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**Jean-Marc Azaïs**[1], **Federico Dalmao** [2] **and Céline Delmas**[3]

[1] *IMT, Université de Toulouse, France. , e-mail:* jean-marc.azais@math.univ-toulouse.fr

[2] *DMEL, CENUR Litoral Norte, Universidad de la República, Uruguay , e-mail:* fdalmao@unorte.edu.uy

[3] *MIAT, INRAE, Université de Toulouse, France. , e-mail:* celine.delmas.toulouse@inrae.fr

**Abstract:** We prove a multivariate central limit theorem for the numbers of critical points above a level with all possible indexes of a non-necessarily isotropic Gaussian random field. In particular, we discuss the non-degeneracy of the limit variance-covariance matrix. We extend, to the non-isotropic framework, known results by Estrade & León and Nicolaescu for the Euler characteristic of an excursion set and for the total number of critical points of Gaussian random fields. Furthermore, we deduce the almost sure convergence of the normalized (by its mean) number of critical points above a level with any given index and, in particular, of the Euler characteristic of an excursion set. Though we use the classical tools of Hermite expansions and Fourth Moment Theorem, our proof of the non-degeneracy of the limit variance-covariance matrix is completely new since we need to consider all chaotic terms.

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## 1. Introduction

The behavior of critical points of random fields has been widely studied from different points of view. The first seminal works date back to the papers by Nosko [30, 31], Lindgren [28], Belyaev and Piterbarg [18, 19], Hasofer [24], the books by Adler [1] and by Piterbarg [37]. They mainly focus on local maxima of Gaussian random fields, their positions and their mean number. In particular, a one-term approximation for the mean number of local maxima above a high level is given by Hasofer [24] and Adler [1] for a stationary Gaussian field. A more accurate asymptotic expansion is then obtained by Delmas [20] and Azaïs and Delmas [10], who also give a good approximation for the distribution of the maximum of a zero-mean stationary Gaussian field. The main applications of such results concern the detection of peaks in a random field. For example, there are works on the detection of activation zones in the human brain [40, 43, 44], on applications in astronomy [41, 42], and on the detection of genes on a chromosome [9, 12, 38]. Local maxima of random fields are also used to define a crest in the modeling of random sea waves, see [14], [15], Ch.11 and references therein.

More recent works study the spreadof the critical points of a Gaussian random field. In particular, it is interesting to know if there is attraction or repulsion among them. The answer can depend on the indexes, i.e., on the number of negative eigenvalues of the Hessian, of the considered critical points. The seminal work by Beliaev, Cammarota and Wigman [16] considers the particular case of the *Random Plane Wave*, a popular Gaussian field defined on $\mathbb{R}^2$. These results have been extended to more general planar random fields in [17] and [27] and to any dimension in [11].

Another direction is to study the number of critical points above some level and with the given indexes. This is the object of the papers by Auffinger, Ben Arous and Černý [8] and by Auffinger and Ben Arous [7], where an exponential behavior of the number of critical points is computed as a function of the index and of the level. These results have direct applications to the minimization of likelihood functions in large dimension since there is a critical region (for the level) in which only minima can be found (at the logarithmic scale).

The last direction is to establish Central Limit Theorems on a family of growing sets. The first paper on this is by Estrade and León [23] and considers the Euler characteristic of the excursion set above the level $u$ of a real-valued isotropic Gaussian random field over a set $\mathcal{T} \subset \mathbb{R}^d$. This quantity is approximated by an alternate sum of the numbers $\mathrm{Crt}_u^k(\mathcal{T})$ of critical points with index $k$ above $u$ in $\mathcal{T}$. More precisely, the modified Euler characteristic is defined by

$$\Phi(\mathcal{T}) := \sum_{k=0}^{d} (-1)^{d-k} \mathrm{Crt}_u^k(\mathcal{T}). \tag{1}$$

The paper uses the Hermite representation of the number of critical points and the Fourth Moment Theorem [33, 34]. In another paper, Nicolaescu [29] studies the total number of critical points

$$\sum_{k=0}^{d} \mathrm{Crt}_{-\infty}^k(\mathcal{T})$$

and establishes a central limit theorem using analogous methods.

Our main aim is to consider the multivariate point of view by studying the asymptotic joint distribution of the numbers of critical points with index $k = 0, 1, \ldots, d$ above a level $u$ in $\mathcal{T}$

$$\big(\mathrm{Crt}_u^0(\mathcal{T}), \ldots, \mathrm{Crt}_u^d(\mathcal{T})\big),$$

and to establish the multivariate central limit theorem. In addition, we check if the limit distribution is degenerated or not depending on whether or not $u = -\infty$. This last point is the most difficult and the most original part of the proof. While classical works [23, 29] prove the positivity of the asymptotic variance using a particular chaos, say that of order 1, 2 or 4, our proof uses a completely new method, since in order to prove the asymptotic positivity of the variance of some linear combination of the numbers of critical points with index $k = 0, 1, \ldots, d$, we use **all the Wiener chaos**. Besides, our proofs are given under weaker assumptions. In particular, we prove that the CLT holds true for general Gaussian stationary random fields and not only for isotropic ones as in other works ([23] or [29]). Actually, a careful reading of these papers shows that isotropy is used to state the finiteness and continuity w.r.t. the level of the second moment of the number of critical points or some related quantity. Here, we avoid the use of isotropy (see Proposition 2.4 below). As a corollary, we prove the almost sure convergence

$$\frac{\mathrm{Crt}_u^k(\mathcal{T})}{\mathbb{E}(\mathrm{Crt}_u^k(\mathcal{T}))} \to 1 \text{ for all k.}$$

The organisation of the paper is as follows. We present the general framework and the main theorem in Section 2. In Section 3, we introduce chaotic expansions, one of our main tools. Section 4 is dedicated to the proof of the main result. Finally, Appendix 4.4 presents auxiliary material and some postponed proofs.

**Notation:** N. D. means *non-degenerate*, $\phi_k$ is the standard normal density in $\mathbb{R}^k$, $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$, $[q] = 1, \ldots, q$, $\simeq$ is the equivalence of functions, $\mathcal{F}$ is the Fourier transform, Var is the variance-covariance matrix of a random vector, Cov is the covariance matrix between two random vectors, we chose some norms that denoted $\|\cdot\|$ as the spaces of matrices and 3 and 4 order tensors.

## 2. Generalities and main result

Consider a real valued random field $X(\cdot)$ defined on $\mathbb{R}^d$, $d \geq 1$. We write $X'(\cdot)$ as the gradient of $X(\cdot)$ and $X''(\cdot)$ as its Hessian matrix. We write $r(\cdot)$ as the covariance function of $X(\cdot)$, that is, $r(t) = \mathbb{E}(X(0)X(t)), t \in \mathbb{R}^d$. We consider the following assumptions.

(A1) : The random field $X(\cdot)$ is centered, stationary and Gaussian with $C^2$ paths, and $\mathrm{Var}(X'(t))$ is N.D.

(A2) : **Geman's condition.** For sufficiently small $B \subset \mathbb{R}^d$, $\mathbb{E}\big((N(v,B))^2\big)$ is continuous as a function of $v$ at zero, where

$$N(v,B) := \#\{t \in B : X'(t) = v\}. \tag{2}$$

(A3) : **Arcones' condition.** Defining

$$\Psi(t) := \max\left\{\Big|\frac{\partial^j r(t)}{\partial t_m}\Big| : m \in \{1,\dots,d\}^j, 0 \le j \le 4\right\}, \tag{3}$$

we have $\Psi(t) \underset{||t||\to\infty}{\to} 0$ and $\Psi \in L^1(\mathbb{R}^d)$.

(A4) : **Spectral condition.** The random field $X(\cdot)$ admits a spectral density which is positive in a neighborhood of 0.

*Remark.* Some remarks are in order. Assumption (A1) establishes minimal regularity and non-degeneracy for the underlying random field. Compared to previous works [13, 23, 29], it is worth pointing out that our work does not ask for isotropy. Besides, note that Assumption (A1) implies by [6], Prop.2.1. that the sample paths of $X(\cdot)$ are almost surely Morse:

$$\mathbb{P}\big\{\exists t \in \mathbb{R}^d : X'(t) = 0,\ \det(X''(t)) = 0\big\} = 0.$$

Assumption (A2) is natural according to previous works [13, 21, 26]. Assumption (A3) provides a simple way to unify the necessary integrability of the covariance function and its derivatives. Finally, Assumption (A4) simplifies the computations involved in the lower bounds for the limit variance.

We define the index $i(t)$ of each critical point $t$ as the number of negative eigenvalues of the Hessian $X''(t)$. We consider the set $\mathcal{T} = [-T,T]^d$, and we define the number of critical points of $X(\cdot)$ above level $u \ge -\infty$ which lie within $\mathcal{T}$ and have index $k = 0,1,\dots,d$:

$$\mathrm{Crt}_u^k(\mathcal{T}) := \#\big\{t \in \mathcal{T} : X(t) > u, X'(t) = 0, i(t) = k\big\}.$$

We consider the normalized variables:

$$\mathcal{C}_u^k(\mathcal{T}) := \frac{\mathrm{Crt}_u^k(\mathcal{T}) - \mathbb{E}(\mathrm{Crt}_u^k(\mathcal{T}))}{\sqrt{T^d}}, \quad k = 0,\dots,d,$$

and in order to study the joint distribution of these r.v.s, we define for $\boldsymbol{\alpha} = (\alpha_0,\alpha_1,\dots,\alpha_d) \in \mathbb{R}^{d+1}$:

$$\mathrm{Crt}_u^{\boldsymbol{\alpha}}(\mathcal{T}) = \sum_{k=0}^{d} \alpha_k \mathrm{Crt}_u^k(\mathcal{T}) \ \text{ and } \ \mathcal{C}_u^{\boldsymbol{\alpha}}(\mathcal{T}) := \frac{\mathrm{Crt}_u^{\boldsymbol{\alpha}}(\mathcal{T}) - \mathbb{E}(\mathrm{Crt}_u^{\boldsymbol{\alpha}}(\mathcal{T}))}{\sqrt{T^d}}. \tag{4}$$

The following theorem is the main result of the paper.

**Theorem 2.1.** *Consider a real-valued random field $X(\cdot)$ defined on $\mathbb{R}^d$ and satisfying Assumptions* (A1)-(A3). *Let $u \in \mathbb{R} \cup \{-\infty\}$ and $\boldsymbol{\alpha} \in \mathbb{R}^{d+1}$. Then*

1. *There exists $V_{\boldsymbol{\alpha}}(u) < \infty$ such that*

$$\lim_{T\to\infty} \mathrm{Var}\big(\mathcal{C}_u^{\boldsymbol{\alpha}}(\mathcal{T})\big) = V_{\boldsymbol{\alpha}}(u).$$

2. *When $u = -\infty$, setting $\boldsymbol{\alpha} = (-1)^d(1, -1, \ldots)$ we have $V_{\boldsymbol{\alpha}} = 0$.*
3. *The distribution of $\mathcal{C}_u^{\boldsymbol{\alpha}}(\mathcal{T})$ converges, as $T \to \infty$, towards the centered normal distribution with variance $V_{\boldsymbol{\alpha}}(u)$.*
4. *In addition, assume* (A4)*. When $u > -\infty$, $\boldsymbol{\alpha} \neq 0$, $V_{\boldsymbol{\alpha}} > 0$.*

*Remark.* When $u > -\infty$, setting $\boldsymbol{\alpha} = (-1)^d(1, -1, \ldots)$, we obtain the results by Estrade & León [23] even under more general conditions. When $u = -\infty$, setting $\boldsymbol{\alpha} = (1, \ldots, 1)$, we obtain the result by Nicolaescu [29] except for the the positivity of the variance.

The following corollary is an immediate consequence of Theorem 2.1.

*Corollary* 2.2. Under the conditions (A1)-(A4) , the random vector $\big(\mathcal{C}_u^0(\mathcal{T}), \ldots, \mathcal{C}_u^d(\mathcal{T})\big)$ converges, as $T \to \infty$ to a multivariate Gaussian distribution. When $u = -\infty$, the limit distribution degenerates. When $u > -\infty$, the limit distribution does not degenerate.

The next corollary upgrades the results in Theorem 2.1 to the almost sure convergence of the Euler Characteristic. Its proof is presented in Appendix C .

*Corollary* 2.3 (Almost sure convergence). Under the conditions (A1)-(A3), for every value of $u$, the random vector

$$\left(\frac{\mathrm{Crt}_u^0(\mathcal{T})}{\mathbb{E}\big(\mathrm{Crt}_u^0(\mathcal{T})\big)}, \ldots, \frac{\mathrm{Crt}_u^d(\mathcal{T})}{\mathbb{E}\big(\mathrm{Crt}_u^d(\mathcal{T})\big)}\right)$$

converges almost surely to $(1, \ldots, 1)$ as $T \to \infty$.

In particular,

$$\frac{\chi_u(\mathcal{T})}{\mathbb{E}\big(\chi_u(\mathcal{T})\big)} \to 1 \text{ a.s.},$$

as $T \to \infty$, where $\chi_u(\mathcal{T})$ is the Euler Characteristic of the excursion above $u$.

*Sufficient conditions for (A2)*

Certain sources are the main references for the finiteness of the second moment of the number of critical points [13] [22]. As a matter of fact, the proofs of these results contain, as a by-product, the proof of the continuity of the second moment w.r.t. the level. Inspired by the tools of these papers, in Appendix D, we provide a proof of the following proposition that extends their results and implies (A2).

*Proposition* 2.4. Suppose that, in addition to (A1), the random field $X(\cdot)$ satisfies:

(A5) For all $\mu \in \mathbb{S}^{d-1}$, $\mu \neq 0$, the random vector $X''(0)\mu$ has a N.D. distribution.

(A6) The integral $\displaystyle\int \frac{\|r^{(4)}(t) - r^{(4)}(0)\|}{\|t\|^d}\, dt$ converges at 0.

Then,

- For all compact $B \subset \mathbb{R}^d$, the second moment of $N(v, B)$ given by (2) is finite.
- For $B$ sufficiently small, the second moment of $N(v, B)$ is continuous as a function of $v$.

## 3. Hermite and Wiener Expansions

In this section, we give the explicit expression of the chaotic expansion for the numbers of critical points. We do it in two steps. The first one is to get a Hermite expansion à la Kratz-León [25]. Secondly, we translate each component in the expansion into a Multiple Wiener-Itô integral. The starting point of our analysis is an integral formula for $\mathrm{Crt}_u^k(\mathcal{T})$. Set $\mathcal{A}_k$ for the set of symmetric $d \times d$ matrices with index $k$, $k = 0, 1, \ldots, d$.

**Lemma 3.1.** *Suppose $X(\cdot)$ is a random field satifying* (A1) *and* (A2). *With the above notation, for $k = 0, 1, \ldots, d$, and $u > -\infty$, we have*

$$\mathrm{Crt}_u^k(\mathcal{T}) = \lim_{\varepsilon \to 0} \frac{1}{W(\varepsilon)} (-1)^k \int_{\mathcal{T}} \det(X''(t)) \mathbf{1}_{\mathcal{A}_k}(X''(t)) \mathbf{1}_{[u,\infty)}(X(t)) \cdot \mathbf{1}_{\|X'(t)\| \leq \varepsilon} dt, \tag{5}$$

*both a.s. and in the $L^2$-sense. Here, $W(\varepsilon)$ is the volume of a ball of radius $\varepsilon$ in $\mathbb{R}^d$.*

*When $u = -\infty$, we have*

$$\mathrm{Crt}^k(\mathcal{T}) = \lim_{\varepsilon \to 0} \frac{1}{W(\varepsilon)} (-1)^k \int_{\mathcal{T}} \det(X''(t)) \mathbf{1}_{\mathcal{A}_k}(X''(t)) \cdot \mathbf{1}_{\|X'(t)\| \leq \varepsilon} dt. \tag{6}$$

Remark that in the second case, the integrand is a product of two independent factors.

### *3.1. Hermite expansion*

The Hessian $X''(t)$ can be viewed either as a symmetric $d \times d$ matrix or as a size $d(d+1)/2$ vector. Set $D = d + \frac{d(d+1)}{2} + 1$,

$$\boldsymbol{X}(t) = \big(X'(t), X''(t), X(t)\big) \in \mathbb{R}^D, \tag{7}$$

and let $\Xi$ be its variance-covariance matrix, i.e., $\Xi = \mathbb{E}(\boldsymbol{X}(t)\boldsymbol{X}(t)^\top)$. Let $\Lambda$ be such that $\Lambda\Lambda^\top = \Xi$. Note that $\Lambda$ can be chosen as a block matrix $\Lambda = \left(\begin{smallmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{smallmatrix}\right)$ due to the independence of $X'(t)$ from $(X''(t), X(t))$. Thus, we can write

$$\boldsymbol{X}(t) = \Lambda \boldsymbol{Y}(t), \tag{8}$$

with $\boldsymbol{Y}(t)$ a standard Gaussian random vector in $\mathbb{R}^D$ for each fixed $t$.

For $\underline{y} \in \mathbb{R}^d$ and $\overline{y} = (x, z) \in \mathbb{R}^{\frac{1}{2}d(d+1)} \times \mathbb{R}$, set $\mathbf{y} = (\underline{y}, \overline{y}) \in \mathbb{R}^D$ and define

$$\tilde{G}_\varepsilon(\mathbf{y}) = G_\varepsilon(\Lambda \mathbf{y}) = \tilde{\delta}_\varepsilon(\underline{y}) \cdot \tilde{f}_k(\overline{y}),$$

with $\delta_\varepsilon(\cdot) := \frac{1}{W(\varepsilon)} \mathbf{1}_{\|\cdot\| \leq \varepsilon}$, $\tilde{\delta}_\varepsilon = \delta_\varepsilon \circ \Lambda_1$, $f_k(x, z) = (-1)^k \det(x) \mathbf{1}_{\mathcal{A}_k}(x) \mathbf{1}_{[u,\infty)}(z)$ and $\tilde{f}_k = f_k \circ \Lambda_2$. Since these functions are in $L^2(\phi_d)$ and $L^2(\phi_{D-d})$ respectively, we can consider their Hermite expansions. Define for $p \in \mathbb{N}$, $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p$ and $\mathbf{y} = (y_1, \ldots, y_p)$:

$$H_{\otimes \mathbf{m}}(\mathbf{y}) := \prod_{i=1}^p H_{m_i}(y_i),$$

where $H_m$ is the Hermite polynomial of degree $m$. Fix $k = 0, \ldots, d$, for $\mathbf{n} = (\underline{n}, \overline{n}) \in \mathbb{N}^d \times \mathbb{N}^{D-d}$, set

$$a_{k,\varepsilon}(\mathbf{n}) := c(\tilde{\delta}_\varepsilon, \underline{n}) c(\tilde{f}_k, \overline{n}),$$

where $c(\tilde{\delta}_\varepsilon, \underline{n})$ and $c(\tilde{f}_k, \overline{n})$ are respectively the $\underline{n}$, $\overline{n}$-th Hermite coefficients of $\tilde{\delta}_\varepsilon$ and $\tilde{f}_k$, that is:

$$\begin{aligned} c(\tilde{\delta}_\varepsilon, \underline{n}) &= \frac{1}{\underline{n}!} \int_{\mathbb{R}^d} \tilde{\delta}_\varepsilon(\underline{y}) H_{\otimes \underline{n}}(\underline{y}) \phi_d(\underline{y}) d\underline{y}; \\ c(\tilde{f}_k, \overline{n}) &= \frac{1}{\overline{n}!} \int_{\mathbb{R}^{D-d}} \tilde{f}_k(\overline{y}) H_{\otimes \overline{n}}(\overline{y}) \phi_{D-d}(\overline{y}) d\overline{y}. \end{aligned}$$

It is easy to see that $c(\tilde{\delta}_\varepsilon, \underline{n})$ converges, so we set

$$d(\underline{n}) = \frac{1}{\underline{n}!}\frac{H_{\otimes \underline{n}}(\underline{0})}{(2\pi)^{d/2}} = \lim_{\varepsilon\to 0} c(\tilde{\delta}_\varepsilon, \underline{n}).$$

The $L^2$ convergence in (5) implies the following proposition. For $q \in \mathbb{N}$, let $\mathcal{J}_q := \{\mathbf{n} \in \mathbb{N}^D : |\mathbf{n}| = q\}$ and, for $\mathbf{n} \in \mathcal{J}_\mathbf{q}$ set $\mathcal{H}_\mathbf{n}(\mathcal{T}) = \int_\mathcal{T} H_{\otimes_\mathbf{n}}(\boldsymbol{Y}(t))dt$ and $a_{\boldsymbol{\alpha}}(\mathbf{n}) = \sum_{k=0}^d \alpha_k a_k(\mathbf{n})$ with $\boldsymbol{\alpha} \in \mathbb{R}^{d+1}$.

*Proposition* 3.2. Let $X(\cdot)$ be a random field that satisfies (A1) and (A2). Then, for $\boldsymbol{\alpha} \in \mathbb{R}^{d+1}$, the following expansion holds in $L^2(\Omega)$:

$$\mathrm{Crt}_u^{\boldsymbol{\alpha}}(\mathcal{T}) = \sum_{q=0}^{\infty}\sum_{\mathbf{n}\in\mathcal{J}_q} a_{\boldsymbol{\alpha}}(\mathbf{n})\mathcal{H}_\mathbf{n}(\mathcal{T}).$$

For fixed $q \geq 1$, we denote

$$s_{q,T}^{\boldsymbol{\alpha}} := \frac{1}{(2T)^{d/2}}\sum_{\mathbf{n}\in\mathcal{J}_q} a_{\boldsymbol{\alpha}}(\mathbf{n})\mathcal{H}_\mathbf{n}(\mathcal{T}),$$

so that $\mathcal{C}_u^{\boldsymbol{\alpha}}(\mathcal{T}) = \sum_{q=1}^\infty s_{q,T}^{\boldsymbol{\alpha}}$. That is, $s_{q,T}^{\boldsymbol{\alpha}}$ is the $q$-th chaotic component of $\mathcal{C}_u^{\boldsymbol{\alpha}}(\mathcal{T})$.

Observe that the expansion separates the geometry (encapsulated in the coefficients $a_{\boldsymbol{\alpha}}(\mathbf{n})$) from the probability (encapsulated in the multi-spectrum $\mathcal{H}_\mathbf{n}(\mathcal{T})$).

### *3.2. Multiple Itô-Wiener Integrals (MWI)*

We use *spectral* stochastic integrals in [32] and [35], Ch.9. We choose to follow the *isonormal Gaussian process* framework in [35], Ch.9, rather than the explicit MWI construction in [32].

Let $\mathcal{H}$ be the set of complex-valued Hermitian square integrable functions w.r.t. Lebesgue measure $d\lambda$ in $\mathbb{R}^d$, that is, $\mathcal{H} = L^2_H(\mathbb{R}^d, d\lambda) = \{\psi : \mathbb{R}^d \to \mathbb{C} : \psi(-\lambda) = \overline{\psi(\lambda)}, \|\psi\|^2_\mathcal{H} = \int_{\mathbb{R}^d} |\psi(\lambda)|^2 d\lambda < \infty\}$. Let $\widehat{B}$ be a complex Hermitian Brownian measure on $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{U}, \mathbb{P})$ such that $\mathcal{U}$ is generated by $\widehat{B}$. The (real-valued) Wiener integral w.r.t. $\widehat{B}$, denoted by

$$I_1^{\widehat{B}}(\psi) = \int_{\mathbb{R}^d} \psi(\lambda) d\widehat{B}(\lambda),$$

is an isometry from $\mathcal{H}$ into $L^2(\Omega)$, see [32] and [35], [Ch.9]. That is, for $\psi_1, \psi_2 \in \mathcal{H}$, we have

$$\mathbb{E}\big(I_1^{\widehat{B}}(\psi_1) I_1^{\widehat{B}}(\psi_2)\big) = \int_{\mathbb{R}^d} \psi_1(\lambda)\overline{\psi_2(\lambda)}d\lambda = \int_{\mathbb{R}^d} \psi_1(\lambda)\psi_2(-\lambda)d\lambda = \big\langle \psi_1, \psi_2 \big\rangle_\mathcal{H}. \tag{9}$$

The $q$-fold multiple Wiener-Itô integral w.r.t. $\widehat{B}$ is defined, for $\psi_j \in \mathcal{H}, j \leq D$ and $\mathbf{n} \in \mathcal{J}_q$, by

$$I_q^{\widehat{B}}\big(\psi_1^{\otimes n_1} \otimes \cdots \otimes \psi_D^{\otimes n_D}\big) = \prod_{j=1}^D H_{n_j}\big(I_1^{\widehat{B}}(\psi_j)\big),$$

where $\otimes$ stands for the tensorial product. Let $\mathcal{H}_q$ denote the domain of $I_q^{\widehat{B}}$, namely, the set of those $\psi : (\mathbb{R}^d)^q \to \mathbb{C}$ such that $\psi(-\lambda_1, \ldots, -\lambda_q) = \overline{\psi(\lambda_1, \ldots, \lambda_q)}$ and $\|\psi\|^2_{\mathcal{H}_q} = \int_{\mathbb{R}^{dq}} |\psi(\lambda_1, \ldots, \lambda_q)|^2 d\lambda_1 \ldots d\lambda_q$ is finite. The latter integral defines the norm and (thus, also) the inner product in $\mathcal{H}_q$.

To describe the isometry induced by $I_q^{\widehat{B}}$, we introduce the symmetrization of a kernel $\psi \in \mathcal{H}_q$:

$$Sym(\psi)(\lambda_1, \dots, \lambda_q) := \frac{1}{q!} \sum_{\pi \in \mathcal{S}_q} \psi(\lambda_{\pi(1)}, \dots, \lambda_{\pi(q)}), \tag{10}$$

with $\mathcal{S}_q$ being the group of permutations of $[q] := \{1, \dots, q\}$. For $\psi \in \mathcal{H}_q$, we have that $I_q^{\widehat{B}}(\psi) = I_q^{\widehat{B}}(Sym(\psi))$. Moreover, for $\psi_1 \in \mathcal{H}_q$ and $\psi_2 \in \mathcal{H}_p$, it holds that

$$\mathbb{E}\big(I_q^{\widehat{B}}(\psi_1) I_p^{\widehat{B}}(\psi_2)\big) = \delta_{pq} q! \big\langle Sym(\psi_1), Sym(\psi_2)\big\rangle_{\mathcal{H}_q}, \tag{11}$$

$\delta_{pq}$ being the Kroenecker symbol. Hence, $I_q^{\widehat{B}}$ is an isometry from $\mathcal{H}_q^s := \{\psi \in \mathcal{H}_q : \psi = Sym(\psi)\}$, with the modified norm $\sqrt{q!}\|\cdot\|_{\mathcal{H}_q}$, onto its image, which is called the $q$-th Wiener chaos $\mathcal{K}_q$. By convention, $\mathcal{K}_0 = \mathbb{R}$.

It is well known, acccording to [35], Ch.8, that the (orthogonal) sum $\bigotimes_{q=0}^{\infty} \mathcal{K}_q = L^2(\Omega)$. In other words, if $X \in L^2(\Omega)$, there exists an unique sequence of (symmetric) kernels $\psi_q \in \mathcal{H}_q^s : q \geq 1$ such that

$$X = \sum_{q=0}^{\infty} I_q^{\widehat{B}}(\psi_q).$$

In particular, $I_0^{\widehat{B}}(\psi_0) = \psi_0 = \mathbb{E}(X)$. The key point is that chaotic r.v. (i.e: Multiple Wiener-Itô Integrals) are well suited to study asymptotic normality. For $r \leq p \wedge q$, denote $\bar{\otimes}_r$ the $r$-th contraction operator: $\phi \in \mathcal{H}_p^s, \psi \in \mathcal{H}_q^s \mapsto \phi \bar{\otimes}_r \psi \in \mathcal{H}_{d(p+q-2r)}$ such that

$$\begin{aligned}
&\phi \bar{\otimes}_r \psi(\lambda_1, \dots, \lambda_{p+q-2r}) \\
&\qquad = \int_{\mathbb{R}^{dr}} \phi(z_1, \dots, z_r; \lambda_1, \dots, \lambda_{p-r}) \psi(-z_1, \dots, -z_r; \lambda_{p-r+1}, \dots, \lambda_{p+q-2r}).dz_1 \dots dz_r.
\end{aligned}$$

Now, we restrict and adapt to our framework Theorem 11.8.1 in [35] which states that, in this setting, joint convergence is equivalent to marginal convergence. Besides, the third item below provides a practical criterion to check it.

**Theorem 3.3.** *([35], Th.11.8.1). Let $Q \geq 1$. For $q \leq Q$, consider the sequence of kernels $(\psi_{q,n})_{n \geq 1}$ with $\psi_{q,n} \in \mathcal{H}_q^s$. Then, as $n \to \infty$ the following conditions are equivalent:*

1. *$\big(I_1^{\widehat{B}}(\psi_{1,n}), \dots, I_Q^{\widehat{B}}(\psi_{Q,n})\big)$ converges in law towards a centered normal random vector with variance $diag(\sigma_1^2, \dots, \sigma_Q^2)$.*
2. *For each $q \leq Q$, $I_q^{\widehat{B}}(\psi_{q,n})$ converges in law towards a centered normal r.v. with variance $\sigma_q^2$.*
3. *For each $q \leq Q$ and $r = 1, \dots, q-1$, the norms of the contractions $\|\psi_{q,n} \bar{\otimes}_r \psi_{q,n}\|_{\mathcal{H}_q}$ converge to $0$.*

As said above, in the second step, we translate $s_{q,T}^{\boldsymbol{\alpha}}, q \geq 1$ into a Wiener-Itô integral. Assumption (A3) implies that $\boldsymbol{X}(\cdot)$ given by (7) (and thus $\boldsymbol{Y}(\cdot)$) has a spectral density. This fact allows us to find explicit (Hermitian) orthonormal kernels $\psi_{t,j} \in \mathcal{H}$ such that

$$Y_j(t) = I_1^{\widehat{B}}(\psi_{t,j}), \quad j = 1, \dots, D. \tag{12}$$

Indeed, let $f$ stand for the spectral density of $X$, thus, we can write

$$X(t) = \int_{\mathbb{R}^d} e^{it\cdot\lambda} \sqrt{f(\lambda)} d\widehat{B}(\lambda).$$

Taking partial derivatives and denoting $\nu(\lambda) := \big((i\lambda_j)_{1\le j\le d}; (-\lambda_j\lambda_k)_{1\le j\le k\le d}; 1\big)$, we get the $D$-dimensional spectral representation

$$\boldsymbol{X}(t) = \int_{\mathbb{R}^d} e^{it\cdot\lambda}\nu(\lambda)\sqrt{f(\lambda)}d\widehat{B}(\lambda).$$

In consequence, using (8), we conclude that,

$$\boldsymbol{Y}(t) = \int_{\mathbb{R}^d} e^{it\cdot\lambda}(\Lambda^{-1}\nu(\lambda))\sqrt{f(\lambda)}d\widehat{B}(\lambda).$$

Thus, taking coordinates, the kernels in (12) are given by

$$\psi_{t,j}(\lambda) = e^{it\cdot\lambda}(\Lambda^{-1}\nu(\lambda))_j\sqrt{f(\lambda)}, \quad j = 1,\dots,D. \tag{13}$$

*Proposition* 3.4. Assume that $X(\cdot)$ satisfies $(A1)-(A3)$. Then, with the notation in Proposition 3.2 we can write

$$s^{\boldsymbol{\alpha}}_{q,T} = I^B_q(g^{\boldsymbol{\alpha}}_{q,T}), \qquad \text{with } \ g^{\boldsymbol{\alpha}}_{q,T} = \frac{1}{(2T)^{d/2}}\sum_{\mathbf{n}\in\mathcal{J}_q} a_{\boldsymbol{\alpha}}(\mathbf{n})\int_{\mathcal{T}}\psi^{\otimes n_1}_{t,1}\otimes\cdots\otimes\psi^{\otimes n_D}_{t,D}\,dt.$$

Finally, to gain symmetry in the expansion, let us reindex it in the following way. To each $\mathbf{n}\in\mathcal{J}_q$, associate the set of indexes

$$\mathcal{I}_{\mathbf{n}} := \Big\{\mathbf{m}\in[D]^q : \sum_{j=1}^q \mathbf{1}_{\{i\}}(m_j) = n_i, \forall i\le D\Big\},$$

and set

$$b_{\boldsymbol{\alpha}}(\mathbf{m}) := \frac{a_{\boldsymbol{\alpha}}(\mathbf{n})}{\#\mathcal{I}_{\mathbf{n}}}.$$

Note that the family $\{\mathcal{I}_{\mathbf{n}} : \mathbf{n}\in\mathcal{J}_q\}$ forms a partition of $[D]^q$.

The idea is to replace $\psi^{\otimes n_1}_{t,1}\otimes\cdots\otimes\psi^{\otimes n_D}_{t,D}$ by the equivalent but more symmetric $\psi_{t,m_1}\otimes\cdots\otimes\psi_{t,m_D}$ where $n_j$ of the $m_i$s equal $j$. For instance, $\psi^{\otimes 3}_{t,1}\otimes\psi^{\otimes 2}_{t,2}$ is to be associated by $\psi_{t,1}\otimes\psi_{t,1}\otimes\psi_{t,1}\otimes\psi_{t,2}\otimes\psi_{t,2}$ (or by any of the equivalent products obtained by a permutation of the indexes).

Now, we can state the final version of the chaotic expansion of the $q$th component of $\mathrm{Crt}^{\boldsymbol{\alpha}}_u(\mathcal{T})$.

*Proposition* 3.5. Under the hypotheses of Proposition 3.4 and with the notation above, for $\mathbf{m} = (m_1,\dots,m_q)$, we have

$$g^{\boldsymbol{\alpha}}_{q,T} = \frac{1}{(2T)^{d/2}}\sum_{\mathbf{m}\in[D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m})\int_{\mathcal{T}}\psi_{t,m_1}\otimes\cdots\otimes\psi_{t,m_q}\,dt.$$

## 4. Proof of Theorem 2.1

The proof of Theorem 2.1 is divided in several steps.

### *4.1. Finiteness of the limit variance*

By the orthogonality in $q$, we have

$$\mathrm{Var}\big(\mathcal{C}^{\boldsymbol{\alpha}}_u(\mathcal{T})\big) = \sum_{q=1}^\infty\sum_{\mathbf{m},\mathbf{n}\in\mathcal{J}_q} a_{\boldsymbol{\alpha}}(\mathbf{m})a_{\boldsymbol{\alpha}}(\mathbf{n})\frac{\mathbb{E}\big[\mathcal{H}_{\mathbf{m}}(\mathcal{T})\mathcal{H}_{\mathbf{n}}(\mathcal{T})\big]}{(2T)^d}. \tag{14}$$

For each fixed $q$, the convergence of the corresponding term follows in the same manner as [23], Eq.s (8)-(11). Indeed, for fixed $q$, the inner sum has a finite number of terms, thus, it converges if the terms $\frac{1}{(2T)^d}\mathbb{E}\big[\mathcal{H}_{\mathbf{m}}(\mathcal{T})\mathcal{H}_{\mathbf{n}}(\mathcal{T})\big]$ do. We have, for fixed $\mathbf{m},\mathbf{n}\in\mathcal{J}_{\mathbf{q}}$

$$\begin{aligned}\frac{\mathbb{E}\big[\mathcal{H}_{\mathbf{m}}(\mathcal{T})\mathcal{H}_{\mathbf{n}}(\mathcal{T})\big]}{(2T)^d} &= \frac{1}{(2T)^d}\iint_{\mathcal{T}^2}\mathbb{E}[H_{\bar{\otimes}_m}(\mathbf{Y}(s))H_{\bar{\otimes}_n}(\mathbf{Y}(t))]dsdt\\ &\underset{T\to\infty}{\longrightarrow}\int_{\mathbb{R}^d}\mathbb{E}[H_{\bar{\otimes}_m}(\mathbf{Y}(0))H_{\bar{\otimes}_n}(\mathbf{Y}(t))]dt,\end{aligned}$$

because of stationarity. Now, Mehler's formula [23], Eq.(8) gives

$$\mathbb{E}[H_{\bar{\otimes}_m}(\mathbf{Y}(0))H_{\bar{\otimes}_n}(\mathbf{Y}(t))]\leq(\texttt{const})\Psi_Y(t)^q,$$

where $\Psi_Y(t)$ is defined as $\Psi(t)$ replacing $r(t)$ by $\Gamma(t):=\mathbb{E}(Y(0)Y(t))$ in (3). Recall that $\boldsymbol{X}(t)=\Lambda\mathbf{Y}(t)$ for every $t\in\mathbb{R}^d$ by (8), thus, there exists $K>0$ such that $\Psi_Y(t)\leq K\Psi(t)$. Hence, $\mathbb{E}[H_{\bar{\otimes}_m}(\mathbf{Y}(0))H_{\bar{\otimes}_n}(\mathbf{Y}(t))]\leq(\texttt{const})\Psi(t)^q$, and provided that $\Psi\in L^1(\mathbb{R}^d)$ by (A3), we deduce that

$$V^q_{\boldsymbol{\alpha}}(\mathcal{T}):=\sum_{\mathbf{m},\mathbf{n}\in\mathcal{J}_q}a_{\boldsymbol{\alpha}}(\mathbf{m})a_{\boldsymbol{\alpha}}(\mathbf{n})\frac{\mathbb{E}\big[\mathcal{H}_{\mathbf{m}}(\mathcal{T})\mathcal{H}_{\mathbf{n}}(\mathcal{T})\big]}{(2T)^d}\underset{T\to\infty}{\to}V^q_{\boldsymbol{\alpha}}(u).$$

We now proceed to the bound for the variance of the tail of the expansion:

$$\sup_T\sum_{q>Q}V^q_{\boldsymbol{\alpha}}(\mathcal{T})=\sup_T\operatorname{Var}\Big(\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u(\mathcal{T}))\Big)\underset{Q\to\infty}{\to}0. \tag{15}$$

Here $\pi_Q$ stands for the orthogonal projection onto $\bigoplus_{q>q}\mathcal{K}_q$. Note that this fact implies the convergence in (14). Set $I_T:=[-T,T)^d\cap\mathbb{Z}^d$ and $s[0,1)^d:=s+[0,1)^d$. Thus,

$$\mathcal{C}^{\boldsymbol{\alpha}}_u(\mathcal{T})=\frac{1}{(2T)^{d/2}}\sum_{s\in I_T}\mathcal{C}^{\boldsymbol{\alpha}}_u(s[0,1)^d).$$

Using the stationarity of $\boldsymbol{Y}$, we get

$$V_{T,Q}:=\operatorname{Var}\big(\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u(\mathcal{T}))\big)=\sum_{s\in I_T}\frac{\#I_T\cap(I_T-s)}{(2T)^d}\,\mathbb{E}\big(\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u([0,1)^d))\cdot\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u(s[0,1)^d))\big).$$

Now, assumption (A3) gives us the right to choose $a>0$ such that $K\Psi(s)\leq\rho<1$ for $s:||s||_\infty\geq a$. Thus, we split $V_{T,Q}=V^1_{T,Q}+V^2_{T,Q}$ where $V^1_{T,Q},V^2_{T,Q}$ stand respectively for the sums over $s:||s||_\infty\leq a+1$ and over $s:||s||_\infty>a+1$.

Consider $V^1_{T,Q}$ first. Clearly there are less than $(2a+2)^d$ terms in the sum and for each one of them, assuming w.l.o.g. that $2T>a+1$, we have $\#I_T\cap(I_T-s)\leq(2T)^d$. Thus, from the Cauchy-Schwarz inequality and stationarity, we have

$$|V^1_{T,Q}|\leq(2a+2)^d\,\mathbb{E}\Big(\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u([0,1)^d))^2\Big),$$

which tends to 0 as $Q\to\infty$ uniformly w.r.t. $T$.

Now, we move to $V^2_{T,Q}$. We have,

$$\mathbb{E}\big(\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u([0,1)^d))\cdot\pi_Q(\mathcal{C}^{\boldsymbol{\alpha}}_u(s[0,1)^d))\big)=\sum_{q>Q}\iint_{([0,1)^d)^2}\mathbb{E}\big[F_q(\boldsymbol{Y}(t))F_q(\boldsymbol{Y}(s+u))\big]dtdu,$$

where $F_q(\cdot) = \sum_{\mathbf{n}\in\mathcal{J}_q} a_{\boldsymbol{\alpha}}(\mathbf{n}) H_{\bar{\otimes}_{\mathbf{n}}}(\boldsymbol{Y}(\cdot))$. Arcones' inequality, Lemma B.1 implies that

$$\mathbb{E}\big[F_q(\boldsymbol{Y}(t))F_q(\boldsymbol{Y}(s+u))\big] \le K^d \Psi^d(s+u-t)||F_q||_2^2.$$

Since $\|s+u-t\|_\infty \ge a$, we have $K\Psi(s+u-t) \le \rho$. Besides, [4], Lem.4.3, implies that $||F_q||_2^2 \le || \sum_{k=0}^d \alpha_k \tilde{f}_k||_2^2 = (\texttt{const}) < \infty$. Thus,

$$|V_{T,Q}^2| \le (\texttt{const}) \sum_{s\in I_T, ||s||_\infty > a} \sum_{q>Q} \rho^{q-1} \iint_{([0,1)^d)^2} \Psi(s+u-t)dtdu \le (\texttt{const})\rho^Q,$$

which tends to 0 as $Q \to \infty$, uniformly w.r.t. $T$. This proves that the limit variance is finite.

### *4.2. Central Limit Theorem*

We know from (15) that $\lim_{Q\to\infty} \sup_T \pi_Q\big(\mathcal{C}_u^{\boldsymbol{\alpha}}(\mathcal{T})\big) = 0$ in the $L^2$-sense. Hence, it suffices to fix $Q$ and to establish the CLT for the partial sums $S_{Q,T}^{\boldsymbol{\alpha}} := \sum_{q=1}^Q s_{q,T}^{\boldsymbol{\alpha}}$ which, following Theorem 3.3, is equivalent to state the CLT for each fixed chaotic component $s_{q,T}^{\boldsymbol{\alpha}} = I_q^{\widehat{B}}(g_{q,T}^{\boldsymbol{\alpha}})$, $q \le Q$. Recall that

$$g_{q,T}^{\boldsymbol{\alpha}} = \frac{1}{(2T)^{d/2}} \sum_{\mathbf{m}\in[D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) \int_{\mathcal{T}} \psi_{t,m_1} \otimes \cdots \otimes \psi_{t,m_q} dt.$$

and that $\Gamma_{ij}(t-s) = \mathbb{E}(Y_i(t)Y_j(s)) = \langle \psi_{t,i}, \psi_{s,j}\rangle_{\mathcal{H}}$, see Proposition 3.5.

According to Theorem 3.3, it suffices to prove that the norm of the contractions $g_{q,T}^{\boldsymbol{\alpha}} \bar{\otimes}_r g_{q,T}^{\boldsymbol{\alpha}}$, $r = 1, \dots, q-1$ tend to zero as $T \to \infty$. See [23], p. 17, for an analogous computation. Since $q$ is fixed, by linearity, the coefficients play no role in this convergence. Thus, let us concentrate on the norm of the contraction of two normalized integrals as those in $g_{q,T}^{\boldsymbol{\alpha}}$.

$$\Big[\frac{1}{(2T)^{d/2}} \int_{\mathcal{T}} \psi_{t,m_1} \otimes \cdots \otimes \psi_{t,m_q} dt\Big] \otimes_r \Big[\frac{1}{(2T)^{d/2}} \int_{\mathcal{T}} \psi_{s,n_1} \otimes \cdots \otimes \psi_{s,n_q} ds\Big]$$
$$= \frac{1}{(2T)^d} \int_{(\mathcal{T})^2} \prod_{k=1}^r \Gamma_{m_k n_k}(t-s) \cdot \bigotimes_{k=r+1}^q \psi_{t,m_k} \otimes \psi_{s,n_k} \, dsdt.$$

Besides,

$$\Big\|\Big[\frac{1}{(2T)^{d/2}} \int_{\mathcal{T}} \psi_{t,m_1} \otimes \cdots \otimes \psi_{t,m_q} dt\Big] \otimes_r \Big[\frac{1}{(2T)^{d/2}} \int_{\mathcal{T}} \psi_{s,n_1} \otimes \cdots \otimes \psi_{s,n_q} ds\Big]\Big\|^2$$
$$= \frac{1}{(2T)^{2d}} \int_{(\mathcal{T})^4} \prod_{k=1}^r \Gamma_{m_k n_k}(t-s) \prod_{k=1}^r \Gamma_{m_k n_k}(t'-s') \prod_{k=r+1}^q \Gamma_{m_k m_k}(t'-t) \prod_{k=r+1}^q \Gamma_{n_k n_k}(s-s') \, dsdtds'dt'$$
$$\le \frac{1}{(2T)^{2d}} \int_{(\mathcal{T})^4} \Psi(t-s)^r \Psi(t'-s')^r \Psi(t'-t)^{q-r} \Psi(s-s')^{q-r} \, dsdtds'dt'$$
$$\le \frac{(\texttt{const})}{(2T)^d} \int_{(\mathbb{R}^d)^3} \Psi(v_1)^r \Psi(v_2)^r \Psi(v_3)^{q-r} \, dv_1 dv_2 dv_3.$$

To get the last bound, we used an isometric change of variables and the fact that $\Psi$ is bounded. Afterwards, we enlarged the domain $\mathcal{T}$ to the whole space $\mathbb{R}^d$. Assumption (A3) easily shows that the latter expression is bounded by $(\texttt{const})/(2T)^d$. The CLT follows.

### *4.3. Positivity of the limit variance $u > -\infty$*

We compute $V_q^{\boldsymbol{\alpha}}(u)$, that is, the limit variance of the $q$-th chaotic component $s_{q,T}^{\boldsymbol{\alpha}} = I_q^{\widehat{B}}(g_{q,T}^{\boldsymbol{\alpha}})$ as $T \to \infty$ (see Proposition 3.5). Now, by the isometric property of the MWI (11), we get

$$V_{q,T}^{\boldsymbol{\alpha}}(u) = \mathbb{E}\big(I_q^{\widehat{B}}(g_{q,T}^{\boldsymbol{\alpha}})^2\big) = q!\|Sym(g_{q,T}^{\boldsymbol{\alpha}})\|_{\mathcal{H}_q}^2,$$

where $Sym(g_{q,T}^{\boldsymbol{\alpha}})$ stands for the symmetrization of $g_{q,T}^{\boldsymbol{\alpha}}$ as in (10). Hence,

$$\begin{aligned}
V_{q,T}^{\boldsymbol{\alpha}}(u) &= \frac{1}{(2T)^d} \sum_{\mathbf{m},\mathbf{m}' \in [D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) b_{\boldsymbol{\alpha}}(\mathbf{m}') \iint_{\mathcal{T}\times\mathcal{T}} \frac{1}{q!} \sum_{\pi,\pi' \in \mathcal{S}_q} \prod_{i=1}^q \big\langle \psi_{t,m_{\pi^{-1}(i)}}, \psi_{s,m'_{\pi'^{-1}(i)}} \big\rangle_{\mathcal{H}} ds dt \\
&= \frac{1}{(2T)^d} \sum_{\mathbf{m},\mathbf{m}' \in [D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) b_{\boldsymbol{\alpha}}(\mathbf{m}') \iint_{\mathcal{T}\times\mathcal{T}} \frac{1}{q!} \sum_{\pi,\pi' \in \mathcal{S}_q} \prod_{i=1}^q \Gamma_{m_{\pi(i)},m'_{\pi'(i)}}(t-s) ds dt \\
&\underset{T\to\infty}{\to} \sum_{\mathbf{m},\mathbf{m}' \in [D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) b_{\boldsymbol{\alpha}}(\mathbf{m}') \int_{\mathbb{R}^d} \frac{1}{q!} \sum_{\pi,\pi' \in \mathcal{S}_q} \prod_{i=1}^q \Gamma_{m_{\pi(i)},m'_{\pi'(i)}}(t) dt.
\end{aligned}$$

In the second line, we used the isometry (9) of the simple stochastic integrals (12). In that line, we also replaced $\pi^{-1}$ by $\pi$ to simplify the notation. Now, from (12) and (13), we have that $\Gamma_{ij}(t) = \mathcal{F}(\tilde{\psi}_i \tilde{\psi}_j f)(t)$ with

$$\tilde{\psi}_j(\cdot) = (\Lambda^{-1}\nu(\cdot))_j$$

and $\mathcal{F}$ being the Fourier transform. As the product of covariances corresponds, via the Fourier transform, to the convolution of the spectral densities, we have

$$\prod_{i=1}^q \Gamma_{m_{\pi(i)},m'_{\pi'(i)}}(t) = \mathcal{F}\Big(\mathop{*}_{i=1}^{q} \tilde{\psi}_{m_{\pi(i)}} \tilde{\psi}_{m'_{\pi'(i)}} f\Big)(t).$$

Thus,

$$\begin{aligned}
V_{q,T}^{\boldsymbol{\alpha}}(u) &\underset{T\to\infty}{\to} \sum_{\mathbf{m},\mathbf{m}' \in [D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) b_{\boldsymbol{\alpha}}(\mathbf{m}') \frac{1}{q!} \sum_{\pi,\pi' \in \mathcal{S}_q} \int_{\mathbb{R}^d} \mathcal{F}\Big(\mathop{*}_{i=1}^{q} \tilde{\psi}_{m_{\pi(i)}} \tilde{\psi}_{m'_{\pi'(i)}} f\Big)(t) dt \\
&= \sum_{\mathbf{m},\mathbf{m}' \in [D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) b_{\boldsymbol{\alpha}}(\mathbf{m}') \frac{1}{q!} \sum_{\pi,\pi' \in \mathcal{S}_q} \mathcal{F}\Big(\mathcal{F}\Big(\mathop{*}_{i=1}^{q} \tilde{\psi}_{m_{\pi(i)}} \tilde{\psi}_{m'_{\pi'(i)}} f\Big)\Big)(0).
\end{aligned}$$

Here, we used the fact that the integral on the whole space of a function coincides with its Fourier transform evaluated at zero. Now, using the Fourier inversion formula, we get that

$$V_{q,T}^{\boldsymbol{\alpha}}(u) \underset{T\to\infty}{\to} \frac{(2\pi)^d}{q!} \sum_{\mathbf{m},\mathbf{m}' \in [D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) b_{\boldsymbol{\alpha}}(\mathbf{m}') \sum_{\pi,\pi' \in \mathcal{S}_q} \mathop{*}_{i=1}^{q} \tilde{\psi}_{m_{\pi(i)}} \tilde{\psi}_{m'_{\pi'(i)}} f(0).$$

The convolution above can be written as.

$$\mathop{*}_{i=1}^{q} \tilde{\psi}_{m_{\pi(i)}} \tilde{\psi}_{m'_{\pi'(i)}} f(0) = \int_{\mathcal{S}} \prod_{i=1}^q (\tilde{\psi}_{m_{\pi(i)}} \tilde{\psi}_{m'_{\pi'(i)}} f)(\lambda_i) d\sigma(\lambda_1, \dots \lambda_q),$$

with $\mathcal{S} := \{(\lambda_1, \dots, \lambda_q) \in \mathbb{R}^{dq} : \lambda_1 + \dots + \lambda_q = 0\}$ and $d\sigma(\cdot)$ the Lebesgue measure on $\mathcal{S}$.

Finally, we arrange the terms to

$$\lim_{T\to\infty} V^{\boldsymbol{\alpha}}_{q,T}(u) = V^{\boldsymbol{\alpha}}_q(u) = \frac{(2\pi)^d}{q!}\int_{\mathcal{S}} \Big[ \sum_{\mathbf{m}\in[D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) \sum_{\pi\in\mathcal{S}_q}\prod_{i=1}^q \tilde{\psi}_{m_{\pi(i)}}(\lambda_i)\Big]^2 \prod_{i=1}^q f(\lambda_i) d\sigma(\lambda_1\dots\lambda_q)$$

$$= \frac{(2\pi)^d}{q!}\int_{\mathcal{S}} P_q(\lambda_1,\dots,\lambda_q)^2 F(\lambda_1,\dots,\lambda_q) d\sigma(\lambda_1\dots\lambda_q), \tag{16}$$

with

$$P_q(\lambda_1,\dots,\lambda_q) := \sum_{\mathbf{m}\in[D]^q} b_{\boldsymbol{\alpha}}(\mathbf{m}) \sum_{\pi\in\mathcal{S}_q}\prod_{i=1}^q \tilde{\psi}_{m_{\pi(i)}}(\lambda_i), \tag{17}$$

and $F(\lambda_1,\dots,\lambda_q) = \prod_{i=1}^q f(\lambda_i)$.

Let us perform an orthonormal change of variables with $\lambda'_1 = \frac{\lambda_1+\dots+\lambda_q}{\sqrt{q}}$, and let $\bar{P}$ and $\bar{F}$ be the expressions of $P$ and $F$ in this new basis. We have

$$V^{\boldsymbol{\alpha}}_q(u) = (\texttt{const})\int_{\mathbb{R}^{d(q-1)}} \bar{P}_q(0,\lambda'_2,\dots,\lambda'_q)^2 \bar{F}(0,\lambda'_2,\dots,\lambda'_q) d\lambda'_2\dots d\lambda'_q.$$

**Lemma 4.1.** *With the above notation, if $V^{\boldsymbol{\alpha}}_q(u) = 0$, if $X(\cdot)$ satisfies in addition* (A4)*, then,*

$$P_q(\lambda_1,\dots,\lambda_q) = \widetilde{P}(\lambda_1+\dots+\lambda_q), \tag{18}$$

*where $\widetilde{P}$ is a polynomial such that $\widetilde{P}(0) = 0$.*

*Proof.* Using (16) and the change of variables above, we get

$$V^{\boldsymbol{\alpha}}_q(u) = \frac{(2\pi)^d}{q!}\int_{\mathbb{R}^{d(q-1)}} \bar{P}(0,\lambda'_2,\dots,\lambda'_q)^2 \bar{F}(0,\lambda'_2,\dots,\lambda'_q) d\lambda'_2\dots d\lambda'_q = 0.$$

The following facts are clear: (i) $\bar{P}(0,\lambda'_2,\dots,\lambda'_q)$ is a polynomial in $q-1$ variables; (ii) Assumption (A4) implies that the function $\bar{F}(0,\lambda'_2,\dots,\lambda'_q)$ is positive in some neigborhood of $0\in\mathbb{R}^{d(q-1)}$; (iii) the polynomial in $(q-1)$ variables $\bar{P}(0,\lambda'_2,\dots,\lambda'_q)$ must vanish, (iv) $\bar{P}$ is a polynomial in $\lambda'_1$. The result follows. □

We observe now that both representations of $P_q(\lambda_1,\dots,\lambda_q)$ given by (17) and (18) are incompatible for $q\geq 2$, unless $P_q$ vanishes. Indeed, note that the kernels $\tilde{\psi}_j : 1\leq j\leq D$ are polynomials of degree 1 or 2 in the coordinates of $\lambda_j : 1\leq j\leq q$, say $\lambda_{j,\ell} : 1\leq\ell\leq d$, so that the degree of the polynomial $P_q$, $\delta$ say, satisfies $q\leq\delta\leq 2q$. Consider a monomial of maximal degree of $\tilde{P}_q$ that can be written as

$$\prod_{\ell=1}^d a_\ell\big(\lambda_{1,\ell}+\dots+\lambda_{q,\ell}\big)^{b_\ell},$$

with $b_1+\dots+b_d = \delta$. Developing the power, we obtain, for example, a monomial of the expression (18), which is

$$\prod_{\ell=1}^d a_\ell\big(\lambda_{1,\ell}\big)^{b_\ell},$$

i.e., a polynomial in the coordinates of the first variable $\lambda_1$ only. This is not possible to obtain from (17) because, when $q\geq 2$, a monomial of maximal degree in (17) is a product of $q$ monomials, of degree at least 1, each depending on a different $\lambda_1, \lambda_2, \dots, \lambda_q$. To end the proof, we observe that $s^{\boldsymbol{\alpha}}_{q,T} = I^{\widehat{B}}_q(g^{\boldsymbol{\alpha}}_{q,T})$ with

$$Sym(g^{\boldsymbol{\alpha}}_{q,T}) = \frac{1}{(2T)^{d/2}} \int_{\mathcal{T}} e^{i\lambda t} P_q(\lambda_1, \dots, \lambda_q) F(\lambda_1, \dots, \lambda_q) dt.$$

Hence, $s^{\boldsymbol{\alpha}}_{q,T}$ must vanish for $q \geq 2$ implying that $\mathrm{Crt}^{\boldsymbol{\alpha}}_u(\mathcal{T})$ lives in the first Wiener chaos $\mathcal{K}_1$. Now, $\mathcal{K}_1$ contains only Gaussian r.v. Hence, we have a contradiction since $\mathrm{Crt}^{\boldsymbol{\alpha}}_u(\mathcal{T})$ is a discrete r.v. This finishes the proof.

### *4.4. Singularity in the case $u = -\infty$*

Let $\chi(\mathcal{T})$ be the Euler characteristic of $\mathcal{T}$. We fix a set $L \subset \{1, \dots, d\}$ of free varying coordinates and we set $\ell = \#L$. Besides, let $\mathcal{T}_L$ stand for the union of the $2^{d-\ell}$ faces obtained by fixing to 0 or $T$ the coordinates not lying in $L$. Define

$$\mu_k(L) := \#\Big\{ v \in \mathcal{T}_L : X'_j(v) = 0, \text{ for } j \in L; \text{ index } (X''_{ij}(v), i, j \in L) = \ell - k \\ ; \text{ the } d-\ell \text{ outwards derivative are positive} \Big\}.$$

We have [2], Sec. 9.4,

$$\chi(\mathcal{T}) = \sum_{L \subset \{1,\dots,d\}} \sum_{k=0}^{\ell} (-1)^k \mu_k(L). \tag{19}$$

In particular, for $L = \{1, \dots, d\}$ we retrieve the modified Euler characteristic as

$$\Phi(\mathcal{T}) := \sum_{k=0}^{d} (-1)^k \mathrm{Crt}^{\ell-k}_{-\infty}(\mathcal{T}).$$

The first remark is that, of course, $\chi(\mathcal{T}) = 1$, the second one is that Theorem 2.1 applied to each one of the faces in $\mathcal{T}_L$, $L \neq \{1, \dots, d\}$ implies that $\mathrm{Var}(\mu_k(L)) \leq (\mathtt{const}) T^{\ell}$. Therefore, $T^{-\frac{d}{2}} \Phi(\mathcal{T}) \to 0$ proving the singularity.

## Appendix A: Wick's formula

Recall Wick's formula for $X_1, X_2, X_3, X_4$ centered and jointly Gaussian,

$$\mathbb{E}[X_1X_2X_3X_4] = \mathbb{E}[X_1X_2]\mathbb{E}[X_3X_4] + \mathbb{E}[X_1X_3]\mathbb{E}[X_2X_4] + \mathbb{E}[X_1X_4]\mathbb{E}[X_2X_3],$$
$$\mathbb{E}[H_2(X_1)X_2X_3] = 2\mathbb{E}[X_1X_2]\mathbb{E}[X_1X_3].$$

This is a consequence of the well known diagram formula, see [29], Lemma A.1.

## Appendix B: Arcones inequality

We recall the following lemma from Arcones ([3], page 2245). Let $X$ be a standard Gaussian vector on $\mathbb{R}^N$ and $h : \mathbb{R}^N \to \mathbb{R}$ a measurable function such that $\mathbb{E}[h^2(X)] < \infty$, and let us consider its $L^2$ convergent Hermite expansion

$$h(\mathbf{x}) = \sum_{q=0}^{\infty} \sum_{|\mathbf{k}|=q} h_{\mathbf{k}} H_{\otimes \mathbf{k}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N.$$

The Hermite rank of $h$ is defined as

$$\operatorname{rank}(h) = \inf\{\tau : \exists\, \mathbf{k} \in \mathbb{N}^N, |\mathbf{k}| = \tau ; \mathbb{E}[(h(X) - \mathbb{E}h(X))H_{\otimes \mathbf{k}}(X)] \neq 0\}.$$

Then, we have the following result.

**Lemma B.1** ([3]). *Let $W = (W_1, \dots, W_N)$ and $Q = (Q_1, \dots, Q_N)$ be two mean-zero Gaussian random vectors on $\mathbb{R}^N$. Assume that*

$$\mathbb{E}[W_j W_k] = \mathbb{E}[Q_j Q_k] = \delta_{j,k},$$

*for each $1 \leq j, k \leq N$. We define*

$$r^{(j,k)} = \mathbb{E}[W_j Q_k].$$

*Let $h$ be a function on $\mathbb{R}^N$ with finite second moment and Hermite rank $\tau$, $1 \leq \tau < \infty$, define*

$$\Psi := \max\left\{ \max_{1 \leq j \leq N} \sum_{k=1}^{N} |r^{(j,k)}|, \max_{1 \leq k \leq N} \sum_{j=1}^{N} |r^{(j,k)}| \right\}.$$

*Then*

$$|\operatorname{Cov}(h(W), h(Q))| \leq \Psi^\tau \mathbb{E}[h^2(W)].$$

□

## Appendix C: Proof of Corollary 2.3

It suffices to consider one particular index $k \in \{0, \dots, d\}$. It is clear that

$$\mathbb{E}\big(\mathrm{Crt}_u^k(\mathcal{T})\big) = (\texttt{const})T^d.$$

The proof of the following lemma is easy using the "sandwich method".

**Lemma C.1.** *Let*

$$A_T = T^{-d} B_T,$$

*where $B_T$ is non-decreasing in $T$. The a.s. convergence of $A_T$ as $T \to \infty$ is equivalent to the convergence on any geometrical sequence $T_n = \rho^n, \rho > 1$.*

Because of Theorem 2.1 we have

$$\operatorname{Var}\big(\mathrm{Crt}_u^k(\mathcal{T}) \leq (\texttt{const})T^d.$$

This is sufficient to ensure the convergence of the series with general term

$$\operatorname{Var}\Big(\frac{\mathrm{Crt}_u^k([-T_n, T_n]^d)}{T_n^d}\Big),$$

giving, by Markov inequality and Borel-Cantelli lemma, the a.s. convergence on every geometrical sequence. This concludes the proof.

**Appendix D: Proof of Proposition 2.4**

We use a time-scaling argument to ensure that $\mathrm{Var}(X'(t)) = -r''(0) = \mathrm{Id}_d$. Let $B$ be a Borel set of $\mathbb{R}^d$ and let $N(u,B)$ be, as above, the number of points included in $B$ such that $X'(t) = u$. By use of the triangular inequality, we see that it suffices to establish the finiteness of the second moment for a sufficient small set $B$. Thus, we can assume that $t$ is sufficiently small. Our main tool is the Kac-Rice formula of order 2 which requires the distribution of $(X'(s), X'(t))$ to be N.D. for all $s, t \in B$ $s \neq t$. this will be proved later on in (22). Then [6]

$$\mathbb{E}\big(N(u,B)\big(N(u,B)-1\big)\big) = \int_{B^2} \mathbb{E}_{\mathcal{C}}\big|\det(X''(s)\det(X''(t)\big|p_{X'(s),X'(t)}(u,u)dsdt. \tag{20}$$

Here, $\mathbb{E}_{\mathcal{C}}$ is the expectation conditional to $\mathcal{C} := \{X'(s) = X'(t) = u\}$.

Our purpose is to give bounds to the integrand in (20). Because of stationarity, we have only to consider the case $s = 0$. We first study the joint density

$$p_{X'(0),X'(t)}(u,u) \leq (\texttt{const})\big(\det \mathrm{Var}(X'(0), X'(t))\big)^{-\frac{1}{2}}.$$

Let us consider a sequence $t_i, i \in \mathbb{N}$, tending to zero and such that

$$\frac{t_i}{\|t_i\|} \to \mu \in \mathbb{S}^{d-1},$$

where $\mathbb{S}^{d-1}$ stands for the unit sphere in $\mathbb{R}^d$. Using the fact that a determinant is invariant by adding to some row (or column) a linear combination of the other rows (or columns), using (A1), we get

$$\begin{aligned}\det\Big(\mathrm{Var}(X'(0), X'(t_i))\Big) &= \|t_i\|^{2d}\det\Big(\mathrm{Var}\big(X'(0), \frac{1}{\|t_i\|}\big(X'(t_i) - X'(0)\big)\big)\Big)\\ &\simeq \|t_i\|^{2d}\det\Big(\mathrm{Var}\big(X'(0), X''(0)\mu\big)\Big),\end{aligned} \tag{21}$$

Using the compactness of the sphere, this implies that as $t \to 0$:

$$p_{X'(0),X'(t)}(u,u) \leq (\texttt{const})\|t\|^{-d}. \tag{22}$$

We consider now the remaining factor in the integrand of (20). We define

$$\mathcal{A}(t,u) = \mathbb{E}_{\mathcal{C}}\big|\det(X''(0))\det(X''(t))\big|. \tag{23}$$

Using the Cauchy-Schwarz inequality, and the symmetry of the roles of 0 and $t$, we have

$$\mathcal{A}(t,u) \leq \mathbb{E}_{\mathcal{C}}\big(\det{}^2(X''(0))\big).$$

Let us use the parameterisation $t = \mu\rho$, $\rho > 0, \mu \in \mathbb{S}^{d-1}$. The condition is now $\mathcal{C} := \{X'(0) = u, \frac{X'(\mu\rho)-X'(0)}{\rho} = 0\}$.

The $2d \times 2d$ variance-covariance matrix of the conditioners defining $\mathcal{C}$ is now

$$\mathrm{Var} := \begin{pmatrix} \mathrm{Id}_d & -\rho B(\rho,\mu) \\ -\rho B(\rho,\mu) & 2B(\rho,\mu) \end{pmatrix},$$

where

$$B(\rho,\mu) = \rho^{-2}\big(\mathrm{Id}_d + r''(\mu\rho)\big)$$

is a $d \times d$ matrix. We have

$$2B = r^{(4)}(0)_{:,:,\mu,\mu} + o(1),$$

where $r^{(4)}(0)_{:,:,\mu,\mu}$ is the matrix with entry $(i,j)$, $i,j = 1,\dots,d$, equal to $r^{(4)}(0)[e_i, e_j, \mu, \mu]$, $e_i$ being the $i$-th basis vector.

The covariance between $X''(0)$ and $X'(t)$ is given by the tri-variant tensor $r^{(3)}(t)$. This implies that the covariance between $X''(0)$ and the conditioners takes the value

$$\mathrm{Cov} := \mathrm{Cov}\Big(X''(0), \big(X'(0), \frac{X'(\mu\rho) - X'(0)}{\rho}\big)\Big) = \Big(0_{d^3}, \frac{1}{\rho}\big(r^{(3)}(\mu\rho)\big)\Big),$$

We now consider another conditioning, namely with respect to

$$\mathcal{C}' := \{X'(0) = u, X''(0)\mu = 0\}.$$

It is trivial to see that

$$\mathbb{E}_{\mathcal{C}'}(\det^2(X''(0))) = 0.$$

The variance-covariance matrix on the conditioners is now

$$\mathrm{Var}' := \begin{pmatrix} \mathrm{Id}_d & 0 \\ 0 & r^{(4)}(0)_{:,:,\mu,\mu} \end{pmatrix}.$$

Because of (A5), this matrix is N.D. The covariance is now

$$\mathrm{Cov}' := \mathrm{Cov}\big(X''(0), \big(X'(0), X''(0)\mu\big)\big) = \Big(0_{d^3}, r^{(4)}(0)_{:,:,:,\mu}\Big),$$

We now use the following argument about the expression of $\mathbb{E}\big(\det^2(M)\big)$ where $M$ is some $d \times d$ Gaussian matrix. It is the expectation of a polynomial of degree $2d$ in the entries of $M$. To compute it, we can decompose each entry as the sum of its expectation and a centered random variable. It is well known that the expectation of the product of $m$ Gaussian centered variables (possibly equal) is zero if $m$ is odd and is given by the Wick formula if $m$ is even as the sum over pairwise groupings of products of the $m/2$ covariances associated at the grouping. In conclusion, $\mathbb{E}\big(\det^2(M)\big)$ is a degree $2d$ polynomial in the $d(d+1)/2$ entries of the expectation and in the $d(d+1)(d(d+1)+2)/8$ entries of the variance-covariance matrix. But these details are not really needed: all that we need to know is that this function is a polynomial and hence, a local Lipchitz function. Suppose for the moment that $\mu$ is fixed. In such a case the distribution under $\mathcal{C}'$ is fixed, giving that

$$\begin{aligned}\mathbb{E}_{\mathcal{C}}(\det^2(X''(0))) &= \mathbb{E}_{\mathcal{C}}(\det^2(X''(0))) - \mathbb{E}_{\mathcal{C}'}(\det^2(X''(0))) \\ &\leq (\texttt{const})_\mu \|\mathrm{Var}_{\mathcal{C}}(X''(0)) - \mathrm{Var}_{\mathcal{C}'}(X''(0))\| + (\texttt{const})_\mu \|\mathbb{E}_{\mathcal{C}}(X''(0)) - \mathbb{E}_{\mathcal{C}'}(X''(0))\|,\end{aligned}$$

where $(\texttt{const})_\mu$ denotes a "constant" depending on $\mu$. Actually, the local Lipchitz condition implies that one can use the same value for the constant $(\texttt{const})_\mu$ on sufficiently small open sets, thus, one can use a continuous version of this constant. The compacity of $\mathbb{S}^{n-1}$ yields an uniform constant $(\texttt{const})$ independent of $\mu$ such that

$$\begin{aligned}\mathbb{E}_{\mathcal{C}}(\det^2(X''(0))) &= \mathbb{E}_{\mathcal{C}}(\det^2(X''(0))) - \mathbb{E}_{\mathcal{C}'}(\det^2(X''(0))) \\ &\leq (\texttt{const}) \|\mathrm{Var}_{\mathcal{C}}(X''(0)) - \mathrm{Var}_{\mathcal{C}'}(X''(0))\| + (\texttt{const}) \|\mathbb{E}_{\mathcal{C}}(X''(0)) - \mathbb{E}_{\mathcal{C}'}(X''(0))\|.\end{aligned}$$

The regression formulas imply that

$$\mathbb{E}_{\mathcal{C}}(X''(0)) = \mathrm{Cov}\ \mathrm{Var}^{-1}\begin{pmatrix} u\mathrm{J}_d \\ 0_d \end{pmatrix},$$
$$\mathrm{Var}_{\mathcal{C}}(X''(0)) = r^{(4)}(0) - \mathrm{Cov}\ \mathrm{Var}^{-1}\mathrm{Cov}^\top,$$
$$\mathbb{E}_{\mathcal{C}'}(X''(0)) = \mathrm{Cov}'(\mathrm{Var}')^{-1}\begin{pmatrix} u\mathrm{J}_d \\ 0_d \end{pmatrix},$$
$$\mathrm{Var}_{\mathcal{C}'}(X''(0)) = r^{(4)}(0) - \mathrm{Cov}'(\mathrm{Var}')^{-1}(\mathrm{Cov}')^\top,$$

where $\mathrm{J}_d$ is the "all one vector" of size $d$. The formulas above need some additional details: for the first one, for example, let $C_{ijk}$ be the entries of Cov and $V_{k,k'}$ those of $\mathrm{Var}^{-1}$, then the entry $ij$ of $\mathbb{E}_{\mathcal{C}}(X''(0))$ is

$$u\sum_{k=0}^{2d}\sum_{k'=0}^{d} C_{ijk}V_{kk'}.$$

For the second one, the entry $i_1, i_2, i_3, i_4$ of $\mathrm{Cov}\ \mathrm{Var}^{-1}\mathrm{Cov}^\top$ is

$$\sum_{k,k'} C_{i_1,i_2,k}V_{k,k'}C_{i_3,i_4,k'}.$$

Recalling that, for fixed $\mu$, $\mathrm{Cov}'$ and $\mathrm{Var}'$ are fixed with $\mathrm{Var}'$, non singular we get

$$\mathbb{E}_{\mathcal{C}}(\det{}^2(X''(0))) \leq (\mathtt{const})\|\mathrm{Var} - \mathrm{Var}'\| + (\mathtt{const})\|\mathrm{Cov} - \mathrm{Cov}'\|. \tag{24}$$

Let us consider the first term in r.h.s. above. It is bounded by

$$\Big[2\rho^{-2}\big(r''(\mu\rho) - r''(0)\big)\Big] - r^{(4)}(0)_{:,:,\mu,\mu}.$$

By application of the Taylor formula with integral remainder, this last term is equal to

$$\int_0^1 W(\xi)\big[r^{(4)}(\xi\mu\rho)_{:,:,\mu,\mu} - r^{(4)}(0)_{:,:,\mu,\mu}\big]d\xi.$$

where $W(\xi)$ is some weight function.

Let us consider the second term. it is bounded by

$$(\mathtt{const})\big(\frac{1}{\rho}\big(r^{(3)}(\mu\rho)\big) - r^{(4)}(0)_{:,:,:,\mu}\big).$$

By direct integration,

$$\frac{1}{\rho}\big(r^{(3)}(\mu\rho)\big) - r^{(4)}(0)_{:,:,:,\mu} = \frac{1}{\rho}\big(r^{(3)}(\mu\rho) - r^{(3)}(0)\big) - r^{(4)}(0)_{:,:,:,\mu} = \int_0^1 r^{(4)}(\xi\mu\rho)_{:,:,:,\mu} - r^{(4)}(0)_{:,:,:,\mu}\ d\xi.$$

Giving that the r.h.s. of (24) is bounded by

$$(\mathtt{const})\int_0^1 W'(\xi)\|r^{(4)}(\xi\mu\rho)_{:,:,:,\mu} - r^{(4)}(0)_{:,:,:,\mu}\|d\xi \leq (\mathtt{const})\int_0^1 \|r^{(4)}(\xi\mu\rho)_{:,:,:,\mu} - r^{(4)}(0)_{:,:,:,\mu}\|d\xi,$$

since $W'(\xi)$ is bounded.

Gathering together the different bounds obtained so far, we get that the r.h.s. in (20) is bounded by

$$(\mathtt{const})\int_B\int_0^1 \frac{\|r^{(4)}(\xi t) - r^{(4)}(0)\|}{\|t\|^d}d\xi dt = (\mathtt{const})\int_0^1\int_B \frac{\|r^{(4)}(\xi t) - r^{(4)}(0)\|}{\|t\|^d}dtd\xi.$$

The same calculation proves the uniform domination of the integrand in the Kac-Rice formula of order 2, this implies the continuity. This concludes the proof of the first assertion.

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