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Abstract

We give conditions for a non-monotone system to preserve the usual vector order of solutions for a subset of initial conditions. Our approach consists of separating terms that meet Kamke's sign condition from other ones in the dynamics, and considering Picard iterations. These conditions amount for the dynamics to preserve a partial order, which is not necessarily induced by a cone. Examples illustrate the results.

Key-words. ordinary differential equations, solutions comparison, monotone systems, partial order.

1 Introduction

Monotone dynamical systems have received a great attention in the literature (see for instance the monograph [14], the review [10] and the references herein). Let us recall that the semi-flows of monotone systems preserve a vector order, and that their asymptotic behaviors present some strong properties (see [8, 9]). In particular, systems $\dot{x} = f(t, x)$ in $\mathbb{R}^n$ that are cooperative preserve the partial order relative to the positive orthant in $\mathbb{R}^n_+$:

$$y_0 \geq x_0 \Rightarrow y(t) \geq x(t),\ t \geq t_0 \tag{1}$$

(where $y(\cdot)$, $x(\cdot)$ are solutions of the initial value problems $y(t_0) = y_0$, $x(t_0) = x_0$ and $\geq$ is considered component-wise). Kamke's condition

$$\frac{\partial f_i}{\partial x_j}(t, x) \geq 0,\ i \neq j \tag{2}$$

characterizes such systems from the single knowledge of the Jacobian matrix of f. This condition can been extended to partial orders relative to the other orthants of $\mathbb{R}^n$, that are $\{x \in \mathbb{R}^n;\ (-1)^{m_i} x_i \geq 0;\ i = 1 \cdots n\}$ where $m_i \in \{0, 1\}$

$$(-1)^{m_i+m_j} \frac{\partial f_i}{\partial x_j}(t, x) \geq 0,\ i \neq j$$

(see [14]), or for even more general positive cones P of $\mathbb{R}^n$

$$\lambda \left(\frac{\partial f}{\partial x}(t, x).y \right) \geq 0,\ y \in \partial P,\ \lambda \in \Lambda(P) \tag{3}$$

where $\Lambda(P)$ is the set of supporting linear forms of P (see [17]). More recently, the cooperativity property with respect to cones has been characterized for non-smooth dynamics [5].

The preservation of vector order for solutions of dynamical systems has important implications in several applications. In particular, this property is at the core of the interval observers techniques (see for instance [6, 12]). Let us stress that the cooperativity property is required for the observers and not necessarily for the original system, as underlined in [2] and further investigated for instance in [4, 15]. However, in some practical problems, one may observe an order preservation of trajectories (relatively to the positive orthant $\mathbb{R}^n_+$) for some subsets of initial conditions of for some variables only, while the system is not cooperative. For instance, the anaerobic digestion model studied in [11] exhibits an augmentation of the biogas variable when increasing the initial organic matter, while the dynamics is not monotone.

The aim of the present work is to characterize theoretically such situations for a class of systems. For this purpose, we shall consider a decomposition of the map f by isolating terms into a partial map h that prevent Kamke's condition to be fulfilled, and give conditions on the maps f and h for the ordering property (1) to hold for a subset of initial conditions. Note that in practice it is not always easy to find a cone P that verify condition (3) (if it exists). This is also a motivation of our work to propose a methodology that could facilitate this search.

For simplicity of the presentation, we shall consider autonomous dynamics only, but extension of the results to non-autonomous ones does not present any particular difficulty and is left to the reader. The paper is organized as follows. In Section 2, we define a partial order in the positive quadrant induced by the decomposition of the dynamics, under some hypotheses, and give preliminaries results that will be used in the following. Section 3 gives our main results about properties of the maps f and h that ensure the preservation of the partial order. Finally, Section 4 illustrates the results and the methodology on examples. In particular, we analyze the anaerobic digestion model [11] from this perspective.

2 Hypotheses and preliminaries

Consider a dynamical system on a domain $\Omega \subset \mathbb{R}^n$

$$\dot{x}(t) = f(x(t)),\ t \geq 0. \tag{4}$$

We assume that f can be written as

$$f(x) = g(x, h(x)),\ x \in \Omega \tag{5}$$

where h a map from Ω to $\mathbb{R}^m$ such that the following hypotheses are satisfied.

H0. The maps g, h are $\mathcal{C}^1$ on $\mathcal{O}_g$, $\mathcal{O}_h$, respectively, where $\mathcal{O}_g$, $\mathcal{O}_h$ are open sets with $\Omega \times h(\Omega) \subset \mathcal{O}_g$, $h(\Omega) \subset \mathcal{O}_h$.

H1. For any continuous function $\phi : \mathbb{R}_+ \mapsto h(\Omega)$, Ω is positively invariant by the non-autonomous system

$$\dot{x}(t) = g(x(t), \phi(t)),\ t \geq 0 \tag{6}$$

which is moreover forward complete [1].

H2. The set Ω is p-convex (i.e. for any x, y, in Ω with $y \geq x$ and $s \in [0, 1]$, $sx + (1-s)y$ is in Ω) and the dynamics $\dot{x} = g(x, z)$ is cooperative on Ω, for any fixed $z \in h(\Omega)$ i.e. Kamke's condition (2) is fulfilled

$$\frac{\partial g_j}{\partial x_k}(x, z) \geq 0,\ j \neq k,\ x \in \Omega.$$

H3. The map $z \mapsto g(x, z)$ is monotone (i.e. component-wise non-decreasing) on $h(\Omega)$, for any $x \in \Omega$ i.e.

$$z, \bar{z} \in h(\Omega),\ \bar{z} \geq z\ \Rightarrow\ g(x, \bar{z}) \geq g(x, z).$$

H4. The components of h have no critical point in Ω i.e.

$$\nabla h_j(x) \neq 0,\ j = 1 \cdots m,\ \ x \in \Omega.$$

Remark 1. *When the dynamics* (4) *is not cooperative but satisfy H2 and H3, the map h is necessarily non monotone.*

Remark 2. *A controlled dynamical system $\dot{x} = g(x, u)$ (where u is the control) with g satisfying H2 and H3 is called in the literature a "monotone control system" (see [1, 3]) and satisfies the property*

$$y_0 \geq x_0,\ v(\cdot) \geq u(\cdot)\ \Rightarrow\ y(t) \geq x(t),\ t \geq 0$$

where $u(\cdot)$, $v(\cdot)$ are Lebesgue-measurable functions of time (or "open-loops") and $x(\cdot)$, resp. $y(\cdot)$ denotes the (absolutely continuous) solution for $x(0) = x_0$ and control $u(\cdot)$, resp. $y(0) = x_0$ and control $v(\cdot)$. Our study amounts to investigate when this property is satisfied for a feedback control $h(x)$ instead of an open-loop control.

[1] A system is forward complete when solutions exist globally, for any positive time.

We then consider a partial order $\succeq$ on Ω, defined as follows

Definition 1. *For x, y in Ω,*

$$y \succeq x \iff y \geq x \text{ and } h(y) \geq h(x) \text{ (component-wise).}$$

We implicitly assume that h is such that the partial order does not degenerate i.e. $y \succeq x \not\Leftrightarrow y = x$. Let us underline that this partial order is not necessarily induced by a cone when the function h is not homogeneous (see for instance the examples in Section 4.2). Otherwise, when h is for instance linear, $h(x) = Ax$ where A is a $m \times n$ matrix, the partial order $\succeq$ is induced by the cone $\mathcal{C} := \{x \in \mathbb{R}^n_+;\ Ax \geq 0\}$ and $y \succeq x$ amounts exactly to write $y - x \in \mathcal{C}$.

As the partial order $\succeq$ is not necessarily induced by a cone, we shall use different techniques than the usual one based on the Jacobian matrix of f (see [14, 17]). In particular, we shall consider an approximation scheme for solutions of (4) which benefits from the structure (5) and the property of monotone control systems recalled in Remark 2. It consists in starting from a solution of (4) for a given initial condition and making iterations of solutions of the non-autonomous dynamics (6) that converge to the solution of (4) for another initial condition. More precisely, for a given time interval $[0, T]$, we associate to any x_0, $\bar{x}_0$ in Ω the sequence of functions ϕ_i, $i = 0, 1, \cdots$ defined as follows

1. the function ϕ_0 is given by
$$\phi_0(t) = h(x(t)),\ t \in [0, T]$$
where $x(\cdot)$ is the solution of (4) for the initial condition $x(0) = x_0$,
2. the functions ϕ_i for $i = 1, \cdots$ are given recursively by
$$\phi_i = \mathcal{O}[\phi_{i-1}],\ i = 1, \cdots$$
where $\mathcal{O}$ is the operator defined on the set Φ of continuous functions $\phi : [0, T] \mapsto h(\Omega)$ as
$$\mathcal{O}[\phi](t) := h(x_\phi(t)),\ t \in [0, T]$$
and x_ϕ is the solution of (6) for the initial condition $x(0) = \bar{x}_0$.

We shall denote by $\bar{x}(\cdot)$ the solution of (4) for the initial condition $x(0) = \bar{x}_0$, and the corresponding function $\bar{\phi}(t) = h(\bar{x}(t))$ for $t \in [0, T]$.

Remark 3. *For sake of simplicity of the presentation, we have assumed completeness of the system. As we consider comparison of solutions on a finite time interval, this assumption can be relaxed considering a time interval $[0, T]$ on which solutions $x(\cdot)$, $\bar{x}(\cdot)$ are defined.*

Note that the sequence of solutions x_{ϕ_i} which alternates the integration of the g dynamics for a given function ϕ_i and the update of the function ϕ_i is similar to splitting methods for solving numerically ordinary differential equations (see for instance Chapter I.8 in [7]). One has the following property about this sequence of solutions.

Lemma 1. *Assume Hypotheses H0-H1 are fulfilled. The sequence (x_{ϕ_i}, ϕ_{i+1}) converges uniformly to $(\bar{x}, \bar{\phi})$ on $[0, T]$.*

Proof. Let $\epsilon > 0$. The map h being continuous, one has

$$M := \sup_{x \in \mathbb{B}(\bar{x}_0, \epsilon))} \|h(x)\| < +\infty.$$

For $T_0 > 0$, consider the set

$$E := \{x(\cdot) \in \mathcal{C}([0, T_0], \Omega);\ x(0) = \bar{x}_0,\ \|x(t) - \bar{x}_0\| \leq \epsilon,\ t \in [0, T_0]\}.$$

For $x(\cdot) \in E$, we define $\mathcal{A}[x](\cdot)$ as the solution of $\dot{y} = g(y, h(x(t)))$, $y(0) = \bar{x}_0$ for $t \in [0, T_0]$ (note that the solution $y(\cdot)$ of this Cauchy problem is unique and well defined thanks to Hypotheses H0 and H1). One has then

$$y(t) = \bar{x}_0 + \int_0^t g(y(\tau), h(x(\tau))d\tau,\ t \in [0, T_0].$$

The map g being $\mathcal{C}^1$, there exists a number $C > 0$ such that

$$\|y(t) - \bar{x}_0\| \leq \int_0^t \|g(y(\tau), h(x(\tau))\| \, d\tau \leq \int_0^t C(1 + \|y(\tau)\| + \|h(x(\tau)\|)d\tau, \ t \in [0, T_0]$$

and one gets

$$\|y(t) - \bar{x}_0\| \leq \int_0^t C(1 + \|y(\tau)\| + M)d\tau = C(1 + \|\bar{x}_0\| + M)t + C\int_0^t \|y(\tau) - \bar{x}_0\| \, d\tau.$$

With Grönwall's Lemma, one obtains

$$\|y(t) - \bar{x}_0\| \leq C(1 + \bar{x}_0 + M)te^{Ct} \leq C(1 + \bar{x}_0 + M)T_0 e^{CT_0}.$$

For $T_0 > 0$ small enough, one has $C(1 + \|\bar{x}_0\| + M)T_0 e^{CT_0} \leq \epsilon$ and thus $y(\cdot) = \mathcal{A}[x](\cdot)$ belongs to E. $\mathcal{A}$ is then well defined as an operator on E.

Take two elements $x(\cdot)$, $\tilde{x}(\cdot)$ in E. One can write for any $t \in [0, T_0]$

$$\begin{aligned} \|\mathcal{A}[\tilde{x}](t) - \mathcal{A}[x](t)\| &\leq \int_0^t \|g(\mathcal{A}[\tilde{x}](\tau), h(\tilde{x}(\tau))) - g(\mathcal{A}[x](\tau), h(x(\tau)))\| \, d\tau \\ &\leq \int_0^t L_g(\|\mathcal{A}[\tilde{x}](\tau) - \mathcal{A}[x](\tau)\| + \|h(\tilde{x}(\tau)) - h(x(\tau))\|)d\tau \end{aligned}$$

where L_g is the Lipschitz constant of the map g on $\mathbb{B}(\bar{x}_0, \epsilon) \times h(\mathbb{B}(\bar{x}_0, \epsilon))$. One has also

$$\|\mathcal{A}[\tilde{x}](t) - \mathcal{A}[x](t)\| \leq \int_0^t L_g(\|\mathcal{A}[\tilde{x}](\tau) - \mathcal{A}[x](\tau)\|)d\tau + T_0 L_g L_h \|\tilde{x} - x\|_{T_0}$$

where L_h is the Lipschitz constant of h on $\mathbb{B}(\bar{x}_0, \epsilon))$ and $\| \, \|_{T_0}$ denotes the infinity norm on $[0, T_0]$. With Grönwall's Lemma, one can write

$$\|\mathcal{A}[\tilde{x}](t) - \mathcal{A}[x](t)\| \leq T_0 L_g L_h e^{L_g T_0} \|\tilde{x} - x\|_{T_0}, \ t \in [0, T_0].$$

One has $T_0 L_g L_h e^{L_g T_0} < 1$ for T_0 small enough, and we conclude that the operator $\mathcal{A}$ is a contraction mapping on E for the $\| \, \|_{T_0}$ norm. By the Banach's fixed point theorem, we deduce that the Picard's iterations

$$x_{\phi_{i+1}}(\cdot) := \mathcal{A}[x_{\phi_i}](\cdot), \ i = 0, \cdots$$

where $x_{\phi_0}(\cdot)$ is the solution of $\dot{x} = g(x, \phi_0(t))$ with $x(0) = \bar{x}_0$, converges uniformly to the unique fixed point $\bar{x}(\cdot)$ of $\mathcal{A}$, that is the solution of

$$\dot{x} = g(x, h(x)) = f(x), \ x(0) = \bar{x}_0$$

on $[0, T_0]$. By continuity of h, the sequence $\phi_{i+1}(\cdot) = h(x_{\phi_i}(\cdot))$ converges uniformly to $\bar{\phi}(\cdot) = h(\bar{x}(\cdot))$.

If $T_0 < T$, consider an integer k such that $\|x_{\phi_i}(T_0) - \bar{x}(T_0)\| \leq \epsilon$ for $i \geq k$, and $T_1 > T_0$. For any $i \geq k$, define the set $E_i := \{x(\cdot) \in \mathcal{C}([T_0, T_1], \Omega); \ x(T_0) = x_{\phi_i}(T_0), \ \|x(t) - \bar{x}(T_0)\| \leq \epsilon, \ t \in [T_0, T_1]\}$ and the operator $\mathcal{A}_i$ defined on E_i by $\mathcal{A}_i[x](\cdot)$ solution of $\dot{y} = g(y, h(x(t)))$, $y(T_0) = x_{\phi_i}(T_0)$ for $t \in [T_0, T_1]$. As before, one has for a small enough T_1, $\mathcal{A}_i[E_i] \subset E_i$ and $\mathcal{A}_i$ contraction on E_i, for any $i \geq k$. Note that one has $x_{\phi_i} = \mathcal{A}_i.\mathcal{A}_{i-1} \cdots \mathcal{A}_{k+1}[x_{\phi_k}]$ on $[T_0, T_1]$ for $i > k$. Therefore, the sequence x_{ϕ_i} converges uniformly on $[T_0, T_1]$ to a unique limit, which is necessarily the solution $\bar{x}$ (and the convergence of ϕ_{i+1} follows).

This argumentation can be repeated on a time interval $[T_1, T_2]$ for a certain $T_2 > T_1$, and so on, defining an increasing sequence T_j. If T_j converges to $T_\infty \leq T$, one can again apply the same argumentation with T_∞ and $\bar{x}(T_\infty)$ and obtain the existence of an interval $[T_\infty, T'_\infty]$ with $T'_\infty > T_\infty$ on which the convergence of x_{ϕ_i} to $\bar{x}$ is obtained, contradicting the definition of the limit T_∞. Finally, we conclude the uniform convergence of the sequence (x_{ϕ_i}, ϕ_{i+1}) to $(\bar{x}, \bar{\phi})$ on the whole time interval $[0, T]$. □

Remark 4. *This approach is of particular interest when the solution of* (6) *can be easily determined for any given function* $\phi(\cdot)$. *This is typically the case when* g *is linear with respect to* x *and* $h(0) = 0$. *For systems of the form*

$$\dot{x} = Ax + Bh(x)$$

where h is a non-linear (Lipschitz) function and A, B are matrices of adequate dimensions, the solution x_ϕ has the explicit expression

$$x_\phi(t) = e^{At}\bar{x}(0) + \int_0^t e^{A(t-\tau)}B\phi(\tau)\,d\tau \ .$$

As 0 is a solution of the system (i.e. for $x_0 = 0$), one can simply take $\phi_0 = 0$ to initiate the sequence of ϕ_i which converges to the solution for the initial condition $\bar{x}_0$. The iterations $\phi_i = \mathcal{O}^i[0]$, $i = 1, \cdots,$ provides then an alternative method to the classical discretization schemes such as Runge-Kutta.

The approximation scheme we consider here is valid for any pair of initial conditions x_0, $\bar{x}_0$ in Ω. However, our objective in the next Section is to consider initial conditions such that $\bar{x}_0 \succeq x_0$.

3 Comparison of solutions

Our objective here is to give conditions on the map h for the solutions of the system (4) to preserve the partial order $\succeq$. We consider two approaches. The first one is based on a geometric condition for domain invariance, while the second one guarantees the monotonicity of the sequence ϕ_i defined in Section 2.

For $x_0, \bar{x}_0$ in Ω, we shall denote by $x(\cdot)$, $\bar{x}(\cdot)$ the solutions of (4) for the initial conditions $x(0) = x_0$, $\bar{x}(0) = \bar{x}_0$.

3.1 Invariance-based approach

Proposition 1. *Assume Hypotheses H0-H4 are satisfied. If the maps g and h satisfy for any $j = 1 \cdots m$ the condition*

$$x, y \in \Omega;\ y \succeq x,\ h_j(y) = h_j(x) \ \Rightarrow\ D_j(x,y) := \nabla h_j(y).g(y,h(y)) - \nabla h_j(x).g(x,h(x)) \geq 0 \qquad (7)$$

then having $\bar{x}_0 \succeq x_0$ in Ω implies $\bar{x}(t) \succeq x(t)$ for any $t \in [0,T]$.

Proof. We consider the system in Ω^2

$$\begin{aligned} \dot{x} &= g(x,h(x)) \\ \dot{y} &= g(y,h(y)) \end{aligned}$$

and show that the set

$$M := \{(x,y) \in \Omega^2;\ y \succeq x\}$$

is positively invariant. Let $N_M(x,y)$ be the normal cone to M at $(x,y) \in M$ (see for instance [16]). By the intersection formula of normal cones, one has the expression

$$N_M(x,y) = \sum_{k,\ y_k = x_k} \mathbb{R}_+ \begin{pmatrix} e_k \\ -e_k \end{pmatrix} + \sum_{j,\ h_j(y)=h_j(x)} \mathbb{R}_+ \begin{pmatrix} \nabla h_j(x) \\ -\nabla h_j(y) \end{pmatrix},\ (x,y) \in \partial M$$

where e_k denotes the k-th basis vector in $\mathbb{R}^n$ (i.e. with $e_{k,k} = 1$, $e_{k,l} = 0$, $l \neq k$).

For $(x,y) \in \partial M$ such that $y_k = x_k$ for some k, one has

$$\begin{aligned} \delta_k := \begin{pmatrix} e_k \\ -e_k \end{pmatrix} . \begin{pmatrix} g(x,h(x)) \\ g(y,h(y)) \end{pmatrix} &= g_k(x,h(x)) - g_k(y,h(y)) \\ &= \Big(g_k(x,h(x)) - g_k(y,h(x))\Big) + \Big(g_k(y,h(x)) - g_k(y,h(y))\Big) \end{aligned}$$

Let consider the scalar function $\varphi(\lambda) := g_k(x + \lambda(y-x), h(x))$ for $\lambda \in [0,1]$ (which is well defined as Ω is p-convex). One has $\varphi(1) = \varphi(0) + \int_0^1 \varphi'(\lambda)d\lambda$, that is

$$g_k(y,h(x)) = g_k(x,h(x)) + \int_0^1 \sum_l \frac{\partial g_k}{\partial x_l}(x + \lambda(y-x), h(x))(y_l - x_l)d\lambda$$

With $y \geq x$ and $y_k = x_k$, one gets $g_k(y,h(x)) \geq g_k(x,h(x))$ with Assumption H2. From Assumption H3, one has $g_k(y,h(y) \geq g_k(y,h(x))$. Therefore, δ_k is non positive.

For $(x, y) \in \partial M$ such that $h_j(y) = h_j(x)$ for some j, one has

$$\begin{pmatrix} \nabla h_j(x) \\ -\nabla h_j(y) \end{pmatrix} \cdot \begin{pmatrix} g(x, h(x)) \\ g(y, h(y)) \end{pmatrix} = \nabla h_j(x).f(x) - \nabla h_j(y).f(y) = -D_j(x, y)$$

which is non positive under condition (7). Therefore, the inward pointing condition

$$\nu. \begin{pmatrix} g(x, h(x)) \\ g(y, h(y)) \end{pmatrix} \leq 0, \ \nu \in N_M(x, y), \ (x, y) \in \partial M$$

is verified for any $(x, y) \in \partial M$, which implies that M is positively invariant (see for instance [19, Chap. 10, XV]). □

Let us make some comments about the condition (7) of Proposition 1. It has some similarities with condition (3) on the boundary of the cone, but is adapted to the context of a partial order that is not induced by a cone. It is a global condition and not a local one, which is the price to pay for a partial order that is not induced by a cone. However, there are only m scalar conditions to check, because the cooperative property of g with respect to x is already exploited. Note that the general formulation (5) allows to have terms $h_j(x)$ $(j = 1 \cdots m)$ in common in several equations of the dynamics, and is thus of particular interest when the number m is relatively small (even when the partial order $\succeq$ defined from h is induced by a cone, this condition can be convenient to check compared to condition (3)). On practical problems, this condition can be quite easy to check, while checking the cooperative property of g with its Jacobian matrix is often straightforward (see examples in Section 4).

3.2 Monotonicity of the sequence ϕ_i

We exploit here the property of monotone control systems (cf Remark 2) to show that the monotonicity of the sequence ϕ_i provides the order preservation of the solutions.

Lemma 2. *Assume Hypotheses H0-H3 are satisfied. If $\bar{x}_0 \geq x_0$ and the sequence ϕ_i is non decreasing i.e. $\phi_{i+1}(t) \geq \phi_i(t)$ (component-wise) for any $t \in [0, T]$ and $i = 0, 1, \cdots$ then one has*

$$\bar{x}(t) \succeq x(t), \ t \in [0, T].$$

Proof. Remark first that the solution $x(\cdot)$ is also solution of the non autonomous system

$$\dot{x} = g(x, \phi_0(t)) \tag{8}$$

for the initial condition $x(0) = x_0$.

Let $x_{\phi_0}(\cdot)$ be the solution of (8) for the initial condition $x(0) = \bar{x}_0$. From Hypothesis H2, the non-autonomous dynamics (8) is cooperative and consequently one has $x_{\phi_0}(t) \geq x(t)$ for any $t \in [0, T]$.

Let $x_{\phi_1}(\cdot)$ be the solution of

$$\dot{x} = g(x, \phi_1(t)) \tag{9}$$

for the initial condition $x(0) = \bar{x}_0$. If $\phi_1 \geq \phi_0$, then by Hypothesis H3, one has $g(x, \phi_1(t)) \geq g(x, \phi_0(t))$ for any $x \in \Omega$ and $t \in [0, T]$. Let us recall that the solutions of two differential equations, whose right-hand sides are component-wise ordered, and one of the equations is cooperative, are ordered (see for instance [18]). The dynamics (8) being cooperative, one then gets $x_{\phi_1}(t) \geq x_{\phi_0}(t)$ for any $t \in [0, T]$.

Recursively, one obtains $x_{\phi_{i+1}}(t) \geq x_{\phi_i}(t)$ for $i = 1, \cdots$ and $t \in [0, T]$. Then one gets from Lemma 1

$$\bar{x}(t) = \lim_{i \to +\infty} x_{\phi_i}(t) \geq x_{\phi_0}(t) \geq x(t), \ t \in [0, T].$$

In the same way, one has

$$\bar{\phi}(t) = h(\bar{x}(t)) = \lim_{i \to +\infty} \phi_i(t) \geq \phi_0(t) = h(x(t)), \ t \in [0, T].$$

□

Remark 5. *$\bar{x}_0 \geq x_0$ and ϕ_i non decreasing imply $h(\bar{x}_0) = \bar{\phi}(0) \geq \phi_0(0) = h(x_0)$, i.e. one has necessarily $\bar{x}_0 \succeq x_0$.*

We give now sufficient conditions to obtain a non-decreasing sequence of functions ϕ_i.

Proposition 2. *Assume Hypotheses H0-H4 are satisfied. If the maps g and h satisfy for any $j = 1\cdots m$ the condition*

$$\begin{array}{l} x, y \in \Omega;\ y \succeq x,\ z \in h(\Omega), h(x) \geq z, \\ \qquad h_j(y) = h_j(x) \ \Rightarrow\ D_j^-(x,y,z) := \nabla h_j(y).g(y,h(x)) - \nabla h_j(x).g(x,z) \geq 0 \end{array} \tag{10}$$

then having $\bar{x}_0 \succeq x_0$ in Ω implies $\bar{x}(t) \succeq x(t)$ for any $t \in [0,T]$, and the sequence of functions ϕ_i is non decreasing.

Proof. Let us consider the set

$$S := \{(x,y) \in \Omega^2,\ h(y) \geq h(x)\}.$$

Under Hypothesis H4, the normal cone N_S to S (see for instance [16]) verifies

$$N_S(x,y) = \sum_{j,\ h_j(y)=h_j(x)} \mathbb{R}_+ \begin{pmatrix} \nabla h_j(x) \\ -\nabla h_j(y) \end{pmatrix},\ (x,y) \in \partial S.$$

We proceed recursively to show that the sequence ϕ_i is non decreasing (component-wise).

For $i = 0$, one has $x_{\phi_0}(t) \geq x(t)$ for any $t \in [0,T]$ (see the proof of Lemma 2). Note that one has $h(x_{\phi_0}(0)) = h(\bar{x}_0) \geq h(x_0) = h(x(0))$ i.e. $(x(0), x_{\phi_0}(0)) \in S$. We show that $(x(\cdot), x_{\phi_0}(\cdot))$ remains in S, as a solution of the dynamics

$$\left\{ \begin{array}{l} \dot{x} = g(x, h(x)) \\ \dot{y} = g(y, h(x)) \end{array} \right.$$

such that $y(t) \geq x(t)$ for $t \in [0,T]$. At $(x,y) \in \partial S$ with $h_j(y) = h_j(x)$ for some $j \in \{1, \cdots, m\}$, one has

$$\begin{array}{rcl} \begin{pmatrix} \nabla h_j(x) \\ -\nabla h_j(y) \end{pmatrix} \cdot \begin{pmatrix} g(x,h(x)) \\ g(y,h(x)) \end{pmatrix} & = & \nabla h_j(x).g(x,h(x) - \nabla h_j(y).g(y,h(x)) \\ & = & -D_j^-(x,y,h(x)). \end{array}$$

Under condition (10), one obtains the inward pointing property

$$y \geq x,\ (x,y) \in \partial S\ \Rightarrow\ v.\begin{pmatrix} g(x,y) \\ g(y,h(x)) \end{pmatrix} \leq 0,\ v \in N_S(x,y)$$

which implies that the set S is invariant by $(x(\cdot), x_{\phi_0}(\cdot))$ (see for instance [19, Chap. 10, XV]). We deduce that $\phi_1(t) - \phi_0(t) = h(x_{\phi_0}(t)) - h(x(t)))$ remains non-negative for any $t \in [0,T]$.

Assume now that one has $x_{\phi_i}(t) \geq x_{\phi_{i-1}}(t)$ and $\phi_{i+1}(t) \geq \phi_i(t)$ for any $t \in [0,T]$ and $i \leq n$.

For $i = n+1$, the recurrence property $\phi_{n+1}(.) \geq \phi_n(.)$ with $x_{\phi_{n+1}}(0) = x_{\phi_n}(0) = \bar{x}_0$ implies as before, from the properties of the map g (Hypotheses H2 and H3), that one has $x_{\phi_{n+1}}(t) \geq x_{\phi_n}(t)$ for any $t \in [0,T]$. Note that one has $h(x_{\phi_{n+1}}(0)) = h(\bar{x}_0) = h(x_{\phi_n}(0))$ i.e. $(x_{\phi_n}(0), x_{\phi_{n+1}}(0))$ belongs to S. Note that $(x_{\phi_n}(\cdot), x_{\phi_{n+1}}(\cdot))$ is a solution of the dynamics

$$\left\{ \begin{array}{l} \dot{x} = g(x, \phi_n(t)) = g(x, h(x_{\phi_{n-1}}(t))) \\ \dot{y} = g(y, h(x)) \end{array} \right.$$

such that $y(t) \geq x(t)$ and $h(x(t)) - h(x_{\phi_{n-1}}(t)) = \phi_{n+1}(t) - \phi_n(t) \geq 0$ for $t \in [0,T]$. As previously, we write

$$\begin{array}{rcl} \begin{pmatrix} \nabla h_j(x) \\ -\nabla h_j(y) \end{pmatrix} \cdot \begin{pmatrix} g(x,\phi_n(t)) \\ g(y,h(x)) \end{pmatrix} & = & \nabla h_j(x).g(x,h(x_{\phi_{n-1}}(t))) - \nabla h_j(y).g(y,h(x)) \\ & = & -D_j^-(x,y,h(x_{\phi_{n-1}}(t))) \end{array}$$

For $y \geq x$ and $t \in [0,T]$ such that $h(x) \geq h(x_{\phi_{n-1}}(t))$ and $(x,y) \in \partial S$ with $h_j(y) = h_j(x)$ for some $j \in \{1, \cdots, m\}$, one has $D_j(y,x,h(x_{\phi_{n-1}}(t)) \geq 0$ from condition (10). One has then, as before, the inward pointing property

$$y \geq x,\ h(x) \geq h(x_{\phi_{n-1}}(t)),\ (x,y) \in \partial S\ \Rightarrow\ v.\begin{pmatrix} g(x,\phi_n(t)) \\ g(y,h(x)) \end{pmatrix} \leq 0,\ v \in N_S(x,y)$$

from which we deduce that the solution $(x_{\phi_n}(\cdot), x_{\phi_{n+1}}(\cdot))$ remains in S, and thus the function $\phi_{n+2}(\cdot) - \phi_{n+1}(\cdot) = h(x_{\phi_{n+1}}(\cdot)) - h(x_{\phi_n}(\cdot))$ is non-negative. The recurrence property is thus proved.

Finally, the result follows from Lemma 2. $\square$

Remark 6. *Analogously, one obtains under Hypotheses H0-H4 a non-increasing sequence of functions* ϕ_i *with* $x_0 \succeq \bar{x}_0$ *when the maps* g, h *satisfy for any* $j = 1 \cdots m$ *the condition*

$$\begin{array}{l} x, y \in \Omega;\ y \succeq x,\ z \in h(\Omega),\ z \geq h(y) \\ \quad h_j(y) = h_j(x) \ \Rightarrow\ D_j^+(x,y,z) := \nabla h_j(y).g(y,z) - \nabla h_j(x).g(x,h(y)) \geq 0 \end{array} \tag{11}$$

(the proof is similar and left to the reader).

We notice that conditions (10) or (11) are similar to condition (7) but are more demanding. It allows to obtain guaranteed approximations of the solution from below or from above, or both providing a guaranteed frame of the solution in the spirit of intervals computing [13]. In the next sub-section, we focus on a class of systems for which these three conditions are equivalent.

3.3 The separate case

We consider here a class of functions f for which we can derive simpler conditions to check. This is illustrated in the example of Section 4.1.

Definition 2. *The formulation* (5) *is in a separate form if there exists maps* $\tilde{g}$, $\tilde{h}$ *from* Ω *to* $\mathbb{R}^n$ *such that*

$$f(x) = \tilde{g}(x) + \tilde{h}(x),\ x \in \Omega \tag{12}$$

where $\tilde{h}$ *has* m *non-identical null components* $\tilde{h}_j$ *for* j *in a subset* J *of* $\{1, \cdots, n\}$.

Then, we replace the former hypotheses H0-H5 by the following ones.

H0s. The maps $\tilde{g}$ and $\tilde{h}$ are $\mathcal{C}^1$ on Ω.

H1s. For any continuous function $\phi : \mathbb{R}_+ \mapsto h(\Omega)$, Ω is positively invariant by the non-autonomous system

$$\dot{x} = \tilde{g}(x) + \phi(t),\ t \geq 0,\ x \in \Omega$$

which is forward complete.

H2s. Ω is p-convex and the map $\tilde{g}$ satisfies Kamke's condition on Ω.

H4s. For any $j \in J$, $\tilde{h}_j$ has no critical point in Ω.

We have the following conditions to ensure monotonicity with respect to $\succeq$.

Corollary 1. *Assume Hypotheses H0s-H2s and H4s are satisfied when* f *is written in the form* (12). *If the maps* $\tilde{g}$ *and* $\tilde{h}$ *satisfy the property*

$$x, y \in \Omega,\ y \succeq x,\ \tilde{h}_j(y) = \tilde{h}_j(x) \ \Rightarrow\ \frac{\partial \tilde{h}_j}{\partial x_k}(y) \geq \frac{\partial \tilde{h}_j}{\partial x_k}(x) \geq 0,\ j,k \in J \tag{13}$$

then, the conclusions of Propositions 1, 2 hold when the condition

$$x, y \in \Omega,\ y \succeq x,\ \tilde{h}_j(y) = \tilde{h}_j(x) \ \Rightarrow\ \tilde{D}_j(x,y) := \nabla \tilde{h}_j(y).\tilde{g}(y) - \nabla \tilde{h}_j(x).\tilde{g}(x) \geq 0,\ j \in J \tag{14}$$

is verified.

Proof. For $j \in J$, the function D_j defined in (7) of Proposition 1 coincides with $\tilde{D}_j$. The function D_j^- defined in (10) of Proposition 2 can be written as

$$D_j^-(x,y,z) := \left[\nabla \tilde{h}_j(y).\tilde{g}(y) - \nabla \tilde{h}_j(x).\tilde{g}(x)\right] + \left[\nabla \tilde{h}_j(y).\tilde{h}(y) - \nabla \tilde{h}_j(x).z\right].$$

The expression in the first brackets is non-negative under condition (14), while the second verifies under condition (13)

$$\nabla \tilde{h}_j(y).\tilde{h}(y) - \nabla \tilde{h}_j(x).z = \sum_{k \in J} \frac{\partial \tilde{h}_j}{\partial x_k}(y).\tilde{h}_k(y) - \frac{\partial \tilde{h}_j}{\partial x_k}(x).z_k \geq \sum_{k \in J} \frac{\partial \tilde{h}_j}{\partial x_k}(x).(\tilde{h}_k(y) - z_k) \geq 0.$$

For $j \notin J$, one has clearly $D_j(x,y) = 0$ and $D_j^-(x,y,z)$=0. Conditions (7), (10) of Propositions 1, 2 are thus fulfilled. $\square$

In a similar way, one can check that conditions of Corollary 1 imply also the condition (11). Condition (13) can be interpreted as a kind of matching condition: the functions $\tilde{h}_j$ have to be non-decreasing only with respect to variables x_k whose dynamics does not satisfy the cooperativity condition (i.e. $\frac{\partial f_k}{\partial x_l}(x) < 0$ for some $l \neq k$ and $x \in \Omega$).

Remark 7. *When the map $\tilde{h}$ is linear $h(x) = Ax$, the conditions of Corollary 1 take the simple form*

$$\begin{aligned} &A_{jk} \geq 0,\ j,k \in J, \\ &x,y \in \Omega,\ y \geq x,\ A_j(x-y) \geq 0 \ \Rightarrow\ \tilde{D}_j(x,y) = A_j(\tilde{g}(y) - \tilde{g}(x))) \geq 0,\ j \in J. \end{aligned}$$

This is illustrated on the example of Section 4.1.

4 Examples

We present first a biological example, that has initially motivated this work, for which we have a separated dynamics and a cone-induced order. Then, we present a more sophisticated example with non separated dynamics and non cone-induced order.

4.1 Separated dynamics and cone-induced order

We consider a model of anaerobic digestion [11] that describes the conversion of organic matter X into substrate S assimilable by bacteria B, in a landfill

$$\left\{ \begin{array}{rcl} \dot{X} & = & -K_h X + \alpha K_d B \\ \dot{S} & = & f_1 K_h X - \frac{1}{Y}\mu(S)B \\ \dot{B} & = & (\mu(S) - K_d)B \end{array} \right.$$

where parameters K_h, K_d are in $\mathbb{R}_+$ and α, Y, f_1 in $(0,1)$. The growth function μ is C^1 increasing with $\mu(0) = 0$. We refer the reader to [11] for more details about the biological meaning of these assumptions. The originality of this model is to have a re-circulation of dead biomass into organic matter. The biogas produced between times 0 and t is proportionate to the quantity.

$$G(t) = \int_0^t \mu(S(\tau))B(\tau)d\tau.$$

This model is clearly non cooperative. However, one observes in simulations

- when increasing initial conditions, the corresponding solutions are ordered in X and B but not necessarily in S,
- when increasing initial organic matter only, the biogas production at any time $t > 0$ increases.

To show these properties as a consequence of the preservation of a partial order $\succeq$ (as defined in Definition 1), we have to consider a set of coordinates with X, B and replace variable S. With $Z = B + YS$, the model writes in the $\xi := (X, Z, B)^\top$ coordinates

$$\left\{ \begin{array}{rcl} \dot{X} & = & -K_h X + \alpha K_d B \\ \dot{Z} & = & Y f_1 K_h X - K_d B \\ \dot{B} & = & (\mu(\frac{Z-B}{Y}) - K_d)B \end{array} \right.$$

on the invariant domain $\Omega = \{\xi \in \mathbb{R}^3_+;\ Z \geq B)\}$. This model is not cooperative either, but we look now for a partial order $\succeq$ preserved by the system. Our method consists in isolating terms that prevent the system to be cooperative and check if conditions of Proposition 1 or 2, or Corollary 1, are fulfilled. Here, the model can be written in the separate form $\dot{\xi} = \tilde{g}(\xi) + \tilde{h}(\xi)$ with

$$\tilde{g}(\xi) = \begin{pmatrix} -K_h X + \alpha K_d B \\ -K_d Z \\ (\mu(\frac{Z-B}{Y}) - K_d)B \end{pmatrix},\ \tilde{h}(\xi) = \begin{pmatrix} 0 \\ Y f_1 K_h X + K_d(Z - B) \\ 0 \end{pmatrix}$$

One can straightforwardly check that conditions of Corollary 1 are satisfied with $J = \{2\}$ and that one has $\partial_Z \tilde{h}_2 > 0$. Take now $\bar{\xi}$, ξ in Ω with $\bar{\xi} \geq \xi$ and $\tilde{h}_2(\bar{\xi}) = \tilde{h}_2(\xi)$ (that is with $Yf_1K_h(\bar{X} - X) + K_d(\bar{Z} - Z - (\bar{B} - B)) = 0$), and write

$$\begin{array}{rcl} \tilde{D}_2(\xi, \bar{\xi}) & = & -Yf_1K_h^2(\bar{X} - X) + \alpha Y f_1 K_h K_d(\bar{B} - B) - K_d^2(\bar{Z} - Z) \\ & & \qquad -K_d(\mu(\bar{S})\bar{B} - \mu(S)B) + K_d^2(\bar{B} - B) \\ & \geq & Yf_1K_h(K_d - K_h)(\bar{X} - X) + K_d(\alpha Y f_1 K_h - \mu(S))(\bar{B} - B) \end{array}$$

(where $\bar{S} = S + \frac{1}{Y}(\bar{Z} - Z - (\bar{B} - B)) = S - f_1K_h(\bar{X} - X) < S$). Then, a sufficient condition to have condition (14) of Corollary 1 to be satisfied is to have

$$K_d \geq K_h \text{ and } \max_{s>0} \mu(s) \leq \alpha Y f_1 K_h. \tag{15}$$

Concerning the biogas production, one gets from the equations of the model

$$G(t) = B(t) - B(0) + K_d \int_0^t B(\tau)d\tau.$$

Therefore, if one has $\bar{X}(0) > X(0)$ with $\bar{S}(0) = S(0)$ and $\bar{B}(0) = B(0)$, one has $\bar{\xi}(0) \succeq \xi(0)$ and thus $\bar{\xi}(t) \geq \xi(t)$ for any $t > 0$ (with $\bar{B}(\cdot)$, $B(\cdot)$ non identical), which implies $\bar{G}(t) > G(t)$ for any $t > 0$.

Let us now illustrate the preservation of this partial order on numerical simulations for parameters given in Table 1 For these values of the parameters, one can straightforwardly check that condition (15)

K_h	K_d	α	Y	f_1	$\mu(S)$
0.176	0.18	0.9	0.9	0.8	$\dfrac{0.1S}{160+S}$

Table 1: Parameters values for Example 4.1

is fulfilled. We considered the three initial conditions

$$(X_0, S_0, B_0) = (3, 50, 100) \leq (\bar{X}_0^a, \bar{S}_0^a, \bar{B}_0^a) = (5, 50, 200) \leq (\bar{X}_0^b, \bar{S}_0^b, \bar{B}_0^b) = (5, 51, 300).$$

In the ξ-coordinates, one has

$$\xi_0 = (3, 145, 100) \leq \bar{\xi}_0^a = (5, 245, 200) \leq \bar{\xi}_0^b = (5, 345.9, 300)$$

and

$$\tilde{h}_2(\xi_0) = 8.48016 \leq \tilde{h}_2(\bar{\xi}_0^a) = 8.7336 \leq \tilde{h}_2(\bar{\xi}_0^a) = 8.8956$$

Therefore, one has $\xi_0^b \succeq \bar{\xi}_0^a \succeq \bar{\xi}_0$, and one can see on Figure 1 that the solutions are ordered in the (X, Z, B)-coordinates, as expected (but they are not ordered for the variable S).

4.2 Non separated dynamics and non cone-induced order

Consider the system on the domain $\Omega = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$

$$\left\{ \begin{array}{l} \dot{x}_1 = (-\alpha + x_1 + 2x_2 - x_3^2)x_1 + \beta x_3^2 \\ \dot{x}_2 = \gamma x_1 - \delta x_3^2 \\ \dot{x}_3 = \left(1 - e^{-x_2 + x_3^2}\right)x_3 - \delta x_3 \end{array} \right. \tag{16}$$

where α, β, δ, γ are positive parameters. One can easily check that the set Ω is forward invariant by this dynamics. The Jacobian matrix is written as

$$\begin{pmatrix} \star & 2x_1 & 2(\beta x_3 - x_1) \\ \gamma & \star & -2\delta x_3 \\ 0 & e^{-x_2 + x_3}x_3 & \star \end{pmatrix}$$

and thus the system is not monotone on Ω. Let us write now the system as follows

$$\left\{ \begin{array}{l} \dot{x}_1 = -\alpha x_1 + (1 - \frac{\gamma}{\delta})x_1^2 + \beta x_3^2 + \frac{1}{\delta}h(x)x_1 \\ \dot{x}_2 = -2\delta x_2 + h(x) \\ \dot{x}_3 = \left(1 - e^{-x_2 + x_3^2}\right)x_3 - \delta x_3 \end{array} \right.$$

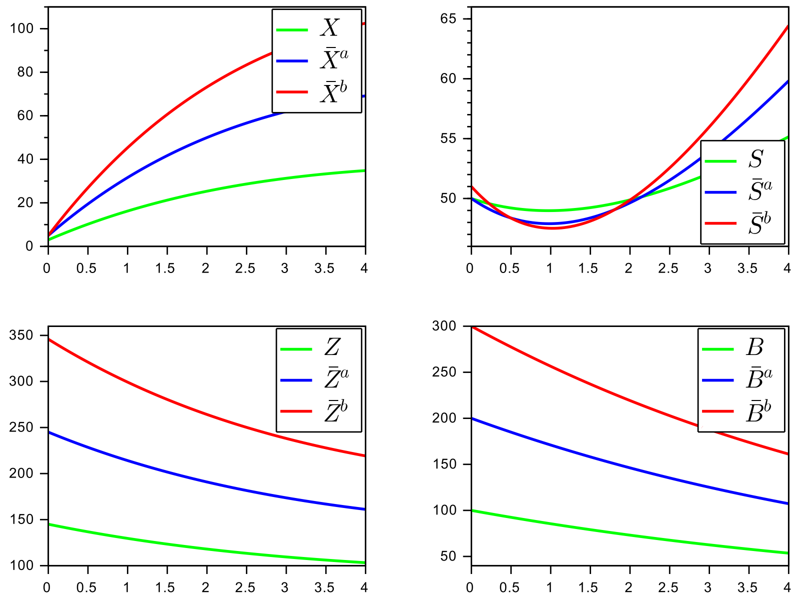

Figure 1: Comparison of some solutions in Example 4.1

where

$$h(x) = \gamma x_1 + \delta(2x_2 - x_3^2).$$

This dynamics is in the form (5) but is not separated. We thus consider the non-autonomous dynamics

$$\dot{x} = g(x, \phi(t)) = \begin{pmatrix} -\alpha x_1 + (1 - \frac{\gamma}{\delta})x_1^2 + \beta x_3^2 + \frac{1}{\delta}\phi(t)x_1 \\ -2\delta x_2 + \phi(t) \\ \left(1 - e^{-x_2 + x_3^2}\right)x_3 - \delta x_3 \end{pmatrix}.$$

One can straightforwardly check that set Ω is also forward invariant whatever is the function ϕ and that the system is cooperative and monotone in ϕ on Ω. Moreover, the gradient of h is always non null. Hypotheses H0-H1-H2-H3-H4 are fulfilled. Note that for this function h, the partial order $\succeq$ given in Definition 1 is not induced by a cone (because $h(x) \geq 0$ does not necessarily implies $h(\tau x) \geq 0$ for any $\tau \geq 0$). To apply Proposition 2, we have to consider for x, y in Ω, $z \in h(\Omega)$ with $y \geq x$ and $h(y) = h(x) \geq z$ the quantity

$$\begin{aligned} D^-(x,y,z) &= \frac{\partial h}{\partial x}(y).g(y, h(x)) - \frac{\partial h}{\partial x}(x).g(x,z) \\ &= \gamma\big(- \alpha(y_1 - x_1) + (1 - \tfrac{\gamma}{\delta})(y_1^2 - x_1^2) + \beta(y_3^2 - x_3^2) + \tfrac{1}{\delta}(h(y)y_1 - z x_1)\big) \\ &+ \ 2\delta\big(- 2\delta(y_2 - x_2) + h(y) - z\big) - 2\delta\Big(\big(1 - e^{-y_2 + y_3^2} - \delta\big)y_3^2 - \big(1 - e^{-x_2 + x_3^2} - \delta\big)x_3^2\Big). \end{aligned}$$

Note that $h(y) = h(x)$ implies

$$2\delta(y_2 - x_2) = \delta(y_3^2 - x_3^2) - \gamma(y_1 - x_1) \text{ and } (y_3^2 - y_2) - (x_3^2 - x_2) = \frac{\gamma}{\delta}(y_1 - x_1) + y_2 - x_2 \geq 0$$

from which one gets (using also $h(y) \geq z$),

$$\begin{aligned} D-(x,y,z) &\geq \gamma\big(- \alpha(y_1 - x_1) + (1 - \tfrac{\gamma}{\delta})(y_1^2 - x_1^2) + \beta(y_3^2 - x_3^2)) + \tfrac{z}{\delta}(y_1 - x_1)\big) \\ &+ \ 2\delta\big(- \delta(y_3^2 - x_3^2) + \gamma(y_1 - x_1)\big). \end{aligned}$$

Then, by rearranging terms, one obtains the inequality

$$D^-(x,y,z) \geq \gamma(2\delta - \alpha + \tfrac{z}{\delta})(y_1 - x_1) + \gamma(1 - \tfrac{\gamma}{\delta})(y_1^2 - x_1^2) + (\gamma\beta - 2\delta^2)(y_3^2 - x_3^2).$$

On the other hand, a straightforward computation gives

$$h(x) = 0 \Rightarrow \frac{\partial h}{\partial x}(x).g(x, h(x)) \geq \gamma(2\delta - \alpha)x_1 + \gamma(1 - \tfrac{\gamma}{\delta})x_1^2 + (\gamma\beta - 2\delta)x_3^2.$$

Therefore, when the parameters satisfy the inequalities

$$\gamma\beta \geq 2\delta \geq \alpha \text{ and } 1 \geq \delta \geq \gamma \tag{17}$$

the set $\tilde{\Omega} = \{x \in \Omega,\ h(x) \geq 0\}$ is forward invariant and one has $D(x, y, z) \geq 0$ for $z \in h(\tilde{\Omega})$. The conclusions of Proposition 2 follow when applied to $\tilde{\Omega}$.

We have run numerical simulations for values of the parameters $\alpha = 0.8$, $\beta = 3$, $\delta = 0.5$, $\gamma = 0.4$ (for which one can check that condition (17) is satisfied) and the three initial conditions $x_0 = (0.1, 0, 1, 0, 2)^\top$, $\bar{x}_0^a = (0.1, 0.1, 0.5)^\top$, $\bar{x}_0^b = (0.1, 0.2, 0.3)^\top$. One computes $h(x_0) = 0.2$, $h(\bar{x}_0^a) = 0.095$, $h(\bar{x}_0^b) = 0.275$. Clearly, x_0, $\bar{x}_0^a$, $\bar{x}_0^b$ belong to $\tilde{\Omega}$ with $\bar{x}_0^a \geq x_0$ and $\bar{x}_0^b \geq x_0$. However, corresponding solutions $x(\cdot)$, $\bar{x}^a(\cdot)$ are not ordered with respect to the $\succeq$ order, while $x(\cdot)$, $\bar{x}^b(\cdot)$ are (see Figure 2). This is in accordance with Proposition 2 because one has $\bar{x}_0^b \succeq x_0$ but not $\bar{x}_0^a \succeq x_0$.

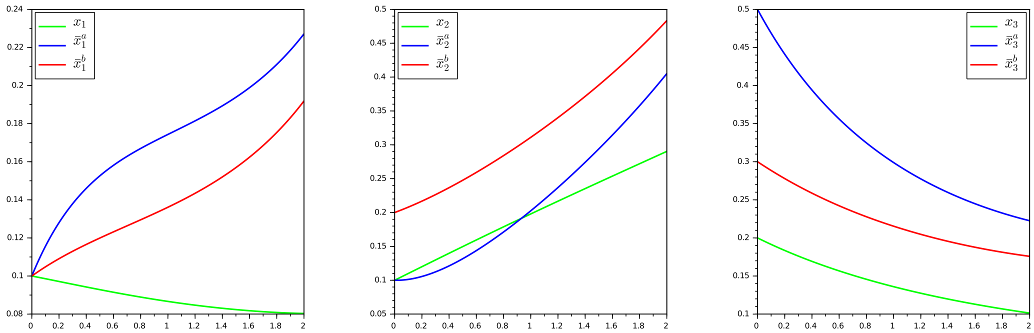

Figure 2: Comparison of solutions $x(\cdot)$, $\bar{x}^a(\cdot)$, $\bar{x}^b(\cdot)$) in Example 4.2

Let us now illustrate the benefit of the guaranteed estimation from below provided by the sequence (x_{ϕ_i}, ϕ_{i+1}) as defined in Section 2. Clearly $x_0 = (0, 0, 0)^\top$ is solution of (16), with $\phi_0 = 0$. Then, for any initial condition $\bar{x}_0 \succeq 0$, one can provide an approximation from below of the solution $\bar{x}(\cdot)$ with the (x_{ϕ_i}, ϕ_{i+1}) sequence. For instance, for $\bar{x}_0 = x_0^a$, Figure 3 depicts the iterations (x_{ϕ_i}, ϕ_{i+1}). As expected, one can see on this figure the non-decreasing behavior and the convergence of this sequence. With only 12 iterations, one obtains an approximation (from below) of ϕ with quite a good accuracy: $\left\|\bar{\phi} - \phi_{10}\right\|_T = 0.0016$ for $T = 2$.

5 Conclusion

In this note, we have provided conditions for a partial order stronger than the usual vector order in $\mathbb{R}^n$ to be preserved by the flow of a dynamical system. We have shown that this property can be related to the monotonic behavior of some Picard iterations, which does not require the partial order to be necessarily induced by a cone. Our approach is based on a separation of terms that satisfy Kamke's condition from other ones, in the expression of the dynamics. For practical problems, it might be difficult to find a partial order for which the dynamics exhibits a monotonicity property. Our approach facilitates this search, as we have shown with examples. However, several ways of making such a separation are possible, and we do not know a systematic way to do it. The study of the best way, in terms of the less restrictive conditions or the largest partially order subset in $\mathbb{R}^n$, could be the matter of a future work. Extensions of this work with respect to other vector order in $\mathbb{R}^n$, and for the design of interval observers, could be also subjects of investigation.

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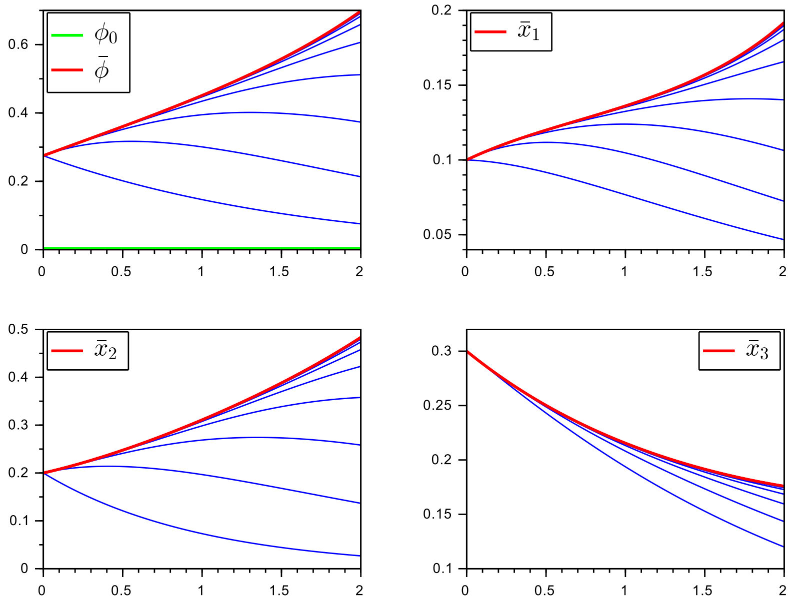

Figure 3: Illustration of the convergence of the sequence (x_{ϕ_i}, ϕ_{i+1}) (in blue) in Example 4.2

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