0, T]$$
and x_ϕ is the solution of (6) for the initial condition $x(0) = \bar{x}_0$.

We shall denote by $\bar{x}(\cdot)$ the solution of (4) for the initial condition $x(0) = \bar{x}_0$, and the corresponding function $\bar{\phi}(t) = h(\bar{x}(t))$ for $t \in [0, T]$.

Remark 2. *For sake of simplicity of the presentation, we have assumed completeness of the system. As we consider comparison of solutions on a finite time interval, this assumption can be relaxed considering a time interval $[0, T]$ on which solutions $x(\cdot)$, $\bar{x}(\cdot)$ are defined.*

Note that the sequence of solutions x_{ϕ_i} which alternates the integration of the g dynamics for a fixed function ϕ_i and the update of the function ϕ_i is in a spirit similar to numerical schemes for computing approximate solutions of differential equations (see for instance [3]), but our purpose here is purely theoretical. One has the following property about this sequence of solutions.

Lemma 1. *Assume Hypotheses H0-H1 are fulfilled. The sequence (x_{ϕ_i}, ϕ_{i+1}) converges uniformly to $(\bar{x}, \bar{\phi})$ on $[0, T]$.*

Proof. Let $\epsilon > 0$. The map h being continuous, one has
$$M := \sup_{x \in \mathbb{B}(\bar{x}_0, \epsilon))} ||h(x)|| < +\infty.$$
For $T_0 > 0$, consider the set
$$E := \{x(\cdot) \in \mathcal{C}([0, T_0], \Omega);\ x(0) = \bar{x}_0,\ ||x(t) - \bar{x}_0|| \leq \epsilon,\ t \in [0, T]\}.$$
For $x(\cdot) \in E$, we define $\mathcal{A}[x](\cdot)$ as the solution of $\dot{y} = g(y, h(x(t)))$, $y(0) = \bar{x}_0$ for $t \in [0, T_0]$ (note that the solution $y(\cdot)$ of this Cauchy problem is unique and well defined thanks to Hypotheses H0 and H1). One has then
$$y(t) = \bar{x}_0 + \int_0^t g(y(\tau), h(x(\tau))d\tau,\ t \in [0, T_0].$$
The map g being $\mathcal{C}^1$, there exists a number $C > 0$ such that
$$||y(t) - \bar{x}_0|| \leq \int_0^t ||g(y(\tau), h(x(\tau))||d\tau \leq \int_0^t C(1 + ||y(\tau)|| + ||h(x(\tau)||)d\tau,\ t \in [0, T_0]$$
and one gets
$$||y(t) - \bar{x}_0|| \leq \int_0^t C(1 + ||y(\tau)|| + M)d\tau = C(1 + ||\bar{x}_0|| + M)t + C\int_0^t ||y(\tau) - \bar{x}_0||d\tau.$$

With Gronwall's Lemma, one obtains

$$||y(t)-\bar{x}_0|| \leq C(1+||\bar{x}_0||+M)te^{Ct} \leq C(1+||\bar{x}_0||+M)T_0e^{CT_0}.$$

For $T_0>0$ small enough, one has $C(1+||\bar{x}_0||+M)T_0e^{CT_0} \leq \epsilon$ and thus $y(\cdot)=\mathcal{A}[x](\cdot)$ belongs to E. $\mathcal{A}$ is then well defined as an operator on E.

Take two elements $x(\cdot)$, $\tilde{x}(\cdot)$ in E. One can write for any $t\in[0,T_0]$

$$\begin{aligned} ||\mathcal{A}[\tilde{x}](t)-\mathcal{A}[x](t)|| &\leq \int_0^t ||g(\mathcal{A}[\tilde{x}](\tau),h(\tilde{x}(\tau)))-g(\mathcal{A}[x](\tau),h(x(\tau)))||d\tau \\ &\leq \int_0^t L_g(||\mathcal{A}[\tilde{x}](\tau)-\mathcal{A}[x](\tau)||+||h(\tilde{x}(\tau))-h(x(\tau))||)d\tau \end{aligned}$$

where L_g is the Lipschitz constant of the map g on $\mathbb{B}(\bar{x}_0,\epsilon)\times h(\mathbb{B}(\bar{x}_0,\epsilon))$. One has also

$$||\mathcal{A}[\tilde{x}](t)-\mathcal{A}[x](t)|| \leq \int_0^t L_g(||\mathcal{A}[\tilde{x}](\tau)-\mathcal{A}[x](\tau)||)d\tau + T_0L_gL_h||\tilde{x}-x||_{T_0}$$

where L_h is the Lipschitz constant of h on $\mathbb{B}(\bar{x}_0,\epsilon))$ and $||\,||_{T_0}$ denotes the infinity norm on $[0,T_0]$. With Gronwall's Lemma, one can write

$$||\mathcal{A}[\tilde{x}](t)-\mathcal{A}[x](t)|| \leq T_0L_gL_he^{L_gT_0}||\tilde{x}-x||_{T_0},\ t\in[0,T_0].$$

One has $T_0L_gL_he^{L_gT_0}<1$ for T_0 small enough, and we conclude that the operator $\mathcal{A}$ is a contraction mapping on E for the $||\,||_{T_0}$ norm. By the Banach's fixed point theorem, we deduce that the Picard's iterations

$$x_{\phi_{i+1}}(\cdot) := \mathcal{A}[x_{\phi_i}](\cdot)),\ i=0,\cdots$$

where $x_{\phi_0}(\cdot)$ is the solution of $\dot{x}=g(x,\phi_0(t))$ with $x(0)=\bar{x}_0$, converges uniformly to the unique fixed point $\bar{x}(\cdot)$ of $\mathcal{A}$, that is the solution of

$$\dot{x}=g(x,h(x))=f(x),\ x(0)=\bar{x}_0$$

on $[0,T_0]$. By continuity of h, the sequence $\phi_{i+1}(\cdot)=h(x_{\phi_i}(\cdot))$ converges uniformly to $\bar{\phi}(\cdot)=h(\bar{x}(\cdot))$.

If $T_0<T$, one can reproduce the same arguments for the initial condition $(T_0,\bar{x}(T_0))$ on a time interval $[T_0,T_1]$ and then on $[T_1,T_2]$ and so on.... One gets an increasing sequence T_i, $i=0\cdots$. If this sequence is bounded, it converges to a certain $T_\infty<+\infty$, and one obtains the uniform convergence of the sequence $(x_{\phi_i}(\cdot),\phi_{i+1}(\cdot))$ to $(\bar{x}(\cdot),\bar{\phi}(\cdot))$ on the time interval $[0,T_\infty)$, where $\bar{x}(T_\infty)$ exists from Hypothesis H1. If $T_\infty<T$, one can apply again the same arguments for the initial condition $(T_\infty,\bar{x}(T_\infty))$ until one obtains the uniform convergence of the sequence $(x_{\phi_i}(\cdot),\phi_{i+1}(\cdot))$ on the time interval $[0,T]$. □

3 Comparison of solutions

We first give a result concerning the comparison of the solutions for the partial order $\succeq$.

Lemma 2. *Assume Hypotheses H0-H3 are satisfied. If $\bar{x}_0\succeq x_0$ and the sequence ϕ_i is non decreasing i.e. $\phi_{i+1}(t)\geq\phi_i(t)$ (component-wise) for any $t\in[0,T]$ and $i=0,1,\cdots$ then one has*

$$\bar{x}(t)\succeq x(t),\ t\in[0,T].$$

Proof. Remark first that the solution $x(\cdot)$ is also solution of the non autonomous system

$$\dot{x}=g(x,\phi_0(t)) \tag{7}$$

for the initial condition $x(0)=x_0$.

Let $x_{\phi_0}(\cdot)$ be the solution of (7) for the initial condition $x(0)=\bar{x}_0$. From Hypothesis H2, the non-autonomous dynamics (7) is cooperative and consequently one has $x_{\phi_0}(t)\geq x(t)$ for any $t\in[0,T]$.

Let $x_{\phi_1}(\cdot)$ be the solution of

$$\dot{x}=g(x,\phi_1(t)) \tag{8}$$

for the initial condition $x(0) = \bar{x}_0$. If $\phi_1 \geq \phi_0$, then by Hypothesis H3, one has $g(x, \phi_1(t)) \geq g(x, \phi_0(t))$ for any $x \in \Omega$ and $t \in [0, T]$. Let us recall that the solutions of two differential equations, whose right-hand sides are component-wise ordered, and one of the equations is cooperative, are ordered (see for instance [11]). The dynamics (7) being cooperative, one then gets $x_{\phi_1}(t) \geq x_{\phi_0}(t)$ for any $t \in [0, T]$.

Recursively, one obtains $x_{\phi_{i+1}}(t) \geq x_{\phi_i}(t)$ for $i = 1, \cdots$ and $t \in [0, T]$. Then one gets from Lemma 1

$$\bar{x}(t) = \lim_{i \to +\infty} x_{\phi_i}(t) \geq x_{\phi_0}(t) \geq x(t), \ t \in [0, T].$$

In the same way, one has

$$\bar{\phi}(t) = h(\bar{x}(t)) = \lim_{i \to +\infty} \phi_i(t) \geq \phi_0(t) = h(x(t)), \ t \in [0, T].$$

□

We are now ready to give our main results.

3.1 Main result

Proposition 1. *Assume Hypotheses H0-H4 are satisfied. If the maps g and h satisfy for any $j = 1 \cdots m$ the condition*

$$\begin{array}{c} x, y \in \Omega, \ z \in h(\Omega), \ y \geq x, \ h(y) \geq h(x) \geq z, \\ h_j(y) = h_j(x) \ \Rightarrow \ D_j(x, y, z) := \nabla h_j(y).g(y, h(x)) - \nabla h_j(x).g(x, z) \geq 0 \end{array} \tag{9}$$

then having $\bar{x}_0 \succeq x_0$ in Ω implies $\bar{x}(t) \succeq x(t)$ for any $t \in [0, T]$.

Proof. Let us consider the set

$$S := \{(x, y) \in \Omega^2, \ h(y) \geq h(x)\}.$$

Under Hypothesis H4, the outer normal to this set is given by the following expression

$$N_S(x, y) = \sum_{j, \ h_j(y) = h_j(x)} \mathbb{R}_+ \begin{pmatrix} \nabla h_j(x) \\ -\nabla h_j(y) \end{pmatrix}, \ (x, y) \in \partial S.$$

We proceed recursively to show that the sequence ϕ_i is non decreasing (component-wise).

For $i = 0$, one has $x_{\phi_0}(t) \geq x(t)$ for any $t \in [0, T]$ (see the proof of Lemma 2). Note that one has $h(x_{\phi_0}(0)) = h(\bar{x}_0) \geq h(x_0) = h(x(0))$ i.e. $(x(0), x_{\phi_0}(0)) \in S$. We show that the solution $(x(\cdot), x_{\phi_0}(\cdot))$ remains in S. If there exists $t > 0$ such that for some $j \in \{1, \cdots, m\}$ one has $h_j(x_{\phi_0}(t)) = h_j(x(t))$ with $h_k(x_{\phi_0}(t)) \geq h_k(x(t))$ for $k \neq j$, one has

$$\begin{aligned} \begin{pmatrix} \nabla h_j(x(t)) \\ -\nabla h_j(x_{\phi_0}(t)) \end{pmatrix} \cdot \begin{pmatrix} g(x(t), \phi_0(t)) \\ g(x_{\phi_0}(t), \phi_0(t)) \end{pmatrix} &= \nabla h_j(x(t)).g(x(t), h(x(t)) - \nabla h_j(x_{\phi_0}(t)).g(x_{\phi_0}(t), h(x(t)) \\ &= -D_j(x(t), x_{\phi_0}(t), h(x(t)). \end{aligned}$$

Under condition (9), one then obtains the inward pointing property

$$(x(t), x_{\phi_0}(t)) \in \partial S \ \Rightarrow \ v. \begin{pmatrix} g(x(t), \phi_0(t)) \\ g(x_{\phi_0}(t), \phi_0(t)) \end{pmatrix} \leq 0, \ v \in N_S(x(t), x_{\phi_0}(t))$$

which implies that the set S is invariant for $(x(\cdot), x_{\phi_0}(\cdot))$ (see for instance [12]). We deduce that $\phi_1(t) - \phi_0(t) = h(x_{\phi_0}(t)) - h(x(t)))$ remains non-negative for any $t \in [0, T]$.

Assume that one has $x_{\phi_i}(t) \geq x_{\phi_{i-1}}(t)$ and $\phi_{i+1}(t) \geq \phi_i(t)$ for any $t \in [0, T]$ and $i \leq n$.

For $i = n + 1$, the recurrence property $\phi_{n+1}(.) \geq \phi_n(.)$ with $x_{\phi_{n+1}}(0) = x_{\phi_n}(0) = \bar{x}_0$ implies as before, from the properties of the map g (Hypotheses H2 and H3), that one has $x_{\phi_{n+1}}(t) \geq x_{\phi_n}(t)$ for any $t \in [0,T]$. Note that one has $h(x_{\phi_{n+1}}(0)) = h(\bar{x}_0) = h(x_{\phi_n}(0))$ i.e. $(x_{\phi_n}(0), x_{\phi_{n+1}}(0))$ belongs to S. As previously, we write

$$\begin{aligned}
&\begin{pmatrix} \nabla h_j(x_{\phi_n}(t)) \\ -\nabla h_j(x_{\phi_{n+1}}(t)) \end{pmatrix} \cdot \begin{pmatrix} g(x_{\phi_n}(t), \phi_n(t)) \\ g(x_{\phi_{n+1}}(t), \phi_{n+1}(t)) \end{pmatrix} \\
&\qquad = \nabla h_j(x_{\phi_n}(t)).g(x_{\phi_n}(t), h(x_{\phi_{n-1}}(t)) - \nabla h_j(x_{\phi_{n+1}}(t)).g(x_{\phi_{n+1}}(t), h(x_{\phi_n}(t)) \\
&\qquad = -D_j(x_{\phi_n}(t), x_{\phi_{n+1}}(t), h(x_{\phi_{n-1}}(t))
\end{aligned}$$

where $h(x_{\phi_n}(t)) - h(x_{\phi_{n-1}}(t)) = \phi_{n+1}(t) - \phi_n(t) \geq 0$. If $h_j(x_{\phi_{n+1}}(t)) - h_j(x_{\phi_n}(t)) = 0$ with $h_k(x_{\phi_{n+1}}(t)) - h_k(x_{\phi_n}(t)) \geq 0$, $k \neq j$ for some j and some $t \in [0,T]$, we get $D_j(x_{\phi_n}(t), x_{\phi_{n+1}}(t), h(x_{\phi_{n-1}}(t)) \geq 0$ from condition (9). One has then, as before, the inward pointing property

$$(x_{\phi_n}(t), x_{\phi_{n+1}}(t)) \in \partial S \;\Rightarrow\; v.\begin{pmatrix} g(x_{\phi_n}(t), \phi_n(t)) \\ g(x_{\phi_{n+1}}(t), \phi_{n+1}(t)) \end{pmatrix} \leq 0, \; v \in N_S(x_{\phi_n}(t), x_{\phi_{n+1}}(t))$$

from which we deduce that the solution $(x_{\phi_n}(\cdot), x_{\phi_{n+1}}(\cdot))$ remains in S, and thus the function $\phi_{n+2}(\cdot) - \phi_{n+1}(\cdot) = h(x_{\phi_{n+1}}(\cdot)) - h(x_{\phi_n}(\cdot))$ is non-negative, q.e.d.

Finally, the result follows from Lemma 2. □

Let us make some comments about conditions of Proposition 1. Condition (9) has some similarities with condition (3) on the boundary of the cone, but is adapted to the context of a partial order that is not induced by a cone. There are m scalar conditions to check, because the cooperative property of g with respect to x is already exploited. Note that the general formulation (5) allows to have terms $h_j(x)$ ($j = 1 \cdots m$) in common in several equations of the dynamics, and is thus of particular interest when the number m is relatively small (even when the partial order $\succeq$ defined from h is induced by a cone, this condition can be convenient to check compared to condition (3)). When h is scalar, condition (9) takes the simpler form

$$x, y \in \Omega, \; z \in h(\Omega), \; y \geq x, \; h(y) = h(x) \geq z \;\Rightarrow\; \nabla h(y).f(y) - \nabla h(x).g(x,z) \geq 0.$$

On practical problems, this condition can be quite easy to check, while checking the cooperative property of g with its Jacobian matrix is usually immediate (see for instance examples in Sections 4.1 and 4.2).

Remark 3. *Note that this result has some interests from a control view point when one deals with a control system $\dot{x} = f(x,u)$ that is cooperative for open-loop controls $u(\cdot)$, and one looks for feedback controls $u = h(x)$ such that some monotony properties of the closed-loop system are preserved.*

3.2 The separate case

We focus now on a way to rewrite the dynamics f for which we can derive simpler conditions to check. This is illustrated on examples of Sections 4.2 and 4.3.

Definition 2. *The formulation (5) is in a separate form if there exists maps $\tilde{g}$, $\tilde{h}$ from Ω to $\mathbb{R}^n$ such that*

$$f(x) = \tilde{g}(x) + \tilde{h}(x), \; x \in \Omega \tag{10}$$

where $\tilde{h}$ has m non-identical null components $\tilde{h}_j$ for j in a subset J of $\{1, \cdots, n\}$.

Then, we replace the former hypotheses H0-H5 by the following ones.

H0s. The maps $\tilde{g}$ and $\tilde{h}$ are $\mathcal{C}^1$ on Ω.

H1s. For any continuous function $\phi : \mathbb{R}_+ \mapsto h(\Omega)$, Ω is positively invariant by the non-autonomous system

$$\dot{x} = \tilde{g}(x) + \phi(t), \; t \geq 0, \; x \in \Omega$$

which is forward complete.

H2s. The map $\tilde{g}$ satisfies the Kamke's condition on Ω.

H4s. For any $j \in J$, $\tilde{h}_j$ has no critical point in Ω.

We have the following conditions to ensure monotony with respect to $\succeq$.

Corollary 1. *Assume Hypotheses H0s-H2s and H4s are satisfied when f is written in the form (10). If the maps $\tilde{g}$ and $\tilde{h}$ satisfy the property*

$$x, y \in \Omega,\ y \succeq x,\ \tilde{h}_j(y) = \tilde{h}_j(x)\ \Rightarrow\ \frac{\partial \tilde{h}_j}{\partial x_k}(y) \geq \frac{\partial \tilde{h}_j}{\partial x_k}(x) \geq 0,\ j, k \in J \tag{11}$$

then, the conclusion of Proposition 1 holds when the condition

$$x, y \in \Omega,\ y \succeq x,\ \tilde{h}_j(y) = \tilde{h}_j(x)\ \Rightarrow\ \tilde{D}_j(x, y) := \nabla \tilde{h}_j(y).\tilde{g}(y) - \nabla \tilde{h}_j(x).\tilde{g}(x) \geq 0,\ j \in J \tag{12}$$

is verified.

Proof. For $j \in J$, the function D_j defined in (9) of Proposition 1 can be written as

$$D_j(x, y, z) := \left[\nabla \tilde{h}_j(y).\tilde{g}(y) - \nabla \tilde{h}_j(x).\tilde{g}(x)\right] + \left[\nabla \tilde{h}_j(y).\tilde{h}(y) - \nabla \tilde{h}_j(x).z\right].$$

The expression in the first brackets is non-negative under condition (12), while the second verifies under condition (11)

$$\nabla \tilde{h}_j(y).\tilde{h}(y) - \nabla \tilde{h}_j(x).z = \sum_{k \in J} \frac{\partial \tilde{h}_j}{\partial x_k}(y).\tilde{h}_k(y) - \frac{\partial \tilde{h}_j}{\partial x_k}(x).z_k \geq \sum_{k \in J} \frac{\partial \tilde{h}_j}{\partial x_k}(x).(\tilde{h}_k(y) - z_k) \geq 0.$$

For $j \notin J$, one has clearly $D_j(x, y, z)$=0. Condition (9) of Proposition 1 is thus fulfilled. □

Condition (11) can be interpreted as a kind of matching condition: the functions $\tilde{h}_j$ have to be non-decreasing only with respect to variables x_k whose dynamics does not satisfy the cooperativity condition (i.e. $\frac{\partial f_k}{\partial x_l}(x) < 0$ for some $l \neq k$ and $x \in \Omega$).

Remark 4. *When the map $\tilde{h}$ is linear $h(x) = Ax$, the conditions of Corollary 1 take the simple form*

$$\begin{aligned} &A_{jk} \geq 0,\ j, k \in J, \\ &x, y \in \Omega,\ y \geq x,\ A_j(x - y) \geq 0\ \Rightarrow\ \tilde{D}_j(x, y) = A_j(\tilde{g}(y) - \tilde{g}(x))) \geq 0,\ j \in J. \end{aligned}$$

This is illustrated on the example of Section 4.3.

4 Examples

We present three examples corresponding to the various situations discussed in Section 3.

4.1 Non separated dynamics and non cone-induced order

Consider the system on the domain $\Omega = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$

$$\left\{ \begin{array}{l} \dot{x}_1 = (-\alpha + x_1 + 2x_2 - x_3^2)x_1 + \beta x_3^2 \\ \dot{x}_2 = \gamma x_1 - \delta x_3^2 \\ \dot{x}_3 = \left(1 - e^{-x_2 + x_3^2}\right)x_3 - \delta x_3 \end{array} \right. \tag{13}$$

where α, β, δ, γ are positive parameters. One can easily check that the set Ω is forwardly invariant by this dynamics. The Jacobian matrix is written as

$$\begin{pmatrix} \star & 2x_1 & 2(\beta x_3 - x_1) \\ \gamma & \star & -2\delta x_3 \\ 0 & e^{-x_2 + x_3}x_3 & \star \end{pmatrix}$$

and thus the system is not monotone on Ω. Let us write now the system as follows

$$\left\{ \begin{array}{l} \dot{x}_1 = -\alpha x_1 + (1 - \frac{\gamma}{\delta})x_1^2 + \beta x_3^2 + \frac{1}{\delta}h(x)x_1 \\ \dot{x}_2 = -2\delta x_2 + h(x) \\ \dot{x}_3 = \left(1 - e^{-x_2 + x_3^2}\right)x_3 - \delta x_3 \end{array} \right.$$

where

$$h(x) = \gamma x_1 + \delta(2x_2 - x_3^2).$$

This dynamics is in the form (5) but is not separated. We thus consider the non-autonomous dynamics

$$\dot{x} = g(x, \phi(t)) = \begin{pmatrix} -\alpha x_1 + (1-\frac{\gamma}{\delta})x_1^2 + \beta x_3^2 + \frac{1}{\delta}\phi(t)x_1 \\ -2\delta x_2 + \phi(t) \\ \left(1 - e^{-x_2+x_3^2}\right)x_3 - \delta x_3 \end{pmatrix}.$$

One can straightforwardly check that set Ω is also forwardly invariant whatever is the function ϕ and that the system is cooperative and monotone in ϕ on Ω. Moreover, the gradient of h is always non null. Hypotheses H0-H1-H2-H3-H4 are fulfilled. Note that for this function h, the partial order $\succeq$ given in Definition 1 is not induced by a cone (because $h(x) \geq 0$ does not necessarily implies $h(\tau x) \geq 0$ for any $\tau \geq 0$). To apply Proposition 1, we have to consider for x, y in Ω, $z \in h(\Omega)$ with $y \geq x$ and $h(y) = h(x) \geq z$ the quantity

$$\begin{aligned} D(x,y,z) &= \frac{\partial h}{\partial x}(y).g(y,h(x)) - \frac{\partial h}{\partial x}(x).g(x,z) \\ &= \gamma\big(-\alpha(y_1-x_1) + (1-\tfrac{\gamma}{\delta})(y_1^2-x_1^2) + \beta(y_3^2-x_3^2) + \tfrac{1}{\delta}(h(y)y_1 - zx_1)\big) \\ &+ \ 2\delta\big(-2\delta(y_2-x_2) + h(y) - z\big) - 2\delta\Big(\big(1-e^{-y_2+y_3^2}-\delta\big)y_3^2 - \big(1-e^{-x_2+x_3^2}-\delta\big)x_3^2\Big). \end{aligned}$$

Note that $h(y) = h(x)$ implies

$$2\delta(y_2-x_2) = \delta(y_3^2-x_3^2) - \gamma(y_1-x_1) \text{ and } (y_3^2-y_2) - (x_3^2-x_2) = \frac{\gamma}{\delta}(y_1-x_1) + y_2 - x_2 \geq 0.$$

Then, by rearranging terms (and using $h(y) \geq z$), one obtains the inequality

$$D(x,y,z) \ \geq \ \gamma(2\delta - \alpha + \tfrac{z}{\delta})(y_1-x_1) + \gamma(1-\tfrac{\gamma}{\delta})(y_1^2-x_1^2) + (\gamma\beta - 2\delta)(y_3^2-x_3^2).$$

On the other hand, a straightforward computation gives

$$h = 0 \ \Rightarrow \ \dot{h} \geq \gamma(2\delta-\alpha)x_1 + \gamma(1-\tfrac{\gamma}{\delta})x_1^2 + (\gamma\beta-2\delta)x_3^2.$$

Therefore, when the parameters satisfy the inequalities

$$\gamma\beta \geq 2\delta \geq \alpha \text{ and } \delta \geq \gamma \tag{14}$$

the set $\tilde{\Omega} = \{x \in \Omega, \ h(x) \geq 0\}$ is forwardly invariant, and one has $D(x,y,z) \geq 0$ for $z \in h(\tilde{\Omega})$. The conclusions of Proposition 1 follow when applied to $\tilde{\Omega}$.

We have run numerical simulations for values of the parameters $\alpha = 0.8$, $\beta = 3$, $\delta = 0.5$, $\gamma = 0.4$ (for which one can check that condition (14) is satisfied) and the three initial conditions $x_0 = (0.1, 0, 1, 0, 2)^\top$, $\bar{x}_0^a = (0.1, 0.1, 0.5)^\top$, $\bar{x}_0^b = (0.1, 0.2, 0.3)^\top$. One computes $h(x_0) = 0.2$, $h(\bar{x}_0^a) = 0.095$, $h(\bar{x}_0^b) = 0.275$. Clearly, x_0, $\bar{x}_0^a$, $\bar{x}_0^b$ belong to $\tilde{\Omega}$ with $\bar{x}_0^a \geq x_0$ and $\bar{x}_0^b \geq x_0$. However, corresponding solutions $x(\cdot)$, $\bar{x}^a(\cdot)$ are not ordered, while $x(\cdot)$, $\bar{x}^b(\cdot)$ are (see Figure 1). This is in accordance with Proposition 1 because one has $\bar{x}_0^b \succeq x_0$ but not $\bar{x}_0^a \succeq x_0$.

4.2 Separated dynamics and non cone-induced order

We simplify the dynamics (13) of the former example as follows

$$\left\{ \begin{array}{l} \dot{x}_1 = -\alpha x_1 + \beta x_3^2 \\ \dot{x}_2 = \gamma x_1 - \delta x_3^2 \\ \dot{x}_3 = \left(1 - e^{-x_2+x_3^2}\right)x_3 - \delta x_3 \end{array} \right. \tag{15}$$

which is also non-monotone on the same invariant domain $\Omega = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$. We write this system as follows

$$\left\{ \begin{array}{l} \dot{x}_1 = -\alpha x_1 + \beta x_3^2 \\ \dot{x}_2 = -2\delta x_2 + \tilde{h}_2(x) \\ \dot{x}_3 = \left(1 - e^{-x_2+x_3^2}\right)x_3 - \delta x_3 \end{array} \right.$$

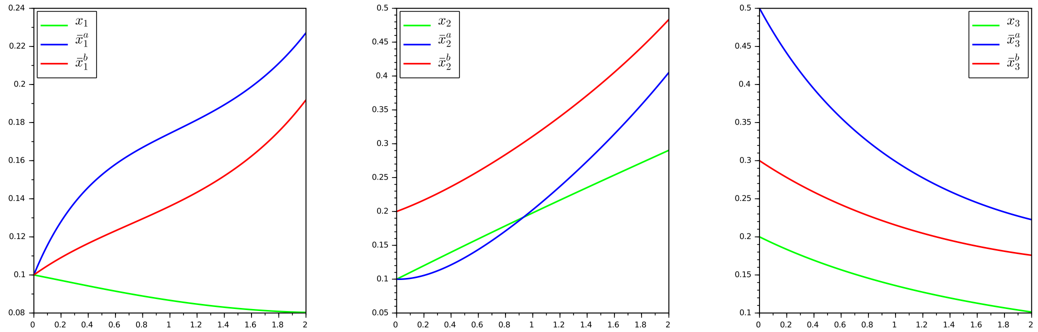

Figure 1: Comparison of solutions $x(\cdot)$, $\bar{x}^a(\cdot)$, $\bar{x}^b(\cdot)$) of system (13)

where

$$\tilde{h}_2(x) = \gamma x_1 + \delta(2x_2 - x_3^2).$$

This dynamics is in the separated form (10) with $J = \{2\}$. One has $\frac{\partial \tilde{h}_2}{\partial x_2} = 2\delta \geq 0$ and condition (11) is thus verified. Clearly Hypotheses H0s-H2s and H4s are satisfied. One has then just to check condition (12) to apply Corollary 1, where

$$\begin{aligned} \tilde{D}_2(x,y) &= \frac{\partial \tilde{h}_2}{\partial x}(y).\tilde{g}(y) - \frac{\partial \tilde{h}_2}{\partial x}(x).\tilde{g}(x) \\ &= \gamma\big(-\alpha(y_1-x_1) + \beta(y_3^2 - x_3^2)\big) + 2\delta\big(-2\delta(y_2-x_2)\big) \\ &\qquad -2\delta\Big(y_3^2\big(1-e^{-y_2+y_3^2}-\delta\big) - x_3^2\big(1-e^{-x_2+x_3^2}-\delta\big)\Big). \end{aligned}$$

Computations are similar to those in the former example but simpler because there is no z variable here. For x, y in Ω with $y \geq x$ and $\tilde{h}_2(y) = \tilde{h}_2(x)$, one simply gets

$$\tilde{D}_2(x,y) \geq \gamma(2\delta - \alpha)(y_1 - x_1) + (\gamma\beta - 2\delta)(y_3^2 - x_3^2)$$

and we conclude that the condition $\tilde{D}_2(x,y) \geq 0$ is verified when the parameters verify the inequalities

$$\gamma\beta \geq 2\delta \geq \alpha. \tag{16}$$

Then, Corollary 1 applies on the whole domain Ω.

We have run numerical simulations for $\alpha = 1$, $\beta = 1$, $\delta = 0.55$, $\gamma = 1.2$ (which satisfy (16)) and the initial conditions $x_0 = (0.1, 0, 1, 1)^\top$, $\bar{x}_0^a = (0.1, 0.1, 3)^\top$, $\bar{x}_0^b = (0.6, 0.6, 1.5)^\top$ (see Figure 2). One computes $h(\bar{x}_0^a) - h(x_0) = -8\delta < 0$, $h(\bar{x}_0^b) - h(x_0) = 0.5\gamma - 0.25\delta > 0$. As before, we obtain that $x(\cdot)$, $\bar{x}^a(\cdot)$ are not ordered, while $x(\cdot)$, $\bar{x}^b(\cdot)$ are.

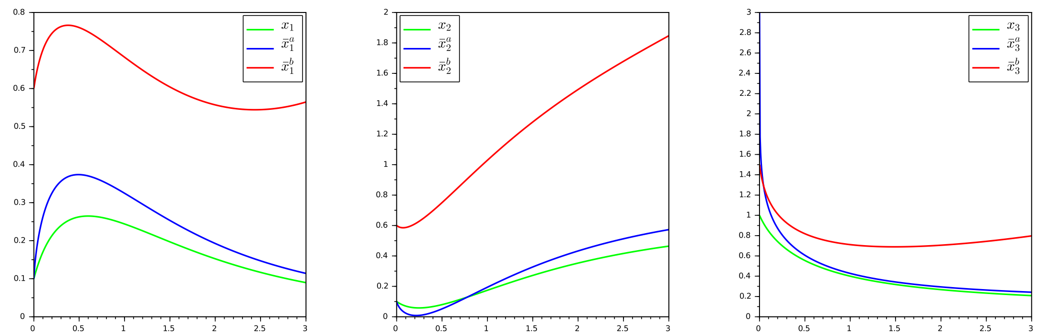

Figure 2: Comparison of solutions $x(\cdot)$, $\bar{x}^a(\cdot)$, $\bar{x}^b(\cdot)$) of system (15)

4.3 Separated dynamics and cone-induced order

We replace the x_3^2 terms by x_3 in the expression (15) of the previous dynamics, which gives the system

$$\left\{\begin{array}{l} \dot{x}_1 = -\alpha x_1 + \beta x_3 \\ \dot{x}_2 = \gamma x_1 - \delta x_3 \\ \dot{x}_3 = \Big(1 - e^{-x_2+x_3}\Big)x_3 - \delta x_3 \end{array}\right. \tag{17}$$

This example is similar to the model studied in [7] but presented here in a different set of coordinates for simplicity of exposition. It can be written

$$\left\{\begin{array}{l} \dot{x}_1 = -\alpha x_1 + \beta x_3 \\ \dot{x}_2 = -\delta x_2 + \tilde{h}_2(x) \\ \dot{x}_3 = \Big(1 - e^{-x_2+x_3}\Big)x_3 - \delta x_3 \end{array}\right.$$

with the linear function

$$\tilde{h}_2(x) = \gamma x_1 + \delta(x_2 - x_3) = a.x$$

where $a = [\gamma, \delta, -\delta]^\top$. As $\tilde{h}$ is linear, the partial order $\succeq$ is induced by a cone and one can use condition (3) to check the monotony of the solutions with respect to $\succeq$. However, following Remark 4, we show that conditions of Corollary 1 are indeed very simple. One has $a_2 = \delta \geq 0$ and

$$\begin{array}{rcl} \tilde{D}_2(x,y) & = & a.(\tilde{g}(y) - \tilde{g}(x)) \\ & = & \gamma\big(-\alpha(y_1 - x_1) + \beta(y_3 - x_3)\big) - \delta^2(y_2 - x_2) \\ & & \qquad +\delta\Big((\delta - 1)(y_3 - x_3) + e^{-y_2+y_3}y_3 - e^{-x_2+x_3}x_3\Big) \end{array}$$

which gives

$$\tilde{D}_2(x,y) \geq (\delta - \alpha)(y_1 - x_1) + (\gamma\beta - \delta)(y_3 - x_3)$$

for $y \geq x$ in Ω such that $a.(y - x) = 0$. Conditions of Corollary 1 are then fulfilled when one has

$$\gamma\beta \geq \delta \geq \alpha. \tag{18}$$

We have run numerical simulations for $\alpha = 1$, $\beta = 1$, $\delta = 1.1$, $\gamma = 1.2$ that satisfy (18)) and the initial conditions $x_0 = (1, 1, 0.1)^\top$, $\bar{x}_0^a = (1, 1, 1.1)^\top$, $\bar{x}_0^b = (1.3, 2, 1.1)^\top$ (see Figure 3). One has $\bar{x}_0^b \geq \bar{x}_0^a \geq x_0$ with $\bar{x}_0^b \succeq x_0$ but not $\bar{x}_0^a \succeq x_0$, and one can see as before that solutions are ordered for initial conditions that verify the partial order $\succeq$.

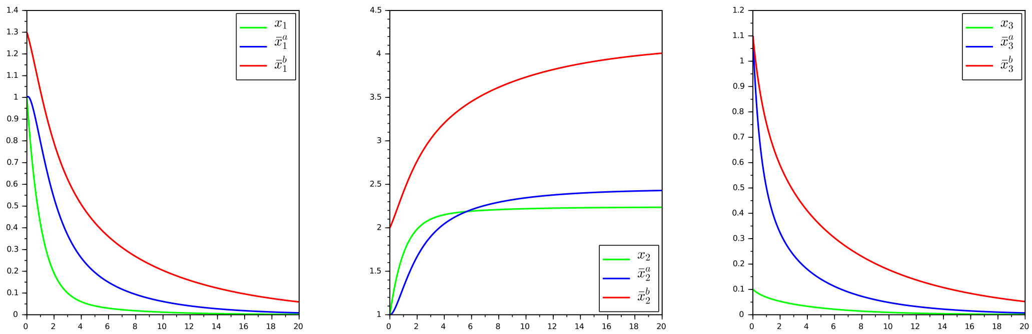

Figure 3: Comparison of solutions $x(\cdot)$, $\bar{x}^a(\cdot)$, $\bar{x}^b(\cdot)$) of system (17)

4.4 Illustration of the methodology

To illustrate our approach, we have computed the sequence ϕ_i for x_0 and $\bar{x}_0 = \bar{x}_0^b$, and the corresponding solutions x_{ϕ_i}, as defined in Section 2. On Figures 4, 5, 6, one can observe the monotonic behavior of these functions and their convergence to $\bar{\phi}$ and $\bar{x}$.

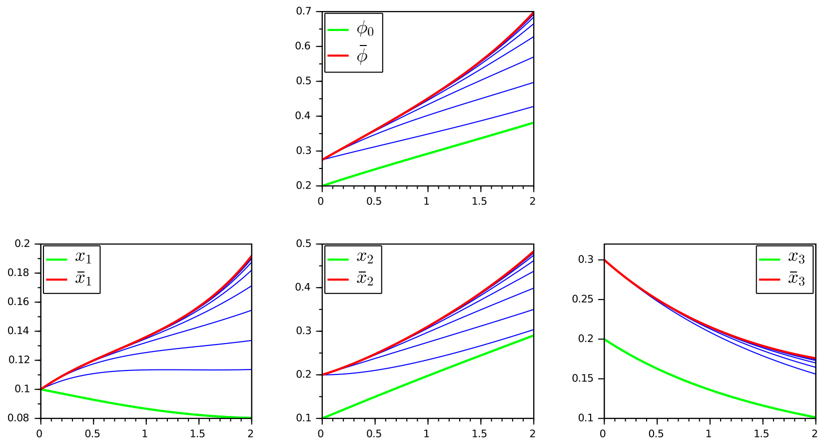

Figure 4: Iterations of ϕ_i and solutions x_{ϕ_i} for system (13) (in blue)

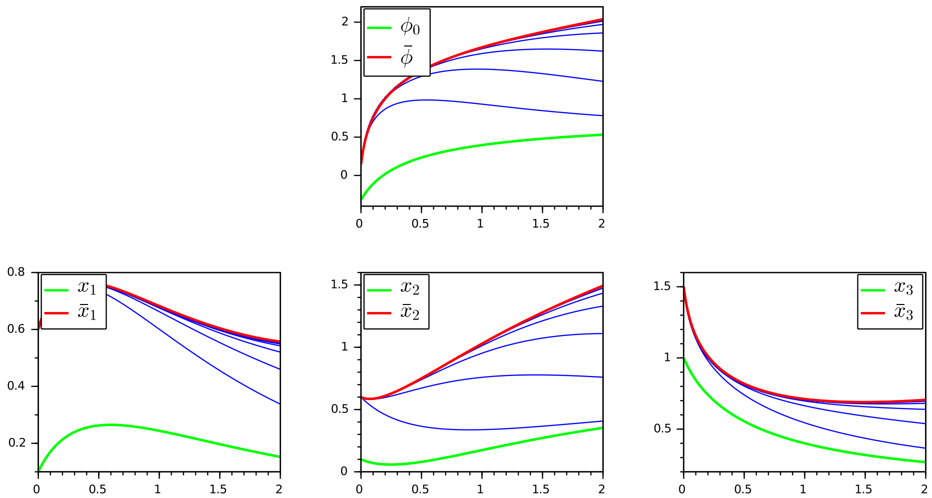

Figure 5: Iterations of ϕ_i and solutions x_{ϕ_i} for system (15) (in blue)

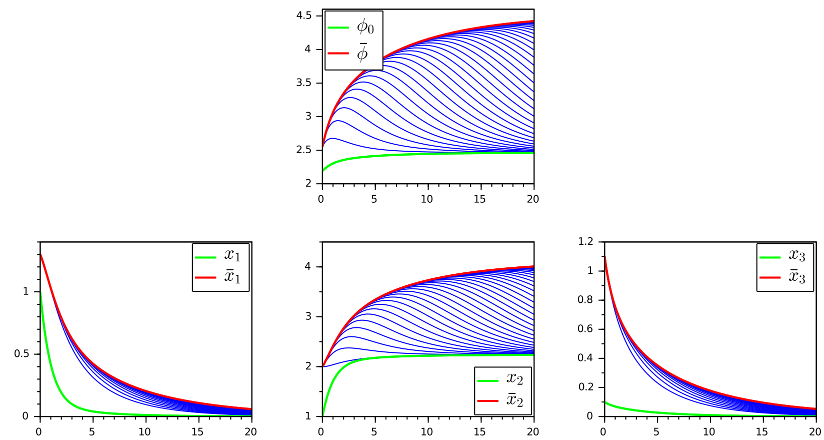

Figure 6: Iterations of ϕ_i and solutions x_{ϕ_i} for system (17) (in blue)

5 Conclusion

In this note, we have provided conditions for a partial order stronger than the usual vector order in $\mathbb{R}^n$ to be preserved by the flow of a dynamical system. We have shown that this property can be related to the monotonic behavior of some Picard iterations, which does not require the partial order to be necessarily induced by a cone. Our approach is based on a separation of terms that satisfy the Kamke's condition from other ones, in the expression of the dynamics. For practical problems, it might be difficult to find a partial order for which the dynamics exhibit a monotony property. Our approach facilitates this search, as we have shown with examples. However, several ways of making such a separation are possible. The study of the best way, in terms of the less restrictive conditions or the largest partially order subset in $\mathbb{R}^n$, could be the matter of a future work. Extension of this work with respect to other vector order in $\mathbb{R}^n$ could be also subject of investigation.

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