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Self-Financing Model for Cabbage Crops with Pest Management

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Abstract

Smallholder farmers rely on their farm earnings to cover operating costs and generate income. That is not an easy task because of the pests, which reduce yields and generate plant protection costs. The farm yield and plant protection depend on the budget capacity of the farmer. In this work, we want to explore conditions for a sustainable and self-financing cabbage farm. We propose then a non-linear mathematical model for cabbage crops by considering the current account of the plantation as a dynamic variable. We assume that this variable increases due to the sale of cabbages, and provides for the seedling purchase, the plant protection costs, and the grower's income. In the first part, we analyze the model without pest management. We determine how the budget must be spent and we show the existence of a double transcritical bifurcation. We quantify the seasonal yield and income, and estimate the damage due to pest herbivory. In the second part, we analyze a slightly simplified version of our model and obtain the existence of a backward bifurcation. Furthermore, we show that botanical pesticides can be used to prevent and eradicate the pest spread with relatively low plant protection costs.

Keywords: Dynamic system, Reproductive number, Self-financing smallholder, Vegetable crop, Botanical pesticide, *Plutella xylostella*

Highlights:

- Bio-economic modeling is more adapted for understanding agricultural issues.
- Cabbage is a robustly sustainable and self-financing crop.
- Using botanical pesticides is an affordable pest management technique.

1 Introduction

Agriculture occupies a crucial place in the development and research programs of many countries. The added value generated by agriculture, forestry, and fishing increased by 73% between 2000 and 2019 with a peak of around USD 3.5 trillion in 2018 [1]. In Africa, this increase is around USD 234 billion. In 2020, primary crops production was around 9.3 billion tonnes. Vegetables represent 12% of primary crops production, that is 1,128 million tonnes [2], cabbage being the fourth most widely cultivated vegetable, with a 6% share of the overall vegetable production [2]. Cabbage (*Brassica oleracea varieta capitata*) is used for human consumption [3, 4] and features in several diet programs [5, 6]. Furthermore, cabbage crops constitute a substantial source of income for African smallholder farmers [7, 8].

FAO estimates that annually up to 40% of the global crop production is lost due to pests, which results in costs over USD 220 billion, including at least USD 70 billion due to invasive insects. As for the diamondback moth (*Plutella xylostella*), it attacks cruciferous crops [9]; it is the most destructive pest of cabbage crops worldwide [10, 11, 12, 13]. The insect can be found in Senegal [10], Benin [14], Ghana [15], South Africa [16], Togo [17], Brazil [18], India [12], Canada [19]. Its development cycle in tropical regions such as India allows for up to 16 generations in a single year [12]. The most significant damage is observed during the third larval stage. The larvae feed on the soft parts of cabbage leaves [10, 9] and can cause the death of the host in the case of heavy infestation [20]. Controlling this pest is hence a major issue.

Pest management based on synthetic pesticides has its limits, as the pest developed resistance to most of the active substances in commonly used pesticides [21, 9, 22]. In addition, an excessive use of synthetic pesticides is harmful to the environment and causes significant health problems for human [17, 23]. In this context, botanical pesticides based on crude extracts from locally-growing plants are a viable alternative [24, 25, 12, 15]. The use of these pesticides is safe for humans and their environment [17, 26, 27]. In addition to their biocide properties, botanical pesticides have repellent and antifeedant effects [28, 15, 27], and attract natural enemies of plutella xylostella [29]. Furthermore, botanical pesticides can be produced locally using simple methods. This has the direct effect of reducing treatment costs and therefore, substantially increasing profits for smallholder farmers [30, 31]. In general, smallholder farms must be self-financing; that is, the income generated by the farm must cover all expenses incurred by the farm, as well as the owner's living expenses.

Bio-economic models have been extensively studied (see [32, 33] and references therein). In these models, the net economic revenue is given by the Gordon's algebraic equation [32] and resources growth following natural laws [34]. This classic approach cannot adequately address our self-financing concern. In fact, the farm yield and plant protection depend on the budget capacity of the farmer. A model that incorporates this budgetary constraint as a dynamic variable is therefore more appropriate [35, 36].

Thus, we developed a bio-economic model with a threefold purpose: to quantify the damage caused by the attack of plutella larvae in a cabbage field; to assess the effectiveness of using botanical pesticides in controlling pest; and to evaluate the sustainability and the profitability of such systems. While taking into account the specific features of the interaction between cabbage and its pest, our model presents how a smallholder farmer can distribute the income from his farm to not only sustain his farming activity but also make a living from it, with or without pest management based on botanical pesticides.

This paper is organized as follows. Section 2 describes the self-financing model formulation and establishes the positivity and boundedness properties of its solutions. In Section 3, we study the uncontrolled model. Different equilibria and their stability conditions are obtained together with some numerical simulations. In Section 4, we study a slightly simplified controlled model, which has the same dynamical properties as the complete model presented in Section 2. Pest management is based on the use of botanical pesticides, both as treatment and prophylaxis. In this setting, the model exhibits a backward phenomenon. Theoretical results are then illustrated by numerical experiments. Finally, the conclusion is drawn in Section 5.

2 Mathematical model and basic properties

In this section, we build the mathematical model and establish some of its basic properties.

2.1 Model formulation

As young cabbages are more susceptible than adult ones [10], the biomass in our model is split into a young susceptible biomass and a resistant adult biomass, denoted by B_y and B_a respectively. We assume a logistic growth for each biomass with growth rates r_{0y} and r_{0a}, and an aging rate γ, where $1/\gamma$ represents the average susceptibility duration. We assume that the cultivated soil land contains enough nutrients to allow normal growth of plants. This situation is observable with volcanic soils which are very fertile, as its is the case of in the West region in Cameroon[37, 38]. Adult biomass is harvested at rate h. We assume that, without investment in replanting, the plantation is inevitably doomed to disappear. In mathematical language, this natural assumption is described by the following conditions:

$$\alpha_y := \gamma - r_{0y} > 0 \text{ and } \alpha_a := h - r_{0a} > 0. \tag{1}$$

In order to evaluate the plantation sustainability, we introduce the plantation current account as a dynamic variable M. This account is credited as a result of sales of mature biomass harvested and sold at unit price q. The money is withdrawn from this account at rate k to invest to buy new plants, to protect the plantation from pests and finally to cover the farmer's living expenses. The young biomass plantation rate is represented by the bounded function $k_1M/(1 + \delta_M M)$, where k_1 and δ_M represent the investment rate in young biomass and the fear cost of biomass competition, respectively. We chose a bounded function for the plantation rate as a large account balance does not necessarily directly translate into larger fields and manpower. The plant protection cost is αM where α represents the investment rate for pesticide acquisition such that:

$$k > k_1 + \alpha. \tag{2}$$

We denote the smallholder farmer's total income by a variable T, and its dynamics is modeled by the following equation:

$$\dot{T} = kM - \frac{k_1 M}{1 + \delta_M M} - \alpha M.$$

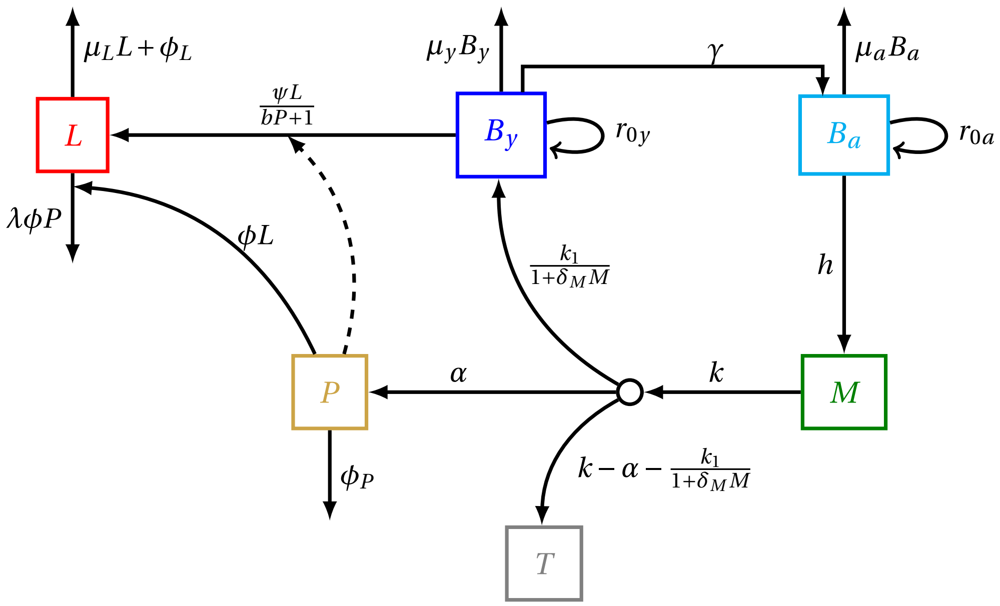

Figure 1: Schematic diagram for system (3). Parameter values are given in Table 1. Young (B_y) and Adult (B_a) biomass growth at the rate r_{0y} and r_{0a}. Young biomass age at the rate γ. Adult biomass harvested at the rate h and sold (at the unit price q). The money from the sale is transferred to the current account (M). This money is withdrawn at the rate k and used for young biomass acquisition (parameters k_1 and δ_M), pesticide acquisition (parameter α) and farmer daily income (T). Larvae (L) feed young biomass (parameter ψ). Botanical pesticide (P) is used for pest management with an antifeedant effect (parameter b). Pesticide is absorbed by larvae (parameter ϕ) which leads to a lethal effect (parameter λ). All biomass, larvae and pesticide undergo mortality (μ_y, μ_a, μ_L, ϕ_L, ϕ_P).

We represent the pest compartment by the variable L. The larvae functional response is modeled by the Holling type I function ψB_y, where ψ is the consumption rate. The growth of larvae is assumed to be logistic-like with the modified growth rate $c\psi B_y L$ and the competition term $\mu_L L^2$ where c and μ_L represent the conversion efficiency and the per capita competition death rate of larvae respectively. In addition, a linear mortality is taken in account with the rate ϕ_L.

The most remarkable features of botanical pesticides are repellent and antifeedant effects. We modified the larvae functional response of cabbage attack by adding a $bP+1$ denominator to represent the antifeedant effect of botanical pesticides. Botanical pesticides purchased and sprayed in the plantation are also absorbed by larvae following a functional response ϕP. This absorption is responsible for an additional larvae mortality rate $\lambda\phi P$ where λ represents the efficacy of pesticides. Furthermore, botanical pesticides decay naturally at a rate ϕ_P.

According to the above considerations, we obtain the following model which describes our self-financing system:

$$\begin{aligned}
\dot{B}_y &= r_{0y}B_y - \mu_y B_y^2 + \frac{r_1 k_1 M}{1+\delta_M M} - \frac{\psi L B_y}{bP+1} - \gamma B_y,\\
\dot{B}_a &= \gamma B_y + r_{0a}B_a - \mu_a B_a^2 - hB_a,\\
\dot{L} &= c\frac{\psi L B_y}{bP+1} - \lambda\phi P L - \mu_L L^2 - \phi_L L,\\
\dot{P} &= a_P\alpha M - \phi P L - \phi_P P,\\
\dot{M} &= qhB_a - kM.
\end{aligned} \tag{3}$$

For model (3), the initial conditions are given by:

$$B_y(0)\geq 0,\ B_a(0)\geq 0,\ L(0)\geq 0,\ P(0)\geq 0,\ \text{and } M(0)\geq 0. \tag{4}$$

A complete description of model parameters is given in Table 1 and model schematic diagram in Fig. 1.

2.2 Positivity and boundedness of solutions

Local existence, uniqueness and non-negativity of the solutions of the Cauchy problem (3–4) are direct consequences of *Cauchy-Lipschitz*'s theorem and a simple analysis of the vector-field on the boundary of the positive orthant $\mathbb{R}_+^5$ [39]. The following theorem establishes the boundedness and global existence of the unique solution of system (3).

Table 1: Description of parameters in system 3. The estimation procedure is explained in Appendix A

Parameters	Descriptions	Values	Units
r_{0y}	Growth rate of young biomass	1.6×10^{-2}	day^{-1}
μ_y	Competition rate of young biomass	1.07×10^{-6}	$\text{kg}^{-1}.\text{day}^{-1}$
r_1	Inverse of the unit purchase price of young biomass	2.17	$\text{USD}.\text{kg}^{-1}$
k_1	Investment rate for young biomass acquisition	0.8	day^{-1}
δ_M	Fear cost of biomass competition	10^{-5} or $1.17 \times 10^{-2\ddagger}$	USD^{-1}
γ	Aging rate of young biomass	0.02	day^{-1}
r_{0a}	Growth rate of adult biomass	1.6×10^{-2}	$\text{kg}.\text{day}^{-1}$
μ_a	Competition rate of adult biomass	1.68×10^{-6}	$\text{kg}^{-1}.\text{day}^{-1}$
h	Harvest rate	0.03	day^{-1}
c	Conversion efficiency	0.6	$\text{larvae}.\text{kg}^{-1}$
ψ	Biomass consumption rate by larvae	4.46×10^{-5}	$\text{larvae}^{-1}.\text{day}^{-1}$
b	Pesticide antifeedant effect	0.01	l^{-1}
λ	Efficacy of botanical pesticide	0.04	$\text{larvae}.\text{l}^{-1}$
ϕ	Uptake rate of botanical pesticide by larvae	0.05	$\text{larvae}^{-1}.\text{day}^{-1}$
μ_L	Competition rate of larvae	2×10^{-6}	$\text{larvae}^{-1}.\text{day}^{-1}$
ϕ_L	Natural mortality rate of larvae	0.07	day^{-1}
a_P	Inverse of the pesticide unit price	2.16	$\text{l}.\text{USD}^{-1}$
α	Investment rate for pesticide acquisition	1.35×10^{-2}	day^{-1}
ϕ_P	Natural decay rate of botanical pesticide	0.1	day^{-1}
q	Unit selling price of adult biomass	0.69	$\text{USD}.\text{kg}^{-1}$
k	Plantation self- financing rate	0.9	day^{-1}

Theorem 1. *The domain* Ω *defined by*

$$\begin{aligned}\Omega = \Big\{&(B_y; B_a; L; P; M) \in \mathbb{R}^5_+ | B_y \leq K_y, B_a \leq \frac{\gamma K_y}{\alpha_a}, L \leq \frac{c\psi K_y}{\mu_L}, \\ &M \leq \frac{qh\gamma K_y}{k\alpha_a}, P \leq \frac{a_p \alpha q h \gamma K_y}{k \alpha_a \phi_P}\Big\}, \quad \textit{where} \quad K_y = \frac{r_1 k_1}{\delta_M \alpha_y},\end{aligned} \tag{5}$$

is positively invariant for the flow of the system (3) *and attractive.*
Furthermore, the solution $(B_y(t); B_a(t); L(t); P(t); M(t))$ *of system* (3) *with initial conditions satisfying assumption* (4) *is uniquely defined for all* $t \geq 0$.

Proof. See Appendix B. ☐

3 First case study: self-financing uncontrolled model

In this section, we assume that the smallholder farmer does not apply any pest management strategy. Under this assumptions, we set $\alpha = 0$ and $P = 0$ in system (3) and obtain the following self-financed uncontrolled model:

$$\begin{aligned}\dot{B_y} &= r_{0y} B_y - \mu_y B_y^2 + \frac{r_1 k_1 M}{1 + \delta_M M} - \psi L B_y - \gamma B_y, \\ \dot{B_a} &= \gamma B_y + r_{0a} B_a - \mu_a B_a^2 - h B_a, \\ \dot{L} &= c\psi L B_y - \mu_L L^2 - \phi_L L, \\ \dot{M} &= qhB_a - kM.\end{aligned} \tag{6}$$

3.1 Equilibrium and stability analysis

3.1.1 Equilibria

Sub-model (6) potentially admits three equilibria:

- the extinction equilibrium $E_0 = (0; 0; 0; 0)$;
- the pest-free equilibrium (PFE) $E_1 = (B_{y_1}; B_{a_1}; 0; M_1)$ when $\mathcal{R}_B > 1$;
- the endemic equilibrium $E^\star = (B_y^\star; B_a^\star; L^\star; M^\star)$ when $\mathcal{R}_B > 1$ and $\mathcal{R}_0 > 1$;

where

$$\mathcal{R}_B = \frac{k_1\gamma r_1 qh}{k\alpha_y\alpha_a} \quad \text{and} \quad \mathcal{R}_0 = \frac{c\psi\left(\alpha_a + \mu_a B_{a_1}\right) B_{a_1}}{\gamma\phi_L}. \tag{7}$$

The biomass reproductive number $\mathcal{R}_B$ indicates if cabbage production is financially sustainable for the farm given a cabbage price and the investment policy. The basic reproductive number $\mathcal{R}_0$ gives the number of larvae produced by one larvae during its lifetime when introduced in a healthy plantation. $\mathcal{R}_0$ can be obtained directly by applying the standard next-generation matrix approach [40].

The existence of E_0 is trivial. Thereafter, we show the feasibility of equilibria E_1 and $E^\star$.

Pest-free equilibrium We obtain the coordinates of E_1 by solving the following algebraic equations:

$$r_1\frac{k_1 M}{1+\delta_M M} - \mu_y B_y^2 - \alpha_y B_y = 0, \tag{8}$$

$$\gamma B_y - \mu_a B_a^2 - \alpha_a B_a = 0, \tag{9}$$

$$qhB_a - kM = 0. \tag{10}$$

Using (9) and (10) in (8) and the fact that $B_a \neq 0$, we obtain the following equation in B_a:

$$G_1(B_a) = \frac{r_1 k_1 qh\gamma}{k+\delta_M qhB_a} - \left(\alpha_y\left(\alpha_a + \mu_a B_a\right) + \frac{\mu_y}{\gamma}\left(\alpha_a + \mu_a B_a\right)^2 B_a\right) = 0. \tag{11}$$

From Equation (11), we have :

(i) $G_1(B_a)' = -\frac{r_1 k_1 \delta_M q^2 h^2 \gamma}{(k+\delta_M qhB_a)^2} - \alpha_y\mu_a - \frac{\mu_y}{\gamma}\left(\left(\alpha_a+\mu_a B_a\right)^2 + 2\mu_a\left(\alpha_a + \mu_a B_a\right)Ba\right) < 0$ for all $B_a \geq 0$;

(ii) $G_1(0) = \alpha_y\alpha_a\left(\mathcal{R}_B - 1\right) > 0$, if and only if $\mathcal{R}_B > 1$;

(iii) $G_1\left(\frac{\gamma K_y}{\alpha_a}\right) < \frac{r_1 k_1 \alpha_a}{\delta_M K_y} - \alpha_a\alpha_y = 0.$

Thus, equation $G_1(B_a) = 0$ has a unique positive solution denoted by B_{a_1} in the interval $\left(0; \frac{\gamma K_y}{\alpha_a}\right)$ if and only if $\mathcal{R}_B > 1$. The remaining components of E_1 are obtained as follows: $B_{y_1} = \frac{(\alpha_a+\mu_a B_{a_1})B_{a_1}}{\gamma}$ and $M_1 = \frac{qhB_{a_1}}{k}$.

Endemic equilibrium The components of $E^\star$ are obtained by solving the following algebraic equations:

$$r_1\frac{k_1 M}{1+\delta_M M} - \mu_y B_y^2 - \psi L B_y - \alpha_y B_y = 0, \tag{12}$$

$$\gamma B_y - \mu_a B_a^2 - \alpha_a B_a = 0, \tag{13}$$

$$c\psi B_y - \mu_L L - \phi_L = 0, \tag{14}$$

$$qhB_a - kM = 0. \tag{15}$$

Plugging (13) in (12) we obtain for all $B_a \in (0; B_{a_1})$:

$$L(B_a) = \frac{G_1(B_a)}{\psi\left(\alpha_a + \mu_a B_a\right)}, \tag{16}$$

where G_1 is given by (11).

Similarly, plugging (13) and (15) in (14), we obtain for $B_a \geq \beta$, where β denotes the positive root of the right hand-side of equation (17):

$$L(B_a) = \frac{c\psi\left(\alpha_a + \mu_a B_a\right) B_a - \gamma\phi_L}{\mu_L\gamma}. \tag{17}$$

Equating (16) and (17), we have $G_1(B_a) = G_2(B_a)$ where

$$G_2(B_a) = \psi\left(\alpha_a + \mu_a B_a\right)\left[\frac{c\psi\left(\alpha_a + \mu_a B_a\right) B_a - \gamma\phi_L}{\mu_L\gamma}\right].$$

It is clear that $G_2(B_a)' > 0$ for all $B_a \geq \beta$, as it ensures that both factors are positive and increasing.

Since $G_1(B_a) > 0$ if and only if $B_a < B_{a_1}$ and $G_2(B_a) > 0$ if and only if $B_a > \beta$ it comes that the two curves G_1 and G_2 intersect with $L(B_a) > 0$ if and only if $\beta < B_{a_1}$. This intersection is unique as $G_1'(.) < 0$ and $G_2'(.) > 0$ in the $[\beta, B_{a_1}]$ interval. Using Equation (7), the condition $\beta < B_{a_1}$ is equivalent to $\mathcal{R}_0 > 1$. Let $B_a^\star$ be the interception point of G_1 and G_2. Substituting B_a by $B_a^\star$ in (13), (15), and (16), we obtain positive values of $B_y^\star$, $M^\star$ and $L^\star$, respectively. Thus, sub-model (6) has a unique endemic equilibrium $E^\star$ if and only if $\mathcal{R}_B > 1$ and $\mathcal{R}_0 > 1$.

3.1.2 Stability

Regarding the local stability of the equilibria of sub-model (6), we have the following main result.

Theorem 2.
- *The extinction equilibrium E_0 is always feasible. It is locally asymptotically stable provided $\mathcal{R}_B < 1$ and unstable whenever $\mathcal{R}_B > 1$.*
- *The PFE E_1 is feasible if and only if $\mathcal{R}_B > 1$; it is then unique. It is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*
- *The endemic equilibrium $E^\star$ is feasible if and only if $\mathcal{R}_B > 1$ and $\mathcal{R}_0 > 1$; it is then unique. Moreover, whenever this equilibrium exists, it is locally asymptotically stable.*

Proof. See Appendix C. □

Moreover, we have a global stability result for the extinction equilibrium, stated in the following theorem.

Theorem 3. *The extinction equilibrium E_0 is globally asymptotically stable in Ω if and only if $\mathcal{R}_B \le 1$.*

Proof. We consider the following positive definite function:

$$V = k\alpha_a B_y + qhr_1k_1B_a + \frac{k\alpha_a}{c}L + r_1k_1\alpha_a M.$$

Differentiating the above function V with respect to time, t, along the solutions of sub-model (6), after some algebraic simplifications, we obtain

$$\begin{aligned}\dot{V} = {} & k\alpha_a\alpha_y\left(\mathcal{R}_B - 1\right)B_y - k\alpha_a\mu_y B_y^2 - qhr_1k_1\mu_a B_a^2 \\ & - \frac{k\alpha_a}{c}\left(\mu_L L^2 + \phi_L L\right) - \frac{r_1k_1\alpha_a\delta_M M^2}{1+\delta_M M}.\end{aligned}$$

Thus, V is a strict Lyapunov function for equilibrium E_0 if $\mathcal{R}_B \le 1$. We conclude that E_0 is globally asymptotically stable in Ω if and only if $\mathcal{R}_B \le 1$. □

Let us now investigate the bifurcations of sub-model (6). The following result holds.

Theorem 4. *Sub-model (6) presents a double forward transcritical bifurcation: the first one at $\mathcal{R}_B = 1$ and the second one at $\mathcal{R}_0 = 1$.*

Proof. The forward transcritical bifurcation at $\mathcal{R}_B = 1$ follows from Theorem 2 and Theorem 3. Indeed, equilibrium E_0 changes from stable ($\mathcal{R}_B \le 1$) to unstable ($\mathcal{R}_B > 1$), while the pest-free equilibrium E_1 becomes positive from E_0 and asymptotically stable.

Similarly, when $\mathcal{R}_B > 1$, equilibrium E_1 changes from stable ($\mathcal{R}_0 < 1$) to unstable ($\mathcal{R}_0 > 1$), while the endemic equilibrium $E^\star$ becomes positive from E_1 and asymptotically stable. □

The typical bifurcation diagrams near bifurcation points $\mathcal{R}_B = 1$ and $\mathcal{R}_0 = 1$ are illustrated in Fig. 2(a) and Fig. 2(b), respectively.

We establish the following main result.

Theorem 5. *The domain*

$$\Omega' = \left\{(B_y; B_a; L; M) \in \mathbb{R}_+^4 | B_y \le B_{y_1}, B_a \le B_{a_1}, M \le M_1\right\} \tag{18}$$

is positively invariant. Furthermore, the pest-free equilibrium $E_1 = (B_{y_1}; B_{a_1}; 0; M_1)$ is asymptotically stable in Ω' if and only if $\mathcal{R}_0 \le 1 < \mathcal{R}_B$.

Proof. See Appendix D □

3.2 Simulations

In this section, we perform some numerical simulations to substantiate the theoretical results. All parameter values are given in Table 1. With these values, the biomass reproductive number $\mathcal{R}_B = 3.21$ and the basic reproductive number $\mathcal{R}_0 = 1.64$.

The stability results obtained show that the asymptotic behavior of sub-model (6) is entirely defined by thresholds $\mathcal{R}_B$ and $\mathcal{R}_0$. The condition for a sustainable agroecosystem is $\mathcal{R}_B > 1$. Fig. 3(a) shows, for an array of purchasing prices of young plants from the nurseries, how the smallholder farmer should fix the unit selling price of its cabbages and what level of sustainability it ensures. Starting from the reference parameter values in Table 1, the selling price can drop or the purchase price can increase, but still the production generates a

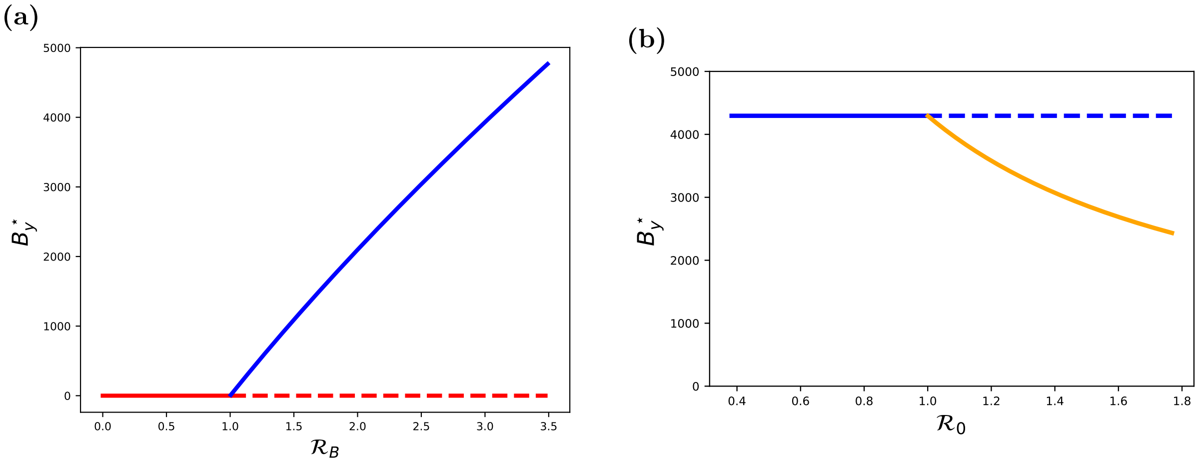

Figure 2: Bifurcation diagrams for sub-model (6), according to the biomass reproductive number $\mathcal{R}_B$ and the basic reproductive number $\mathcal{R}_0$ defined in (7). (a) $\mathcal{R}_B$-bifurcation diagram, obtained by varying parameter q and (b) $\mathcal{R}_0$-bifurcation diagram, obtained by varying parameter ϕ_L, with $\mathcal{R}_B = 3.21 > 1$. All remaining parameter values are given in Table 1. The red, blue and orange lines represent the extinction E_0, pest-free E_1 and endemic $E^\star$ equilibria, respectively. A solid line corresponds to a stable equilibrium, a dashed line to an unstable equilibrium.

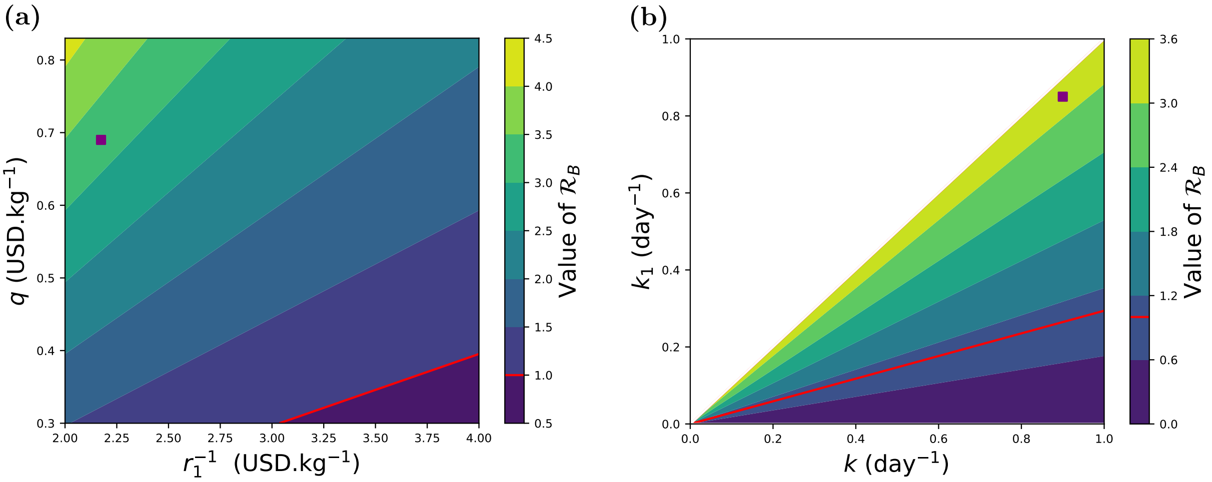

Figure 3: Variation of the biomass reproductive number $\mathcal{R}_B$ as a function of: (a) the unit purchase price of young biomass r_1^{-1} and the unit selling price of adult biomass q; (b) the plantation self-financing and living expense rate k and the investment rate for young biomass k_1. The values of the remaining parameters are given in Table 1. The purple squares represent the parameter values Table 1. The red line represents $\mathcal{R}_B = 1$. The price ranges in sub-figure (a) are realistic values extracted from [30] for q and [41] for r_1^{-1}. The white zone on sub-figure (b) represents the region where $k_1 > k$ which contradicts our modeling hypothesis (2).

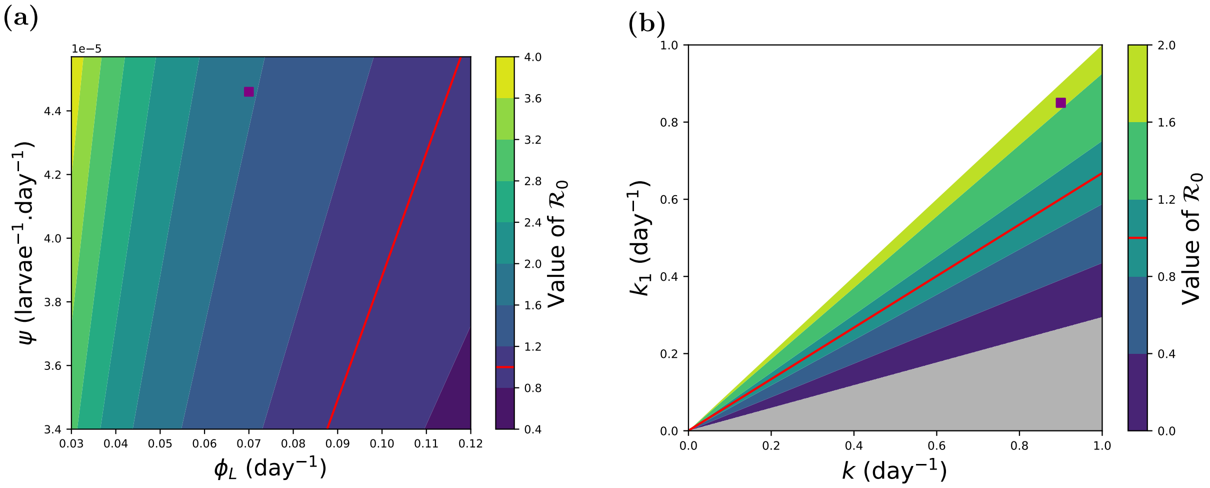

Figure 4: Variation of the basic reproductive number $\mathcal{R}_0$ as a function of: (a) the mortality rate of larvae ϕ_L and the biomass consumption rate of larvae ψ; (b) the self- financing rate k and the investment rate for young biomass k_1. The values of the remaining parameters are given in Table 1. The purple squares represent the parameter values in Table 1. The red line represents $\mathcal{R}_0 = 1$. The price ranges in sub-figure (b) are realistic values extracted from [30] for q and [41] for r_1^{-1}. The white zone on sub-figure (b) represents the region where $k_1 > k$ which contradicts our modeling hypothesis (2); the gray area represents the region where $\mathcal{R}_B \leq 1$ corresponding to the situation where the system converges globally towards the trivial equilibrium.

positive result. To get below the sustainability threshold, both need to change drastically within the realistic price ranges. Cabbage is then a robustly sustainable crop.

In addition, Fig. 3(b) reveals that, with stable market prices (the parameters of cabbage and pest dynamics being set to their reference values in Table 1), the purchase of new seedlings must represent more than 21% (slope of the red line in Fig. 3(b)) of the total expense in order to satisfy the sustainability condition. Finally, in the absence of pests, cabbage is an easy crop to produce, with excellent yields per hectare for a very affordable investment.

Maintaining a healthy plantation without any pest management strategy is a difficult task. Fig. 4(a) shows that the basic reproduction number $\mathcal{R}_0$ is greater than 1 for most realistic values of the pest traits. This leads to the persistence of infestation. Nevertheless, it may be possible to control the pest by under-exploiting the farmland (low k_1 values to drop below the red line, but high enough to remain above the gray area) so that larvae disappear progressively, as shown in Fig. 4(b). Unfortunately, this solution is not feasible from an economic and food security point of view, as the yield and the revenues then drastically decrease.

Fig. 5 illustrates the stability results of sub-model (6), with the reference parameter values in Table 1. This situation corresponds to a farm without any pest management. As $\mathcal{R}_0 = 1.64 > 1$, the endemic equilibrium (orange star) is stable, while the pest-free equilibrium (red dot) is unstable. The convergence towards the endemic equilibrium occurs even with an initial condition close to the pest-free equilibrium (yellow curve).

Let us define the seasonal yield and income at the equilibrium by, respectively,

$$\mathcal{Y} = h \times \overline{B_a} \times A_t \quad \text{and} \quad \mathcal{I} = q \times h \times \overline{B_a} \times A_t, \tag{19}$$

where A_t denotes the average duration of the cropping season and $\overline{B_a}$ is the level of adult biomass at the stable equilibrium. Considering that $A_t = 90$ days, the seasonal yield is around 7.6 tonnes for an income of 5,220 USD, which is consistent with the values per hectare reported by [30].

In Fig. 6, we present the dynamics of state variables B_y (Fig. 6(a)) and L (Fig. 6(b)). The convergence towards the endemic equilibrium is slow and requires at least 1500 days, with oscillations that are typical for a prey-predator system. In addition, the green curve describes a typical behavior of a prey-predator model: as the host biomass starts at a very low level, the pest population decreases, until the biomass reaches a satisfactory level and then the pest outbreak begins.

According to Fig. 4(a), an increase of the pest mortality rate ϕ_L implies an decrease of the basic reproductive number $\mathcal{R}_0$. By considering an increased value $\phi_L = 0.12$ compared to reference value in Table 1, $\mathcal{R}_0 = 0.96 < 1$, so the pest-free equilibrium becomes stable. This is illustrated in Fig. 7 with the same initial conditions as in Fig. 5. At the pest-free equilibrium, the seasonal yield is around 11.1 tonnes, corresponding to an income of 7,750 USD, which is consistent with [30] and [42].

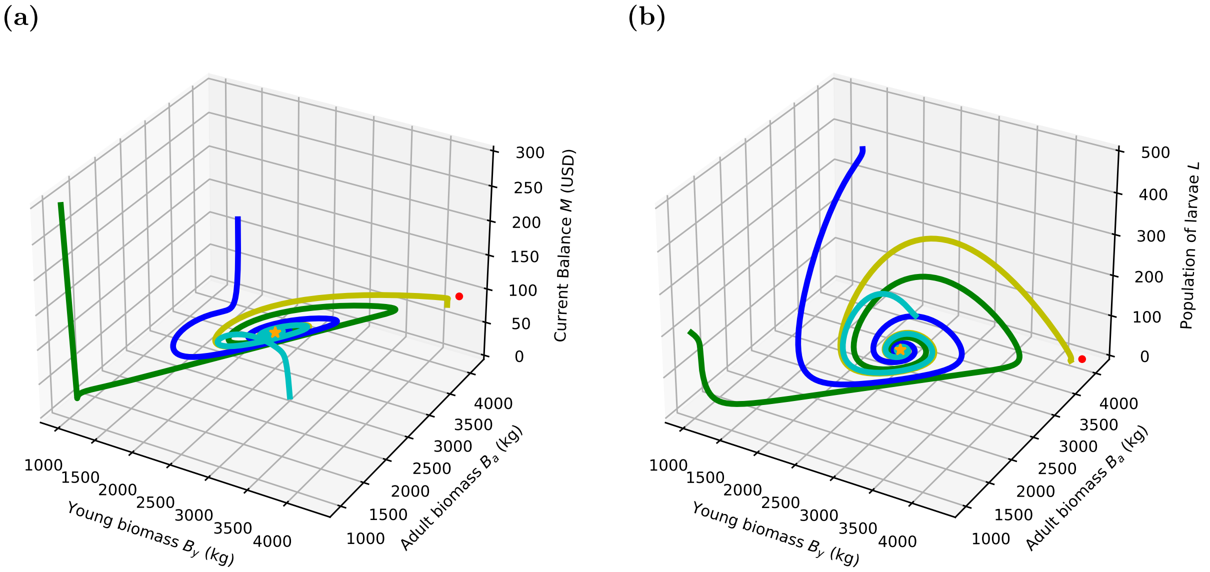

Figure 5: Phase portrait of sub-model (6) in sub-spaces (a) $B_y - B_a - M$ and (b) $B_y - B_a - L$. The orange star is the stable endemic equilibrium $E^\star$ and the red dot is the unstable pest-free equilibrium E_1. Parameter values are given in Table 1. The initial conditions $(B_y(0); B_a(0); L(0); M(0))$ are: $(1000; 1000; 200; 300)$ in green, $(4195.29; 4009.54; 0.88; 84.52)$ in yellow, $(2000; 3000; 500; 200)$ in blue and $(3000; 1000; 0.8; 0)$ in cyan.

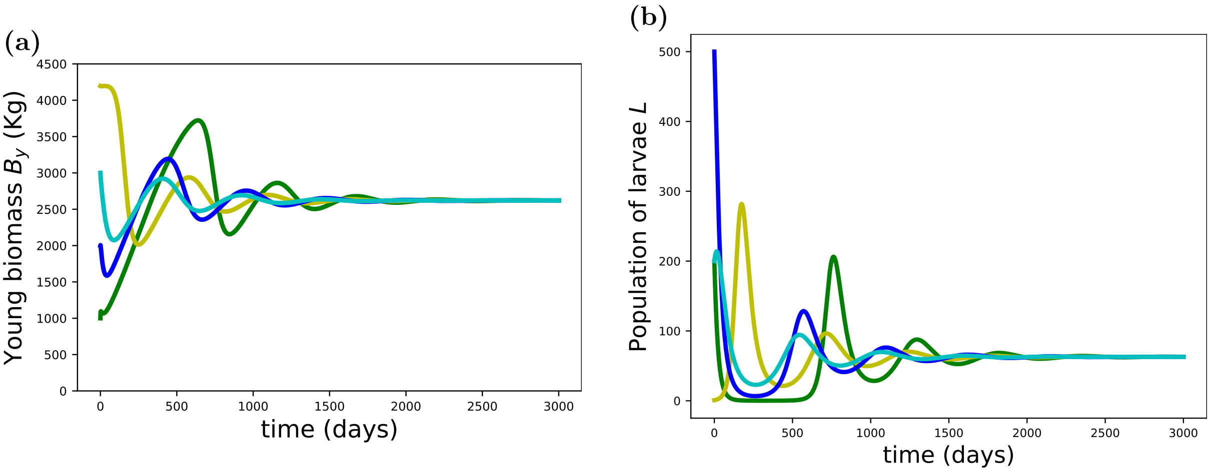

Figure 6: Dynamics of state variables (a) B_y and (b) L of sub-model (6). Parameter values and initial conditions are the same as in Fig. 5.

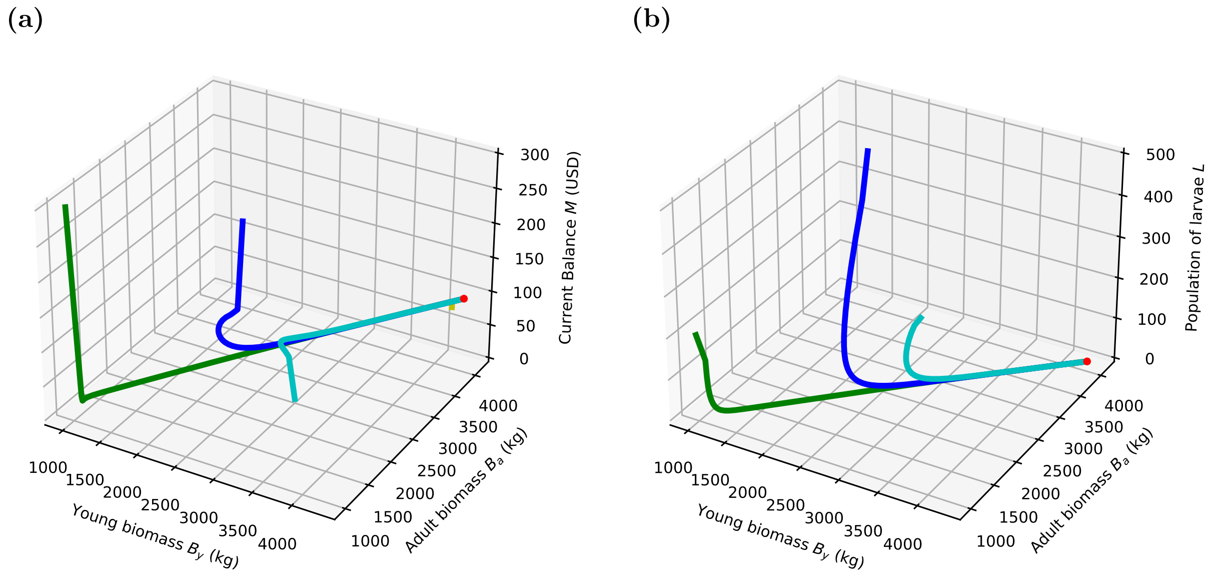

Figure 7: Phase portrait of sub-model (6) in sub-spaces (a) $B_y - B_a - M$ and (b) $B_y - B_a - L$. The red dot is the stable pest-free equilibrium E_1. Parameter values are given in Table 1 excepted $\phi_L = 0.12$ leading to $\mathcal{R}_0 = 0.96 < 1$. The initial conditions are the same as Fig. 5.

4 Second case study: self-financing controlled model

In this section, we perform a mathematical analysis of a slightly simplified version of model (3), in which we neglect the antifeedant action of the botanical pesticides ($b = 0$). Moreover, we assume that the farmer has enough experience to space the plants properly, such that one we can neglect the competition in the two biomass compartments ($\mu_y = \mu_a = 0$). The other values of parameters remain unchanged, as given in Table 1, except $\delta_M = 1.17 \times 10^{-2}$, which is adjusted to obtain reasonable estimations of the cabbage biomass.

Under the above considerations, we obtain the following slightly modified model:

$$\begin{aligned}
\dot{B}_y &= r_{0y}B_y + \frac{r_1 k_1 M}{1+\delta_M M} - \psi L B_y - \gamma B_y,\\
\dot{B}_a &= \gamma B_y + r_{0a}B_a - hB_a,\\
\dot{L} &= c\psi L B_y - \lambda\phi P L - \mu_L L^2 - \phi_L L,\\
\dot{P} &= a_P \alpha M - \phi P L - \phi_P P,\\
\dot{M} &= qhB_a - kM.
\end{aligned} \tag{20}$$

4.1 Equilibrium and stability analysis

4.1.1 Equilibrium

We first determine the equilibria of sub-model (20). The extinction equilibrium $E_{c_0} = (0; 0; 0; 0; 0)$ always exists. The unique pest-free equilibrium

$$\begin{aligned}
E_{c_1} &= \left(\frac{\alpha_a k\left(\mathcal{R}_B - 1\right)}{\gamma\delta_M qh}; \frac{k\left(\mathcal{R}_B - 1\right)}{\delta_M qh}; 0; \frac{a_p\alpha\left(\mathcal{R}_B - 1\right)}{\phi_P\delta_M}; \frac{\mathcal{R}_B - 1}{\delta_M}\right)\\
&= \left(B_{y_{c_1}}; B_{a_{c_1}}; 0; P_{c_1}; M_{c_1}\right)
\end{aligned}$$

exists whenever $\mathcal{R}_B > 1$, with $\mathcal{R}_B$ given by (7).

Endemic equilibria In order to study the existence of the endemic equilibrium, we recall that $\alpha_y = \gamma - r_{0y}$ and $\alpha_a = h - r_{0a}$, and introduce the new threshold

$$\mathcal{R}_e = \frac{c\psi\alpha_a B_{a_{c_1}}}{\gamma\left(\lambda\phi P_{c_1} + \phi_L\right)} \tag{21}$$

representing the effective reproductive number.

We want to solve the system:

$$
\begin{aligned}
r_1\frac{k_1M}{1+\delta_M M}-\psi LB_y-\alpha_y B_y &= 0, && (22)\\
\gamma B_y-\alpha_a B_a &= 0, && (23)\\
c\psi B_y-\lambda\phi P-\mu_L L-\phi_L &= 0, && (24)\\
a_P\alpha M-\phi_P P-\phi PL &= 0, && (25)\\
qhB_a-kM &= 0. && (26)
\end{aligned}
$$

Relations (23) and (26) give:

$$B_y=\frac{\alpha_a B_a}{\gamma}\quad \text{and}\quad M=\frac{qhB_a}{k}. \tag{27}$$

Introducing (27) in (22), we have:

$$L(B_a)=\frac{1}{\psi\alpha_a}\left(\frac{r_1k_1qh\gamma}{k+\delta_M qhB_a}-\alpha_a\alpha_y\right), \tag{28}$$

which is positive if and only if $B_a<B_{a_{c_1}}$.

Using both (28) and (26) in (25), we get:

$$P(B_a)=\frac{a_p\alpha qhB_a}{k\left(\phi_p+\phi\psi\alpha_a L(B_a)\right)}=\frac{D_1\left(k+\delta_M qhB_a\right)B_a}{D_2+D_3B_a}, \tag{29}$$

with

$$
\begin{aligned}
D_1 &= \frac{a_p\alpha\psi\alpha_a qh}{k}>0,\\
D_2 &= k\phi\alpha_a\alpha_y\left(\mathcal{R}_B-1\right)+\phi_p\psi\alpha_a k>0,\\
D_3 &= \delta_M qh\left(\phi_P\psi\alpha_a-\alpha_a\alpha_y\phi\right).
\end{aligned}
$$

The equivalence between the two definitions of P implies that $D_2+D_3B_a>0$ for all $B_a\in(0;B_{a_{c_1}})$.

Finally, using (27), (28) and (29) to replace B_y, L and P by functions of B_a in (24), one has

$$G_3(B_a)=\frac{c\psi\alpha_a}{\gamma}B_a-\lambda\phi P(B_a)-\mu_L L(B_a)-\phi_L=0 \tag{30}$$

that we will study on the biologically feasible interval $(0;B_{a_{c_1}})$ to determine the endemic equilibria.

The derivative of L in (28) with respect to B_a gives

$$L'(B_a)=-\frac{r_1k_1q^2h^2\gamma\delta_M}{\psi\alpha_a(k+\delta_M qhB_a)^2}<0,$$

and its second derivative yields

$$L^{''}(B_a)=\frac{r_1k_1q^3h^3\gamma\delta_M^2}{\psi\alpha_a(k+\delta_M qhB_a)^3}>0. \tag{31}$$

Similarly, the first derivative of P in (29) gives

$$P'(B_a)=D_1\frac{\delta_M qhD_3B_a^2+2\delta_M qhD_2B_a+D_2k}{(D_2+D_3B_a)^2},$$

and

$$P^{''}(B_a)=\frac{2D_1D_2\delta_M r_1k_1\gamma\phi q^2h^2}{(D_2+D_3B_a)^3}.$$

Because $D_2+D_3B_a>0$ for all $B_a\in(0;B_{a_{c_1}})$, we then have $P^{''}(B_a)>0$.

Thus,

$$G_3^{''}(B_a)=-\lambda\phi P^{''}(B_a)-\mu_L L^{''}(B_a)<0. \tag{32}$$

It means that $G_3':B_a\mapsto G_3'(B_a)$ is a decreasing function on $\left(0;B_{a_{c_1}}\right)$. Remarking that $G_3(0)<0$ since $P(0)=0$ and $L(0)>0$ when $\mathcal{R}_B>1$, we can now distinguish two cases.

1. If $G_3'(0)\leq 0$, the function G_3 defined by (30) does not have a positive solution on $\left(0;B_{a_{c_1}}\right)$.
2. If $G_3'(0)>0$, we distinguish two sub-cases.

(a) If $G_3'(B_{a_{c_1}}) \geq 0$ then G_3 admits a unique positive root on $\left(0; B_{a_{c_1}}\right)$ if and only if $G_3(B_{a_{c_1}}) > 0$.

(b) If $G_3'(B_{a_{c_1}}) < 0$ then, there exists a unique $\zeta \in \left(0; B_{a_{c_1}}\right)$ such that $G_3'(\zeta) = 0$. By remaking that G_3 is a decreasing function on $\left(\zeta; B_{a_{c_1}}\right)$, we always have $G_3(\zeta) > G_3\left(B_{a_{c_1}}\right)$. And then:

  - if $0 > G_3(\zeta) > G_3\left(B_{a_{c_1}}\right)$, then G_3 has no positive root on $\left(0; B_{a_{c_1}}\right)$;
  - if $G_3(\zeta) > 0 > G_3\left(B_{a_{c_1}}\right)$, then G_3 has two positive roots on $\left(0; B_{a_{c_1}}\right)$;
  - if $G_3(\zeta) > G_3\left(B_{a_{c_1}}\right) \geq 0$ or $G_3(\zeta) = 0$, then the function G_3 has a unique positive root on $(0; B_{a_{c_1}})$.

We can express the above conditions on G_3 in terms of the effective reproductive number $\mathcal{R}_e$ given by (21) as follows.

Since

$$\begin{aligned} G_3(B_{a_{c_1}}) &= \frac{c\psi\alpha_a}{\gamma} B_{a_{c_1}} - \lambda\phi P(B_{a_{c_1}}) - \mu_L L(B_{a_{c_1}}) - \phi_L \\ &= \frac{c\psi\alpha_a}{\gamma} B_{a_{c_1}} - \lambda\phi P_{c_1} - \phi_L, \end{aligned}$$

it is obvious that

$$G_3(B_{a_{c_1}}) > 0 \Longleftrightarrow \mathcal{R}_e > 1. \tag{33}$$

We will now analyze the threshold situations between the different sub-cases of case 2.(b). These situations are characterized by $G_3(\zeta) = 0$ and $G_3(B_{a_{c_1}}) = 0$. Due to concavity, these cannot be satisfied simulataneously for a unique set of parameters and we have $G_3(\zeta) > G_3(B_{a_{c_1}})$, so that

$$\frac{c\psi\alpha_a}{\gamma}\zeta - \mu_L L(\zeta) - \lambda\phi P(\zeta) > \frac{c\psi\alpha_a}{\gamma} B_{a_{c_1}} - \mu_L L(B_{a_{c_1}}) - \lambda\phi P(B_{a_{c_1}}) \tag{34}$$

In the following, we will analyze for which ϕ_L values $G_3(\zeta) = 0$ or $G_3(B_{a_{c_1}}) = 0$ are satisfied, all other parameters staying unchanged. Among other things, the choice of ϕ_L as varying parameter has the advantage of impacting neither the value of ζ nor that of $B_{a_{c_1}}$ so that inequality (34) still holds.

If we consider that $G_3(\zeta) = 0$ holds for some ϕ_L^s, we get

$$\phi_L^s = \frac{c\psi\alpha_a}{\gamma}\zeta - \mu_L L(\zeta) - \lambda\phi P(\zeta) \tag{35}$$

Similarly, $G_3(B_{a_{c_1}}) = 0$ for some ϕ_L^0 gives

$$\phi_L^0 = \frac{c\psi\alpha_a}{\gamma} B_{a_{c_1}} - \lambda\phi P_{c_1}. \tag{36}$$

Injecting (35) and (36) into (34) then yields

$$\phi_L^s > \phi_L^0.$$

Noting in (21) that $\phi_L \mapsto \mathcal{R}_e(\phi_L)$ is a decreasing function, we obtain $\mathcal{R}_e\left(\phi_L^s\right) < \mathcal{R}_e\left(\phi_L^0\right)$ with, as we have seen in (33)

$$\mathcal{R}_e\left(\phi_L^0\right) = 1 \text{ and } \mathcal{R}_e\left(\phi_L^s\right) = \frac{c\psi\alpha_a B_{a_{c_1}}}{\gamma\left(\lambda\phi_P P_{c_1} + \phi_L^s\right)}.$$

Thus, considering a given ϕ_L value, and since G_3 is a decreasing function of ϕ_L, we make multiple use of

$$0 > G_3(\zeta) \Leftrightarrow G_3(\zeta)_{|\phi_L^s} > G_3(\zeta)_{|\phi_L} \Leftrightarrow \phi_L > \phi_L^s$$

and

$$0 > G_3\left(B_{a_{c_1}}\right) \Leftrightarrow G_3\left(B_{a_{c_1}}\right)_{|\phi_L^0} > G_3\left(B_{a_{c_1}}\right)_{|\phi_L} \Leftrightarrow \phi_L > \phi_L^0.$$

And then,

- $0 > G_3(\zeta) > G_3\left(B_{a_{c_1}}\right) \Leftrightarrow \phi_L > \phi_L^s > \phi_L^0 \Leftrightarrow \mathcal{R}_e(\phi_L) < \mathcal{R}_e\left(\phi_L^s\right) < 1$ (region above blue curve in Fig. 8).
- $G_3(\zeta) > 0 > G_3\left(B_{a_{c_1}}\right) \Leftrightarrow \phi_L^s > \phi_L > \phi_L^0 \Leftrightarrow \mathcal{R}_e\left(\phi_L^s\right) < \mathcal{R}_e(\phi_L) < 1$ (region between the blue and red curves in Fig. 8).
- $G_3(\zeta) > G_3\left(B_{a_{c_1}}\right) > 0 \Leftrightarrow \phi_L^s > \phi_L^0 > \phi_L \Leftrightarrow \mathcal{R}_e\left(\phi_L^s\right) < 1 < \mathcal{R}_e(\phi_L)$ (region below the red curve in Fig. 8).

In the next, $\mathcal{R}_e$ is exactly the threshold given by (21). The existence of endemic equilibria can be summarized as follows.

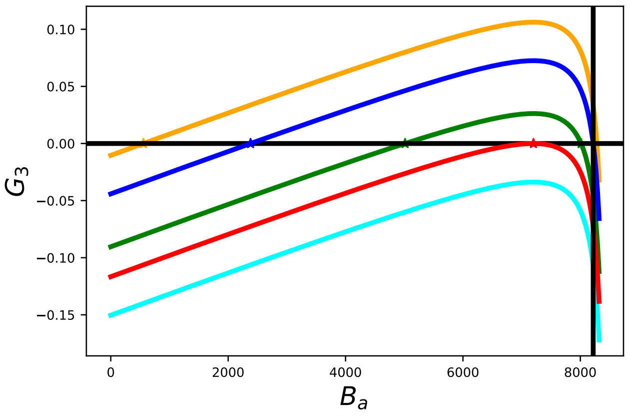

Figure 8: Plots of function $G_3(B_a)$ for different values of ϕ_L: $\phi_L = 0.01$ day^{-1} (orange curve), $\phi_L^0 \approx 0.04$ (blue curve), 0.09 day^{-1} (green curve), $\phi_L^s \approx 0.12$ day^{-1} (red curve) and 0.15 day^{-1} (cyan curve). They correspond to hypothesis (H_3) of Theorem 6: $G_3'(0) = 1.86 \times 10^{-5} > 0$ and $G_3'(B_{a_{c_1}}) = -7.73 \times 10^{-4} < 0$. All remaining parameter values are given in Table 1. The stars are the zeros of G_3 in $\left(0; B_{a_{c_1}}\right)$ and the correspond to the endemic equilibria $E_c^\star$ and $E_c^{\star\star}$, whenever they exist.The vertical black line represents the line $B_a = B_{a_{c_1}}$ (PFE).

Theorem 6. *Assume that* $\mathcal{R}_B > 1$. *Then the existence of endemic equilibria of sub-model* (20) *depends on* $G_3'(0)$ *and* $G'(B_{a_{c_1}})$, *with* G_3 *defined in* (30)*:*

(H_1) $G_3'(0) \leq 0$*: sub-model* (20) *has no endemic equilibrium.*

$(H2)$ $G_3'(0) > 0$ *and* $G_3'(B_{a_{c_1}}) \geq 0$ *: sub-model* (20) *has a unique endemic equilibrium if and only if* $\mathcal{R}_e > 1$.

$(H3)$ $G_3'(0) > 0$ *and* $G_3'(B_{a_{c_1}}) < 0$*:*

- *(i) If* $\mathcal{R}_e < \mathcal{R}_e(\phi_L^s)$, *then sub-model* (20) *has no endemic equilibrium.*
- *(ii) If* $\mathcal{R}_e(\phi_L^s) < \mathcal{R}_e < 1$, *then sub-model* (20) *has two endemic equilibria,* $E_c^\star$ *and* $E_c^{\star\star}$ *such that* $B_{a_c}^\star < B_{a_c}^{\star\star}$.
- *(iii) If* $\mathcal{R}_e = \mathcal{R}_e(\phi_L^s)$ *or* $\mathcal{R}_e \geq 1$, *then sub-model* (20) *has a unique endemic equilibrium* $E_c^\star$.

Fig 8 illustrates the situation described above for different values of ϕ_L. The red curve represents the function G_3 at the critical value $\phi_L^s \approx 0.12$ day^{-1}. Below this value, there is no root (cyan curve). Above this value, G_3 can have two (green curve) or one (orange curve) roots depending on the value of ϕ_L.

4.1.2 Stability

Following a similar approach as in Section 3, we establish the following results for the trivial and pest-free equilibria.

Theorem 7. *The extinction equilibrium,* E_{c_0}, *of sub-model* (20), *is globally asymptotically stable if* $\mathcal{R}_B < 1$. *Otherwise, if* $\mathcal{R}_B > 1$, *the pest-free equilibrium* E_{c_1} *exists. It is locally asymptotically stable if* $\mathcal{R}_e < 1$ *and unstable if* $\mathcal{R}_e > 1$.

Moreover, the system presents a forward transcritical bifurcation at $\mathcal{R}_B = 1$.

Proof. We suppose $\mathcal{R}_B < 1$ and we consider the following positive definite function,

$$V = qh\gamma B_y + qh\alpha_y B_a + \frac{qh\gamma}{c}L + \frac{k\alpha_a\alpha_y(1-\mathcal{R}_B)}{a_P\alpha}P + \alpha_a\alpha_y M.$$

Differentiating the above function V with respect to time, t, along the solutions of sub-model (20), after some algebraic simplifications, we obtain

$$\begin{aligned}\dot{V} = &-\frac{qh\gamma}{c}\left(\mu_L L^2 + \phi\lambda PL + \phi_L L\right)\\ &-\frac{k\alpha_a\alpha_y(1-\mathcal{R}_B)}{a_P\alpha}\left(\phi PL + \phi_P P\right)\\ &+\left[k\alpha_a\alpha_y(1-\mathcal{R}_B) + \frac{qh\gamma r_1 k_1}{1+\delta_M M} - k\alpha_a\alpha_y\right]M.\end{aligned}$$

And,

$$\begin{aligned}&k\alpha_a\alpha_y\left(1-\mathcal{R}_B\right)+\frac{qh\gamma r_1k_1}{1+\delta_M M}-k\alpha_a\alpha_y\\ &<k\alpha_a\alpha_y\left(1-\mathcal{R}_B\right)+qh\gamma r_1k_1-k\alpha_y\alpha_a=0.\end{aligned}$$

Thus, V is a strict Lyapunov function for equilibrium E_{c_0} if $\mathcal{R}_B<1$. We conclude that E_{c_0} is globally asymptotically stable in Ω if $\mathcal{R}_B<1$.

We assume that $\mathcal{R}_B>1$. The characteristic polynomial evaluated at the PFE E_{c_1} is given by:

$$\begin{aligned}Q_{c_1}(\chi)=&\left(\chi-\left(c\psi By_{c_1}-\lambda\phi P_{c_1}-\phi_L\right)\right)\left(\chi+\phi_P\right)\\ &\left(\chi^3+B_2\chi^2+B_1\chi+B_0\right),\end{aligned}$$

where

$$\begin{aligned}B_2&=k+\alpha_y+\alpha_a,\\ B_1&=\alpha_y\alpha_a+k\left(\alpha_a+\alpha_y\right),\\ B_0&=k\alpha_y\alpha_a-\frac{k_1qh\gamma r_1}{\left(1+\delta_M M_{c_1}\right)^2}.\end{aligned}$$

We conclude as in Appendix (C).

The proof of forward transcritical bifurcation at $\mathcal{R}_B=1$ is trivial. □

Remark 1. *Under the hypothesis* (H_1) *or* (H_2) *of Theorem 6, we can prove, by adapting the proof of Theorem 5, that all the solutions starting in* Ω'' *tend to the pest-free equilibrium* E_{c_1} *if* $\mathcal{R}_e\leq 1<\mathcal{R}_B$*, with* Ω''

$$\Omega''=\left\{(B_y;B_a;L;P;M)\in\mathbb{R}_+^5|B_y\leq B_{y_{c_1}},B_a\leq B_{a_{c_1}},M\leq M_{c_1}\right\}. \tag{37}$$

If hypothesis (H_2) *holds, the system presents a transcritical bifurcation at the point* $\mathcal{R}_e=1$*.*

We explored the stability of the endemic equilibria, whenever they existed, numerically. Based on the parameter values in Table 1 (with $\delta_M=1.17\times10^{-2}$ per USD), we varied parameter ϕ_L. We found that $E_c^\star$ is stable and $E_c^{\star\star}$ is unstable. Thus, a bistability behavior occurs when $\mathcal{R}_e\left(\phi_L^s\right)<\mathcal{R}_e<1$ (corresponding to $\phi_L\in(\phi_L^0;\phi_L^s)$). The backward bifurcation diagram is shown in Fig. 9(a). In practice, the smallholder farmer cannot influence the natural mortality of larvae ϕ_L.

Varying α, the investment rate for pesticide acquisition, is more interesting from a practical point of view. Determining the threshold α^s corresponding to the lower bound for the existence of endemic equilibria, is not as straightforward as with parameter ϕ_L. So we proceeded numerically and obtained $\alpha^s\approx0.086$ day^{-1} and $\mathcal{R}_e(\alpha^s)\approx0.2$. The threshold $\alpha^0=\phi_P\left(c\psi B_{y_{c_1}}-\phi_L\right)/(\lambda\phi a_pM_{c_1})$, obtained by solving $\mathcal{R}_e=1$ in terms of α. As previously, bi-stability behavior occurs when $\mathcal{R}_e\left(\alpha^s\right)<\mathcal{R}_e<1$. The backward bifurcation diagram is shown in Fig. 9(b). We observe that α has more significant effect to the equilibria values than ϕ_L.

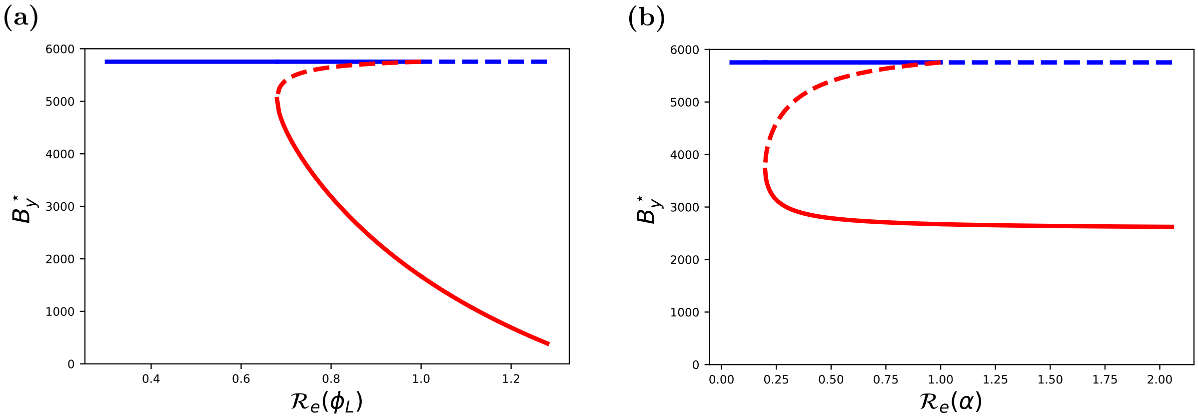

Figure 9: Backward bifurcation diagram for sub-model (20), obtained by varying parameter: (a) ϕ_L, with $\mathcal{R}_e\left(\phi_L^s\right)\approx0.68$ and (b) α, with $\mathcal{R}_e\left(\alpha^s\right)\approx0.2$. All remaining parameter values are given in Table 1. The blue line represents pest-free E_{c_1}. The solid and dashed red lines represent endemic equilibria $E_c^\star$ and $E_c^{\star\star}$. A solid line corresponds to a stable equilibrium, a dashed line to an unstable equilibrium.

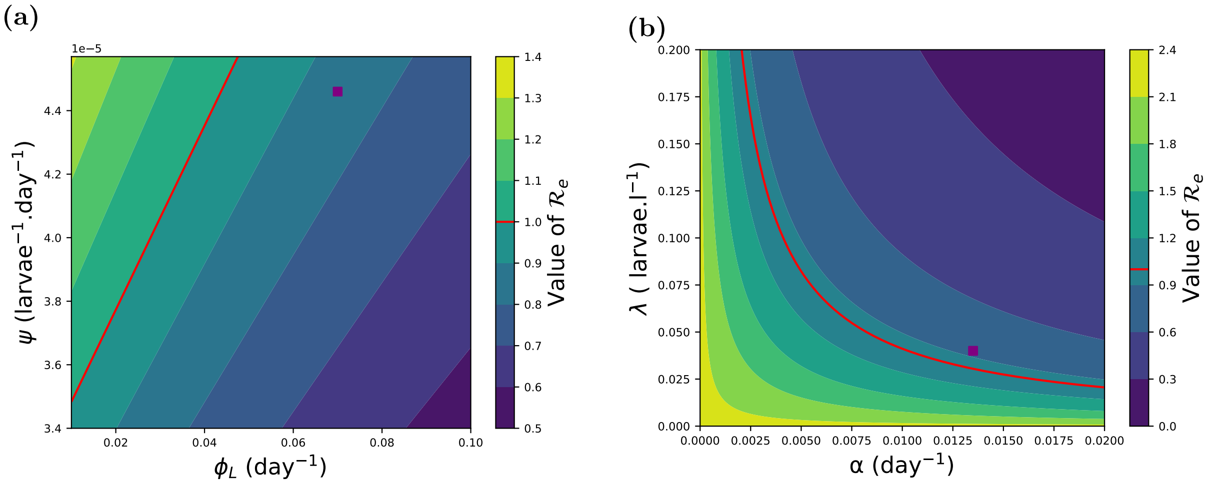

Figure 10: Variation of the effective reproductive number $\mathcal{R}_e$ as a function of: (a) the natural mortality of larvae ϕ_L and the biomass consumption rate by larvae ψ; (b) the investment rate for plant protection α and the efficacy of pesticide λ. The values of the remaining parameters are given in Table 1. The purple squares represent the parameter values in Table 1. The red line represents the level 1

4.2 Simulations

In this section, we carry out numerical simulations of sub-model (20). We recall that all parameters values are given in Table 1 (with $\delta_M = 1.17 \times 10^{-2}$ per USD). The biomass reproductive number $\mathcal{R}_B = 3.21$ and the effective reproductive number $\mathcal{R}_e = 0.85$.

Fig. 10(a) reveals that a high consumption or a low natural mortality rate makes pest management difficult. Fig. 10(b) reveals that the success of pest control depends on both the efficacy of the pesticide and the financial resources allocated. For a botanical pesticide with relatively low efficacy to eradicate larvae from the plantation, massive (and fairly unreasonable) investment is required. Conversely, a highly effective pesticide will require very little financial resources to fight the infestation. Thus, the smallholder farmer needs to choose its products wisely for a cost-efficient plant protection.

Hypothesis (H_1) of Theorem 6 corresponds to a situation where the botanical pesticides is extremely powerful ($\lambda \geq 7.5$ larvae/l). In practice, this situation cannot happen. Hypothesis (H_2) of the same theorem corresponds to a situation where the smallholder farmer invests almost nothing in botanical pesticides (for example $\alpha = 5 \times 10^{-4}$ day^{-1}). Thus, the larvae act like there is no pesticides and sub-model (20) has the same dynamics as sub-model (6). Therefore, simulations were done under hypothesis (H_3) of Theorem 6.

One can compute the cost of plant protection using the metric

$$\alpha \times \overline{M} \times A_t, \tag{38}$$

where $\overline{M}$ is the value of the component M of the stable equilibrium and $A_t = 90$ days still denotes the average time of the agricultural season.

When the pest management strategy works nicely (i.e. the pest-free equilibrium is stable), using (19), we have a global yield around 22.2 tonnes for a total income of 15, 320 USD. Using (38), the cost of plant protection is around 230 USD. These two metrics agree with the work in [30]. Conversely, if the pest strategy does not work nicely (i.e. a endemic equilibrium is stable), the expected yield will be around 10.4 tonnes for an income of 7, 180 USD and a cost protection of only 107 USD. As shown in Fig. 11, in the bistabilty region, one cannot predict the behavior of the system. In fact, with a low level of larvae, the system tends towards endemic equilibrium, while with a high level of larvae, it tends towards pest-free equilibrium (blue and cyan curves).

We observe numerically that the original (3) and simplified (20) models present the same dynamics but with different equilibrium values, as shown in Appendix E.

5 Conclusion

With food security a growing problem, many governments have put agriculture and rural development at the heart of their public policies. This action is hampered by the presence of pests, which destroy crops and cause huge economic losses.

The aim of this study was to quantify the damage caused by the attack of *Plutella* larvae in a cabbage plantation, assess the effectiveness of using botanical pesticides as a pest management strategy, and evaluate the sustainability and profitability of such agroecosystem. To achieve our goals, we built an original mathematical

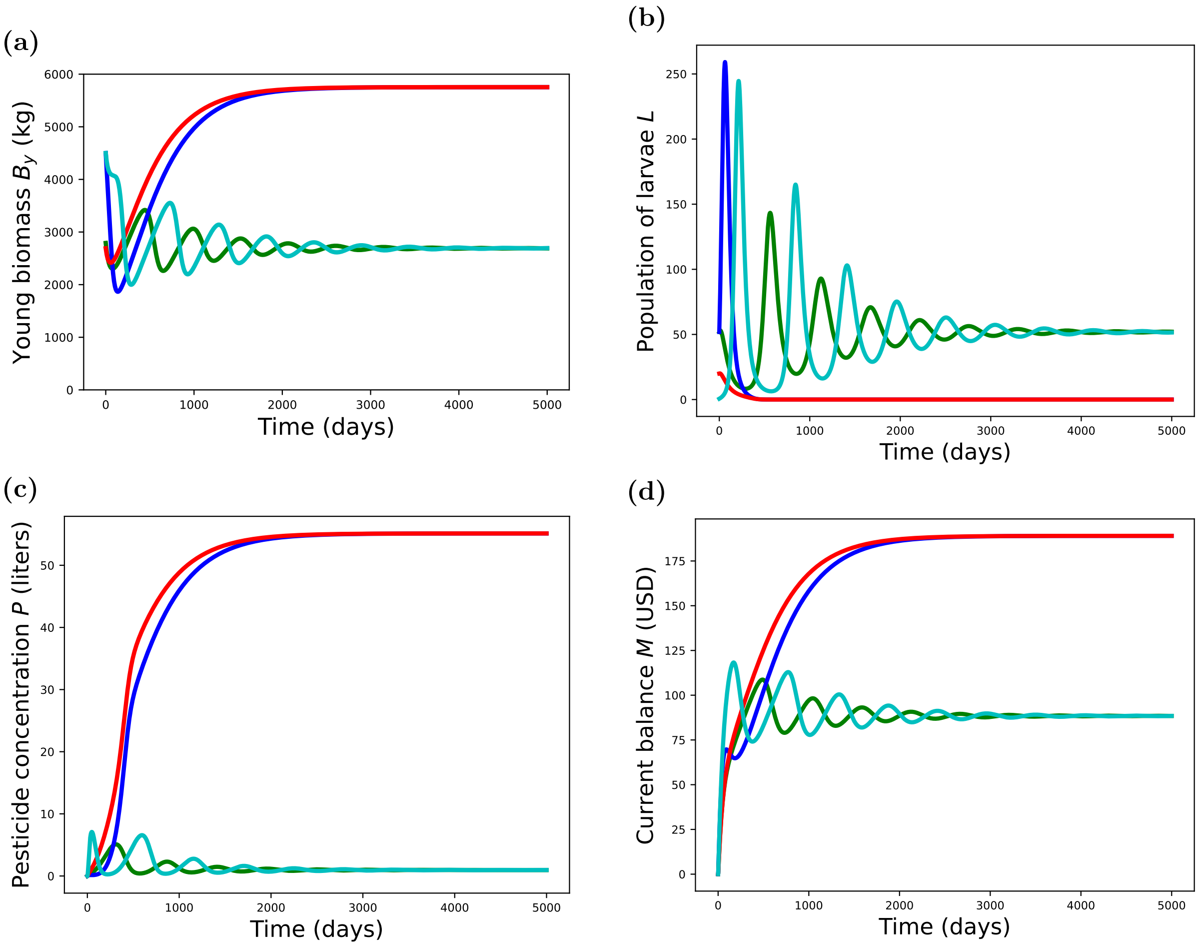

Figure 11: Dynamics of state variables (a) B_y, (b) L, (c) P and (d) M of sub-model (20). Parameter values are given in Table 1. The initial conditions are: $(2791.24; 0; 51.89; 0; 0)$ in green, $(4500; 0; 51.89; 0; 0)$ in blue, $(2691.24; 0; 20; 0; 0)$ in red and $(4500; 0; 0.65; 0; 0)$ in cyan. The effective reproductive number $\mathcal{R}_e$ verifies $\mathcal{R}_e(\alpha^s) \approx 0.2 < \mathcal{R}_e = 0.85 < 1$. Bistability occurs: according to the initial conditions, the system either converges towards the pest-free (red and blue) or the endemic (cyan and green) equilibrium.

model that describes the agro-economic environment associated with cabbage culture. Indeed, we divided the cabbage biomass into two groups to account for the larvae preference for young cabbage and natural resistance of adult cabbage due to the firmness of leaves acquired with age. In order to take into account the self-financing constraints of the farm, we introduced the plantation current account as a dynamical variable in the model. This budget is allocated for purchasing new plants, botanical pesticides, and paying daily expenditures of the smallholder farmer.

In this work, we showed that the farm sustainability depends on the biomass reproductive number $\mathcal{R}_B$. If $\mathcal{R}_B$ is smaller than one, the system will inevitably disappear. Conversely, $\mathcal{R}_B$ values greater than one guarantee the sustainability of the entire ecosystem. Except in extreme market situations (i.e. very high purchase price of young plants $r_1^{-1} > 3$ USD/kg and very low selling price of mature cabbages $q < 0.4$ USD/kg) or when the farmer investment in the farm is particularly low (i.e. $k_1/k < 0.21$), analysis of this threshold shows that cabbage is an very sustainable crop. The mathematical analysis of two variants of the model proposed was carried out in details.

In the first case study, we considered a self-financing uncontrolled model; that is a model without any pest management. We showed that when the condition $\mathcal{R}_B > 1$ is satisfied, the threshold $\mathcal{R}_0$ governs the dynamics of the resulting system. The system presents a transcritical bifurcation at $\mathcal{R}_0 = 1$. When $\mathcal{R}_0 < 1$, the larvae disappear naturally without any control action and the production is around 11.1 tonnes per hectare. If $\mathcal{R}_0 > 1$, the presence of larvae leads to losses greater than 30% of the overall production.

In the second case study, we studied a slightly modified controlled model (20), obtained from the main model (3) by neglecting the antifeedant effect of botanical pesticide and the competition in the biomass compartments. We computed the effective reproductive number $\mathcal{R}_e$ and noticed that it is smaller than $\mathcal{R}_0$, suggesting that integrated pest management acts efficiently against the infestation. A botanical pesticide is a judicious choice because it is effective in both for prophylaxis and treatment(as reported by [24]) with a reasonable treatment cost for the smallholder farmer. Unfortunately, the threshold $\mathcal{R}_e$ does not solely govern the dynamics of the system.

Our model (both the original model and the slightly modified model) exhibits a backward bifurcation revealing that having $\mathcal{R}_e$ less than one is not enough to guarantee the elimination of larvae in the plantation. Eradication occurs when the control is strong enough (i.e. $\mathcal{R}_e < \mathcal{R}_e(\alpha^s)$) which is not necessarily the best option for the smallholder farmer, because important investment in plant protection will reduce his net profit. This bistability phenomenon is dangerous because, depending on the initial condition, the system can converge either towards the pest-free equilibrium or towards an endemic equilibrium. However, it is possible to avoid this bi-stability by investing substantially in plant protection ($\alpha \geq 0.09$ day^{-1}).

Another approach of modeling pesticide application in the plantation is to use feedback control strategy (as proposed by [43]) or combine prophylaxis and feedback control. These extensions give rise to some optimization challenges which represent an avenue of our further works.

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A Parameter estimation

Some parameters were estimated from the literature. The values of the conversion rate c and uptake rate of botanical pesticide by larvae were taken in [43]. [30] reported that undamaged heads fetched 0.56 USD and 0.83 USD per kg for the major and minor seasons respectively. We chose an average cabbage price $q = 0.69$ USD per kg [30].

When replanted, the cabbage consists of 3 to 5 leaves [44, 15]. The weight at transplanting is therefore assumed to be 70g. We assume a growth period for the cabbage of 90 days (three months) for an average weight of 0.7 kg per head [15]. [10] report that there is a strong correlation between the age of the plant and the age of the diamondback moth larvae. In fact, plants less than 55 days old are favored by immature larvae (stages 2 and 3). Knowing also that larvae populations are more voracious at stage 3, we set the aging rate of young biomass $\gamma = 1/55 \approx 0.02$ day^{-1} and the harvest rate $h = 1/(90 - 55) \approx 0.03$ day^{-1}.

One hectare can support around 35000 cabbage plants [30]. Then, the carrying capacity of a 1-hectare field is $35,000 \times 0.7 = 24,500$ kg, split into $K_a = 24,500 \times 35/90 = 9,528$ kg of adult biomass and $K_y = 24,500 \times 55/90 = 14,972$ kg of young biomass. Assuming that the weight of a young cabbage at 55 days after replanting is 0.4 kg, we compute the growth rate of adult biomass $r_{0a} = \ln(0.7/0.4)/35 \approx 1.6 \times 10^{-2}$ day^{-1}. Moreover, we assume that $r_{0y} \approx r_{0a} = 1.6 \times 10^{-2}$ day^{-1}. Thus, the competition rates are given by $\mu_a = 1.6 \times 10^{-2}/9528 \approx 1.68 \times 10^{-6}$ kg^{-1}.day^{-1}, and, $\mu_y = 1.6 \times 10^{-2}/14972 \approx 1.07 \times 10^{-6}$ kg^{-1}.day^{-1}.

According to [12], the infestation is noticeable at more than 20 larvae per plant. In addition, [45] reported that an increase of 5 larvae leads to an estimated increase in consumption of 19 g and 32 g for an exposure of 4 and 7 weeks respectively. Hence, we assume that the biomass consumption rate by larvae ψ has values between $0.032/((20+5) \times 7 \times 7) \approx 2.27 \times 10^{-5}$ and $0.019/((20+5) \times 7 \times 4) \approx 4.57 \times 10^{-5}$ larvae^{-1}.day^{-1}. The larvae development stage lasts about 9 to 32 days [12, 11]. Hence, we assume that the mortality rate of larvae ϕ_L is between 0.03 and 0.1 day^{-1}. Assuming a carrying capacity of 30 larvae per plant [12, 13], the carrying capacity of the plantation is $35,000 \times 30 = 1,050,000$ larvae. The competition of larvae μ_L is given by formula $c\psi K_y/1,050,000$. Thus, the range of μ_L is $[1.94 \times 10^{-6}; 3.91 \times 10^{-6}]$.

According to [30], a hectare is treated between 7 and 8 times over a growing season. It is hence assumed that the natural depletion of botanical pesticide is $\phi_p = 8/90 \approx 0.1$ day^{-1}. In addition, around 500 liters of pesticide per hectare are used [24] for a total cost around 306 USD [30]. Thus, the inverse of the pesticide price is $a_p \approx 500/231 \approx 2.16$ l.USD^{-1}.

According to [41], the unit price of a young cabbage plant is around 0.16 USD per plant, which weighs 70g. Thus, the inverse of price of young biomass is estimated at $r_1 = 0.07/0.16 \approx 0.46$ kg.USD^{-1}.

B Proof of Theorem 1

Let $(B_y(t); B_a(t); L(t); P(t); M(t))$ be a local solution of model (3) defined on $(0; T_+)$ where $T_+ > 0$. Using the first equation of model (3), we have

$$\begin{aligned}\dot{B}_y &= r_{0y}B_y - \mu_y B_y^2 + \frac{r_1k_1M}{1+\delta_M M} - \frac{\psi L B_y}{bP+1} - \gamma B_y \\ &\leq \frac{r_1k_1}{\delta_M} - \alpha_y B_y.\end{aligned}$$

Applying Gronwall's lemma [46], we obtain

$$B_y(t) \leq \frac{r_1k_1}{\delta_M\alpha_y} + \left(B_y(0) - \frac{r_1k_1}{\delta_M\alpha_y}\right)\exp(-\alpha_y t). \tag{39}$$

Thus, as the right-hand side is monotic from $B_y(0)$ to $\frac{r_1k_1}{\delta_M\alpha_y}$,

$$B_y(t) \leq \max\left(K_y; B_y(0)\right) \equiv B_{y_{\max}}, \tag{40}$$

where $K_y = \frac{r_1k_1}{\delta_M\alpha_y}$.

Using the remaining equations of model (3), we prove similarly that,

$$\begin{aligned}B_a(t) &\leq \max\left(\frac{\gamma B_{y_{\max}}}{\alpha_a}; B_a(0)\right) \equiv B_{a_{\max}}, \\ M(t) &\leq \max\left(\frac{qhB_{a_{\max}}}{k}; M(0)\right) \equiv M_{\max}, \\ P(t) &\leq \max\left(\frac{\alpha a_p M_{\max}}{\phi_P}; P(0)\right) \equiv P_{\max}.\end{aligned} \tag{41}$$

Furthermore,

$$\begin{aligned}\dot{L} &= c\frac{\psi L B_y}{bP+1} - \lambda\phi P L - \mu_L L^2 - \phi_L L \\ &\leq c\psi B_{y_{max}} L - \mu_L L^2 \\ L(t) &\leq \frac{L(0)\frac{c\psi B_{y_{\max}}}{\mu_L}}{L(0) + \left(\frac{c\psi B_{y_{\max}}}{\mu_L} - L(0)\right)\exp(-c\psi B_{y_{\max}} t)}.\end{aligned} \tag{42}$$

Then, we obtain

$$L(t) \leq \max\left(L(0); \frac{c\psi B_{y_{\max}}}{\mu_L}\right) \equiv L_{\max}. \tag{43}$$

We assume that the solution $(B_y(t); B_a(t); L(t); M(t))$ starts in the domain Ω which is described in Theorem 2. we deduce easily that $B_{y_{\max}} = K_y$. Then, it follows from similar developments around relations (41) and (43) that Ω is positively invariant.

We focus our attention on the attractiveness of Ω. Let us show that for any solution $(B_y(t); B_a(t); L(t); P(t); M(t))$ started outside of

$$\begin{aligned}\Omega_\epsilon := \big\{&(B_y; B_a; L; P; M) \in \mathbb{R}_+^5 | B_y \leq K_y + \epsilon, B_a \leq K_a + \epsilon, \\ &L \leq K_L + \epsilon, M \leq K_M + \epsilon, P \leq K_P + \epsilon\big\}\end{aligned} \tag{44}$$

with $K_y = \frac{r_1k_1}{\delta_M\alpha_y}$, $K_a = \frac{\gamma(K_y+\epsilon)}{\alpha_a}$, $K_M = \frac{qh(K_a+\epsilon)}{k}$, $K_P = \frac{a_p\alpha(K_M+\epsilon)}{\phi_P}$, $K_L = \frac{c\psi(K_y+\epsilon)}{\mu_L}$, and for any $\epsilon > 0$, enters Ω_ϵ in finite time. If we first suppose that $B_y(0) > K_y + \epsilon$, we have

$$\begin{aligned}\dot{B}_y &= r_{0y}B_y - \mu_y B_y^2 + \frac{r_1k_1M}{1+\delta_M M} - \frac{\psi L B_y}{bP+1} - \gamma B_y \\ &\leq \frac{r_1k_1}{\delta_M} - \alpha_y B_y < \frac{r_1k_1}{\delta_M} - \alpha_y\left(K_y + \epsilon\right) < -\alpha_y\epsilon\end{aligned}$$

which implies that

$$B_y(t) < B_y(0) - \alpha_y\epsilon t.$$

Writing the previous inequality at the time

$$t_\epsilon = \left(B_y(0) - K_y - \epsilon\right) / \left(\alpha_y\epsilon\right),$$

we get $B_y(t_\epsilon) < K_y + \epsilon$. Hence, $B_y(t)$ reaches $K_y + \epsilon$ in finite time $t_y < t_\epsilon$ and will stay below this value afterwards.

Similarly, if we assume that $B_a(0) > K_a + \epsilon$ with $B_y(t) \leq K_y + \epsilon$ (which may be valid at time 0 or after the previously described convergence time t_y), the second equation of model (3) gives

$$\dot{B}_a \leq \gamma \left(K_y + \epsilon\right) - \alpha_a B_a < -\alpha_a \epsilon.$$

And B_a will reach $K_a + \epsilon$ in finite t_a and stay below this value afterwards.

Now, if we suppose that $M(0) > K_M + \epsilon$ with $B_a(t) \leq K_a + \epsilon$ (which may be valid at time 0 or after the previously described convergence time t_a) the dynamic of M in model (3) gives $\dot{M} \leq qh\left(K_a + \epsilon\right) - kM < -k\epsilon$. Then M will reach $K_M + \epsilon$ in finite t_m and stay below this value afterwards. Repeating the same approach for P, we can conclude at the existence of convergence time t_p at which P reaches $K_P + \epsilon$ and stays below this value afterwards. Finally, we suppose $L(0) > c\psi\left(K_y + \epsilon\right)/\mu_L$ with $B_y(t) \leq K_y + \epsilon$. Using the third equation of model (3), we have

$$\dot{L} \leq \left(c\psi\left(K_y + \epsilon\right) - \mu_L L\right) L < -\mu_L \epsilon L.$$

Hence, $L(t) < L(0)\exp(-\mu_L \epsilon t)$. At the time

$$t_\epsilon = \ln\left(L(0)/\left(K_L + \epsilon\right)\right)/(\mu_L \epsilon),$$

we have $L(t_\epsilon) < K_L + \epsilon$. We deduce that L reaches $K_L + \epsilon$ in finite $t_l < t_\epsilon$ and stay below this value afterwards. We conclude that Ω_ϵ is attractive for any $\epsilon > 0$. Hence, Ω is attractive.

We have just shown that the solution $(B_y(t); B_a(t); L(t); P(t); M(t))$, for any initial satisfying (4) is contained in a compact subset of $\mathbb{R}^5$. Using a classical result of dynamical system, we conclude that the solution $(B_y(t); B_a(t); L(t); P(t); M(t))$ is globally defined in $\mathbb{R}_+$.

C Proof of Theorem 2

The jacobian matrix of sub-model (6) is given by

$$\begin{pmatrix} -\alpha_y - 2\mu_y B_y - \psi L & 0 & -\psi B_y & \dfrac{r_1 k_1}{\left(1 + \delta_M M\right)^2} \\ \gamma & -\alpha_a - 2\mu_a B_a & 0 & 0 \\ c\psi L & 0 & c\psi B_y - 2\mu_L L - \phi_L & 0 \\ 0 & qh & 0 & -k \end{pmatrix}. \tag{45}$$

The characteristic polynomial evaluated at the extinction equilibrium E_0 is

$$Q_0(\chi) = \left(\chi + \phi_L\right)\left(\chi^3 + A_2\chi^2 + A_1\chi + A_0\right),$$

with

$$A_2 = k + \alpha_y + \alpha_a, \qquad A_1 = \alpha_y\alpha_a + k\left(\alpha_a + \alpha_y\right),$$
$$R_B = \frac{k_1\gamma r_1 qh}{k\alpha_y\alpha_a}, \qquad A_0 = k\alpha_a\alpha_y\left(1 - \mathcal{R}_B\right).$$

We remark that $-\phi_L$ is an eigenvalue of Q_0. The other eigenvalues are solutions of the polynomial function $\chi \mapsto \chi^3 + A_2\chi^2 + A_1\chi + A_0$.

If $\mathcal{R}_B > 1$, Descartes rule reveals that Q_0 has a unique positive root. Using the Hartman-Grobman theorem in [46], we conclude that E_0 is unstable. Assuming $\mathcal{R}_B < 1$. We use the well-known Routh-Hurwitz stability criterion for stability by verifying conditions: $A_2, A_1, A_0 > 0$ and $A_2A_1 - A_0 > 0$.

The positivity of A_2, A_1 and A_0 are obvious. A direct computation gives

$$\begin{aligned} &A_2A_1 - A_0 = \left(k + \alpha_y + \alpha_a\right)\left(\alpha_y\alpha_a + k\left(\alpha_a + \alpha_y\right)\right) \\ &- k\alpha_a\alpha_y\left(1 - \mathcal{R}_B\right) = k^2\left(\alpha_y + \alpha_a\right) \\ &+ \left(\alpha_y + \alpha_a\right)\left(\alpha_y\alpha_a + k\left(\alpha_a + \alpha_y\right)\right) + r_1k_1qh\gamma > 0. \end{aligned}$$

We conclude that E_0 is locally asymptotically stable.

We assume that $\mathcal{R}_B > 1$. The characteristic polynomial evaluated at the PFE E_1 is given by

$$Q_1(\chi) = \left(\chi - \left(c\psi B_{y_1} - \phi_L\right)\right)\left(\chi^3 + B_2\chi^2 + B_1\chi + B_0\right)$$

where

$$\begin{aligned}
B_2 &= k + \alpha_y + \alpha_a + 2\mu_a B_{a_1} + 2\mu_y B_{y_1} \\
B_1 &= \left(\alpha_y + 2\mu_y B_{y_1}\right)\left(\alpha_a + 2\mu_a B_{a_1}\right) \\
&\quad + k\left(\alpha_a + \alpha_y + 2\mu_a B_{a_1} + 2\mu_y B_{y_1}\right) \\
B_0 &= k\left(\alpha_y + 2\mu_y B_{y_1}\right)\left(\alpha_a + 2\mu_a B_{a_1}\right) - \frac{k_1 q h \gamma r_1}{\left(1+\delta_M M_1\right)^2}.
\end{aligned}$$

It is clear that $B_2, B_1 > 0$. Using Equation 11 and the fact that $1 + \delta_M M_1 > 1$, we have

$$\begin{aligned}
B_0 &> \frac{k}{1+\delta_M M_1}\left(\left(\alpha_y + 2\mu_y B_{y_1}\right)\left(\alpha_a + 2\mu_a B_{a_1}\right)\right. \\
&\quad \left. - \Big(\alpha_y\left(\alpha_a + \mu_a B_{a_1}\right) + \mu\left(\alpha_a + \mu_a B_{a_1}\right)^2 B_{a_1}\Big)\right) \\
&> \frac{k}{1+\delta_M M_1}\left(\left(\alpha_y + \mu\left(\alpha_a + \mu_a B_{a_1}\right) B_a\right)\left(\alpha_a + \mu_a B_{a_1}\right)\right. \\
&\quad \left. - \Big(\alpha_y\left(\alpha_a + \mu_a B_{a_1}\right) + \mu\left(\alpha_a + \mu_a B_{a_1}\right)^2 B_{a_1}\Big)\right) = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
B_2 B_1 - B_0 &= k^2\left(\alpha_a + \alpha_y + 2\mu_a B_{a_1} + 2\mu_y B_{y_1}\right) \\
&+ \left(\alpha_y + \alpha_a + 2\mu_a B_{a_1} + 2\mu_y B_{y_1}\right) \\
&\left(\left(\alpha_y + 2\mu_y B_{y_1}\right)\left(\alpha_a + 2\mu_a B_{a_1}\right)\right. \\
&\left. + k\left(\alpha_a + \alpha_y + 2\mu_a B_{a_1} + 2\mu_y B_{y_1}\right)\right) > 0.
\end{aligned}$$

According to Routh-Hurwitz stability criterion, all the eigenvalues of the polynomial function $\chi \mapsto Q_1(\chi)$ have negative real parts. Thus, the local stability of E_1 depends only of the sign of $c\psi B_{y_1} - \phi_L$. By noticing that $c\psi B_{y_1} - \phi_L = \phi_L\left(\mathcal{R}_0 - 1\right)$, it comes from Hartman-Grobman theorem that E_1 is locally asymptotically stable provided $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

Now supposing $\mathcal{R}_B > 1$ and $\mathcal{R}_0 > 1$. The characteristic polynomial evaluated at $E^\star$ is given by

$$Q_2(\chi) = \chi^4 + C_3\chi^3 + C_2\chi^2 + C_1\chi + C_0 \tag{46}$$

where

$$\begin{aligned}
J_{11} &= \alpha_y + 2\mu_y B_y^\star + \psi L^\star \\
J_{22} &= \alpha_a + 2\mu_a B_a^\star \\
C_3 &= \mu_L L^\star + J_{11} + k + J_{22} \\
C_2 &= c\psi^2 L^\star B_y^\star + \mu_L L^\star J_{11} + k J_{22} \\
&\quad + \left(k + J_{22}\right)\left(\mu_L L^\star + J_{11}\right) \\
C_1 &= \left(k + J_{22}\right)\left(c\psi^2 L^\star B_y^\star + \mu_L L^\star J_{11}\right) \\
&\quad + k J_{22}\left(\mu_L L^\star + J_{11}\right) - \frac{q h \gamma r_1 k_1}{\left(1+\delta_M M^\star\right)^2} \\
C_0 &= k J_{22}\left(c\psi^2 L^\star B_y^\star + \mu_L L^\star J_{11}\right) - \mu_L L^\star \frac{q h \gamma r_1 k_1}{\left(1+\delta_M M^\star\right)^2}.
\end{aligned}$$

We want to use again the Routh-Hurwitz stability criterion. It consists in verifying the conditions : $C_3,\ C_2,\ C_1,\ C_0 > 0$, $C_3 C_2 - C_0 > 0$ and $\left(C_3 C_2 - C_1\right) C_1 - C_3^2 C_0 > 0$.

It is clear that $C_3,\ C_2 > 0$. In addition, using Equation (12)

$$\begin{aligned}
C_1 &= \left(k + J_{22}\right)\left(c\psi^2 L^\star B_y^\star + \mu_L L^\star J_{11}\right) \\
&\quad + k J_{22}\left(\mu_L L^\star + J_{11}\right) - \frac{q h \gamma r_1 k_1}{\left(1+\delta_M M^\star\right)^2} \\
&> k J_{11} J_{22} - \frac{q h \gamma r_1 k_1}{\left(1+\delta_M M^\star\right)^2} \\
&> \frac{k}{1+\delta_M M^\star}\left(\left(\alpha_y + \frac{\mu_y}{\gamma}\left(\alpha_a + \mu_a B_a^\star\right) B_a^\star + \psi L^\star\right)\right. \\
&\left(\alpha_a + \mu_a B_a^\star\right) - \left(\alpha_y + \psi L^\star\right)\left(\alpha_a + \mu_a B_a^\star\right) \\
&\left. - \frac{\mu_y}{\gamma}\left(\alpha_a + \mu_a B_a^\star\right)^2 B_a^\star\right) = 0
\end{aligned}$$

and

$$\begin{aligned}C_0 &= kJ_{22}\left(c\psi^2 L^{\star} B_y^{\star} + \mu_L L^{\star} J_{11}\right) - \mu_L L^{\star} \frac{qh\gamma r_1 k_1}{\left(1+\delta_M M^{\star}\right)^2} \\ &> \mu_L L^{\star}\left(kJ_{11}J_{22} - \frac{qh\gamma r_1 k_1}{\left(1+\delta_M M^{\star}\right)^2}\right) > 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}C_3C_2 - C_1 = (k+J_{22})\,&((k+J_{22})\left(\mu_y L^{\star} + J_{11}\right) + kJ_{22}) \\ &+ \left(\mu_y L^{\star} + J_{11}\right)\left(c\psi^2 L^{\star} B_y^{\star} + \mu_L L^{\star} J_{11}\right. \\ &\qquad \left. + (k_1 + J_{22})\left(\mu_L L^{\star} + J_{11}\right)\right) + \frac{qh\gamma r_1 k_1}{\left(1+\delta_M M^{\star}\right)^2} > 0\end{aligned}$$

and,

$$\begin{aligned}&\left(C_3C_2 - C_1\right)C_1 - C_3^2 C_0 > \left(k^2 + J_{22}^2\right)\left(\mu_L^{\star} + J_{11}\right)\left(k + J_{22}\right) \\ &+ \left(kJ_{22} + k^2 + J_{22}^2\right)\left(\mu_L L^{\star} + J_{11}\right)^2 > 0.\end{aligned}$$

We conclude by using Routh-Hurwitz stability criterion that the endemic equilibrium $E^{\star}$, whenever it exists, is always locally asymptotically stable.

D Proof of Theorem 5

We assume that the solution $(B_y(t); B_a(t); L(t); M(t))$ starts in the domain Ω' which is described in Theorem 5.

If $B_a = B_{a_1}$, we have $\dot{B_a} = \gamma B_y - \mu_a B_{a_1}^2 - \alpha_a B_{a_1} = \gamma\left(B_y - B_{y_1}\right) \leq 0$. then $B_a(t) \leq B_{a_1}$ for all $t \geq t_0$. Similarly, if $M = M_1$, we obtain $\dot{M} = qhB_a - kM_1 = qh\left(B_a - B_{a_1}\right)$. Hence, $M(t) \leq M_1$ for all $t \geq t_0$. Finally, if $B_y = B_{y_1}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}B_y}{\mathrm{d}t} &= r_{0y}B_{y_1} - \mu_y B_{y_1}^2 - \psi L B_{y_1} - \gamma B_{y_1} + r_1\frac{k_1 M}{1+\delta_M M} \\ &\leq r_{0y}B_{y_1} - \mu_y B_{y_1}^2 - \gamma B_y + r_1\frac{k_1 M}{1+\delta_M M} \\ &\leq r_1 k_1\left(-\frac{M_1}{1+\delta_M M_1} + \frac{M}{1+\delta_M M}\right) \\ &\leq r_1 k_1\frac{M - M_1}{\left(1+\delta_M M\right)\left(1+\delta_M M_1\right)} \\ &\leq 0.\end{aligned}$$

Then $B_y(t) \leq B_{y_1}$. We conclude that Ω' is positively invariant.

Before dealing with the stability in Ω', we first prove the following half-way result.

Theorem 8. *The pest-free equilibrium E_1 is globally asymptotically stable in the sub-region $\mathbb{R}_+^3 \setminus \{0\}$ of hyperplan $L = 0$ if and only if $\mathcal{R}_0 \leq 1 < \mathcal{R}_B$.*

Proof. This proof is adapted from a result given in [47].

We consider

$$X_1 = g(X_1) \tag{47}$$

with

$$X_1 = \begin{pmatrix} B_y \\ B_a \\ M \end{pmatrix}, \quad \text{and} \quad g(X_1) = \begin{pmatrix} -\mu_y B_y^2 - \alpha_y B_y + r_1\frac{k_1 M}{1+\delta_M M} \\ \gamma B_y - \mu_a B_a^2 - \alpha_a B_a \\ qhB_a - kM \end{pmatrix}.$$

System (47) has two equilibria: 0 and $X_1^E = (B_{y_1}; B_{a_1}; M_1)$ on $\mathbb{R}_+^3$ if and only if $1 < \mathcal{R}_B$.

$\mathbb{R}_+^3$ is positively invariant. Furthermore, the Jacobian matrix of g evaluated at X_1 is given by

$$\mathcal{J}_g(X_1) = \begin{pmatrix} -\alpha_y - 2\mu_y B_y & 0 & \frac{r_1 k_1}{(1+\delta_M M)^2} \\ \gamma & -\alpha_a - 2\mu_a B_a & 0 \\ 0 & qh & -k \end{pmatrix}. \tag{48}$$

$\mathcal{J}_g(X_1)$ is a cooperative irreducible matrix. Then, the flow ϕ_t of system (47) is strongly monotone in $\mathbb{R}^3_+$. In addition, for every $\epsilon > 0$, the set $Z = [0; z_\epsilon]$ with $z_\epsilon = (K_y + \epsilon; K_a + \epsilon; K_M + \epsilon)^T$, is positively invariant and attractive for the flow ϕ_t as Z is the projection of Ω_ϵ on this subspace (see (44)).

According to Perron-Frobenius Theorem, the stability modulus s of $\mathcal{J}_g(0)$, is a positive real eigenvalue of $\mathcal{J}_g(0)$ and there is a positive vector v such that $\mathcal{J}_g(0)v = sv$.

Let x be a positive vector of $\mathbb{R}^3_+$. Using attractiveness property of Z, we have $y = \phi_{t_1}(x) \in Z$ for large t_1.

Then, there is $\eta := \eta(y) > 0$ such that $\phi_t(y) \in [\eta v; z_\epsilon]$ which is a positive invariant order preserving interval and we have $g(z_\epsilon) < 0$ and $g(\eta v) \approx \mathcal{J}_g(0)\eta v = s\eta v > 0$. Using Proposition 3.2.1 in [47], $\phi_t(z_\epsilon)$ and $\phi_t(\eta v)$ converge monotonically both to X_1^E.

By monotonicity,

$$\eta v < \phi_t(\eta v) < \phi_t(y) < \phi_t(z_\epsilon) < z_\epsilon, \quad t > 0.$$

Letting $t \to +\infty$ leads to the w-limit set of y is $\{X_1^E\}$. Then $\phi_t(x)$ converges to X_1^E for all positive vector x of $\mathbb{R}^3_+$.

Furthermore, B_y can not stay equal to 0. Indeed, if $B_y = 0$ and $M > 0$ at a time t_0, we have $\dot{B}_y = r_1 k_1 M/(1 + \delta_M M) > 0$. In addition, if $B_y = M = 0$ and $B_a = 0$ at a time t_0, one has $\dot{M} = qhB_a > 0$ so that M instantaneously becomes positive and it follows that $\dot{B}_y = r_1 k_1 M/(1 + \delta_M M) > 0$. One can reproduce the same approach for variables B_a and M. Thus, except if $x(t_0) = 0$, x instantaneously enters the positive orthant $\mathcal{R}^3_+$ and is amenable to the previous developments.

Thus, X_1^E is asymptotically stable everywhere in $\mathbb{R}^3_+ \setminus \{0\}$. $\square$

For convenience, we write model (6) in the following vector form.

$$\dot{X}_1 = F(X_1; X_2) \quad \text{and} \quad \dot{X}_2 = G(X_1; X_2)$$

with

$$X_1 = \begin{pmatrix} B_y \\ B_a \\ M \end{pmatrix}, \quad X_2 = L$$

$$F(X_1; X_2) = \begin{pmatrix} r_{0y} B_y - \mu_y B_y^2 - \psi L B_y - \gamma B_y + r_1 \frac{k_1 M}{1 + \delta_M M} \\ \gamma B_y + r_{0a} B_a - \mu_a B_a^2 - h B_a \\ qhB_a - kM \end{pmatrix}$$

$$G(X_1; X_2) = c\psi L B_y - \mu_L L^2 - \phi_L L.$$

In this case, the pest-free equilibrium E_1 becomes $(X_1^E; 0)$.

According to Theorem 8, X_1^E is globally asymptotically stable for $\dot{X}_1 = F(X_1; 0)$.

In addition, we remark that

$$G(X_1; X_2) = AX_2 - \hat{G}(X_1; X_2)$$

where, for all variables in Ω', one has

$$A = \frac{\partial G}{\partial X_2}\left(X_1^E, 0\right) = \phi_L\left(\mathcal{R}_0 - 1\right)$$

$$\text{and } \hat{G}(X_1; X_2) = c\psi\left(B_{y_1} - B_y\right) + \mu_L L^2 \geq 0.$$

Thus, according to [48], the PFE E_1 is asymptotically stable in Ω' if and only if $\mathcal{R}_0 \leq 1 < \mathcal{R}_B$.

We comment that the uniform persistence of sub-model (6) can be obtained in Ω' when $\mathcal{R}_0 \leq 1 < \mathcal{R}_B$ [49].

E Other simulations

In this appendix, we simulate sub-model (20) and model (3) for different values of parameter α, which represents the investment rate in plant protection. All remaining parameters are given in Table 1.

Figures Fig. 12 and Fig. 13 are similar to Fig. 11 (in which $\alpha = 1.35 \times 10^{-2}$ day^{-1}) and depict the dynamics of sub-model (20). Fig. 12 presents a situation where the pest-free equilibrium is always stable and the system does not have an endemic equilibrium. It corresponds to a high investment in plant protection ($\alpha = 0.09$ day^{-1}), associated with a cost of 1,530 USD computed according to (38). It generates a seasonal yield of 22.2 tonnes for an income of 15,320 USD, computing according to (19) at the pest-free equilibrium.

Fig. 13 illustrates the situation where there is a unique endemic stable equilibrium and an unstable pest-free equilibrium. It corresponds to a low investment in plant protection ($\alpha = 0.01$ day^{-1}), associated with a cost of 79 USD, and it generates a seasonal yield of 10.3 tonnes for an income of 7,111 USD at the endemic equilibrium.

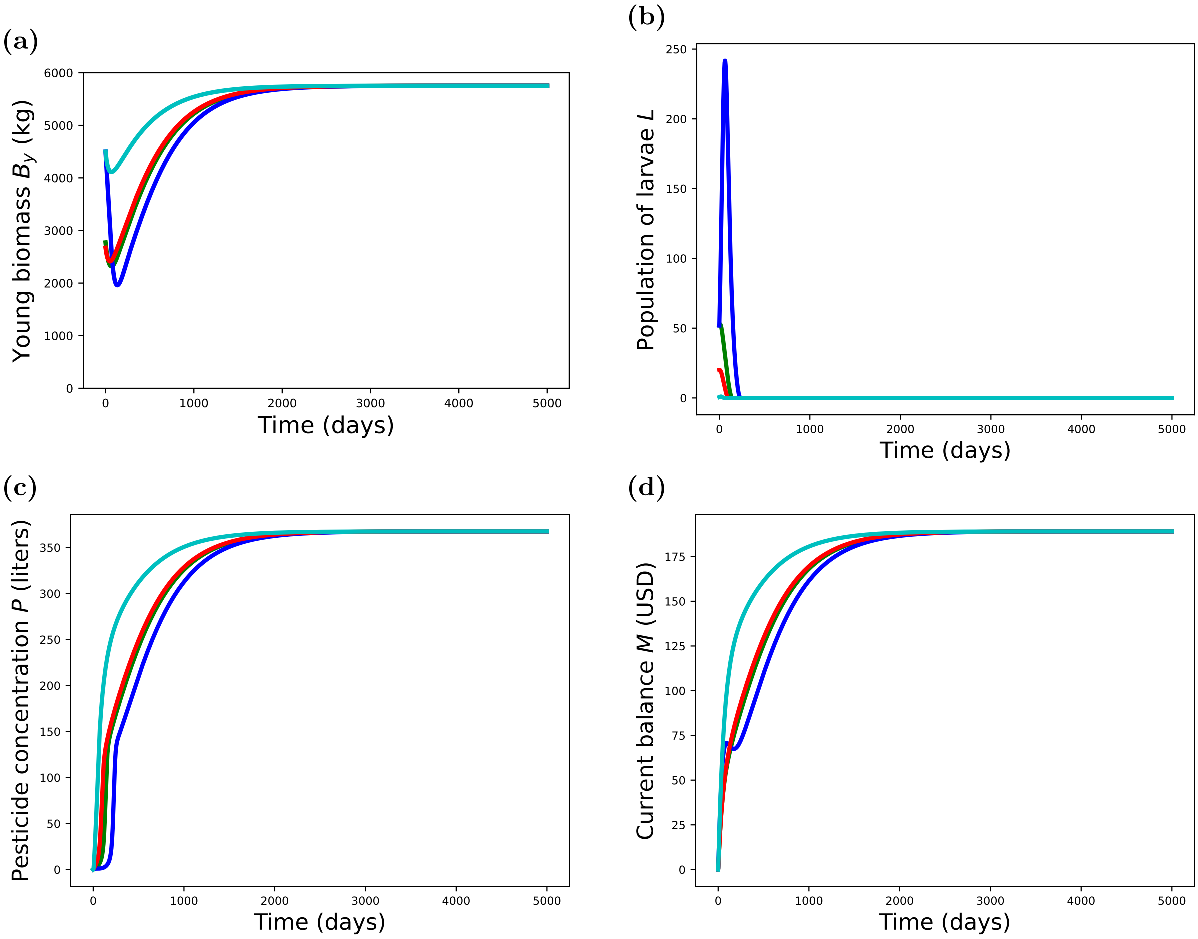

Figure 12: Dynamics of state variables (a) B_y, (b) L, (c) P and (d) M of sub-model (20). The initial conditions and parameter values are the same as in Fig. 11 except for $\alpha = 0.09$ day^{-1} leading to an effective reproductive number $\mathcal{R}_e = 0.19 < \mathcal{R}_e(\alpha^s) \approx 0.2 < 1$. For all initial conditions, the system converges towards the pest-free equilibrium.

The effective reproductive number of original model (3) is given by

$$\mathcal{R}'_e = \frac{c\psi\alpha_a B_{a1}}{\gamma\left(\lambda\phi P_1 + \phi_L\right)\left(1 + bP_1\right)},$$

with $P_1 = a_p\alpha M_1/\phi_P$. The equivalence of the threshold α^0 for the original model (3), obtained by solving $\mathcal{R}'_e(\alpha) = 1$, is given by $\alpha^{0'} = \phi_P\Delta/(2a_P M_1\lambda b\phi)$, with $\Delta = -\left(\lambda\phi + b\phi_L\right) + \sqrt{\left(\lambda\phi + b\phi_L\right)^2 + 4\lambda b\phi\left(c\psi B_{y_1} - \phi_L\right)}$.

Using parameter values in Table 1, we have $\alpha^{0'} \approx 7.34 \times 10^{-3}$ day^{-1}. The lower bound for the existence of endemic equilibria $\alpha^{s'} \approx 0.022$ day^{-1}. Fig. 14 shows model (3), for different values of parameter α. Comparing these dynamics with those of sub-model (20) depicted in panel (a) of Fig. 11, 12 and 13, we observe that both models have similar dynamics. In fact, we observe a bistability phenomenon for $\alpha \in (7.35 \times 10^{-3}; 0.022)$ day^{-1} (see Fig. 14(b) and Fig. 14(c)). When $\alpha \geq 0.022$ day^{-1}, the solutions tend to the pest-free equilibrium of (3) (see Fig. 14(d)), and when $\alpha \leq 7.34 \times 10^{-3}$ day^{-1}, they converge towards the endemic equilibrium of (3) (see Fig. 14(a)). Furthermore, $\alpha^0 >> \alpha^{0'}$ means that a pesticide with a double action is more effective and costless than a classical one. Such results encourage vulgarizing the use of botanical pesticides.

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Declaration of competing interest

The authors declare that they have no competing interests that are directly or indirectly related to this work.

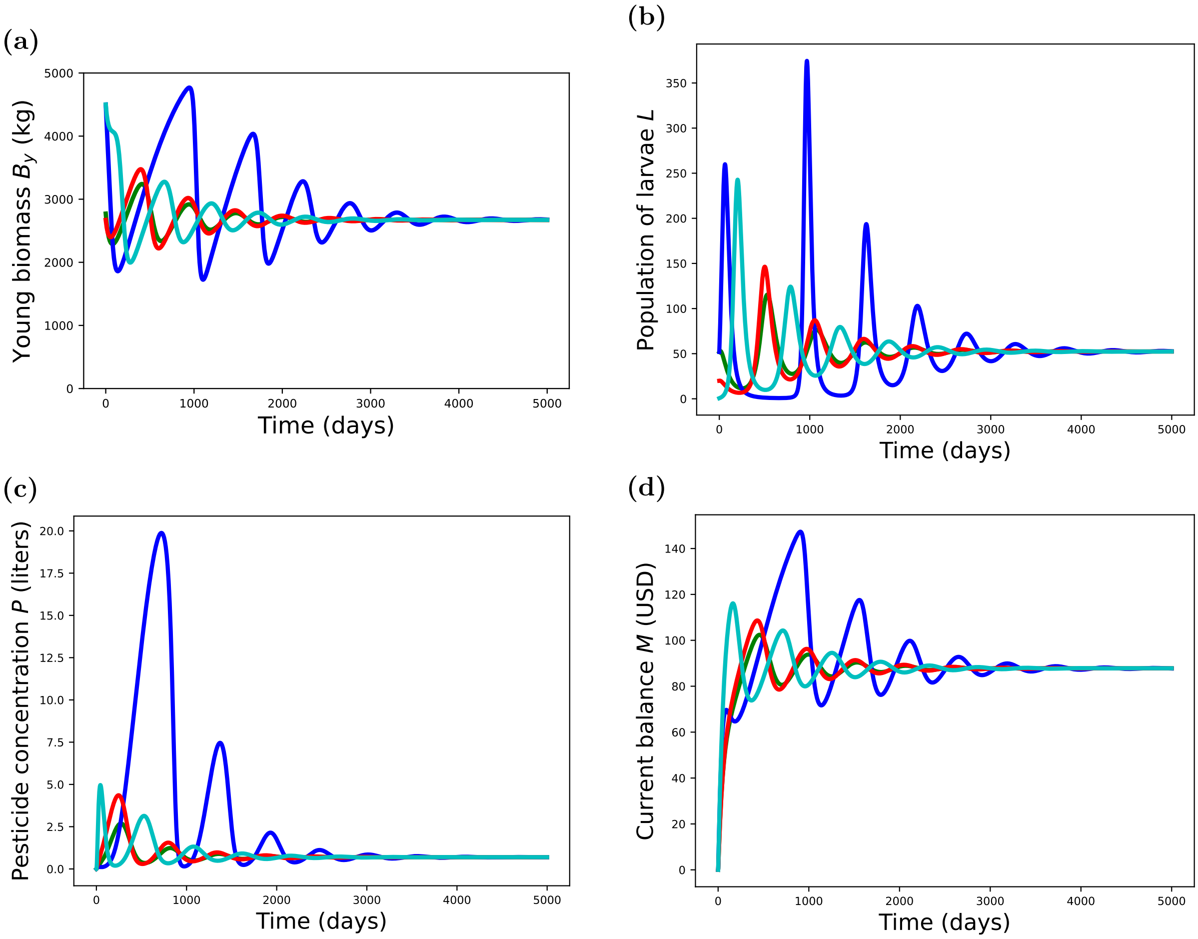

Figure 13: Dynamics of state variables (a) B_y, (b) L, (c) P and (d) M of sub-model (20). The initial conditions and parameter values are the same as in Fig. 11 except for $\alpha = 0.01$ day^{-1} leading to an effective reproductive number $\mathcal{R}_e = 1.02 > 1$. For all initial conditions, the system converges towards the endemic equilibrium.

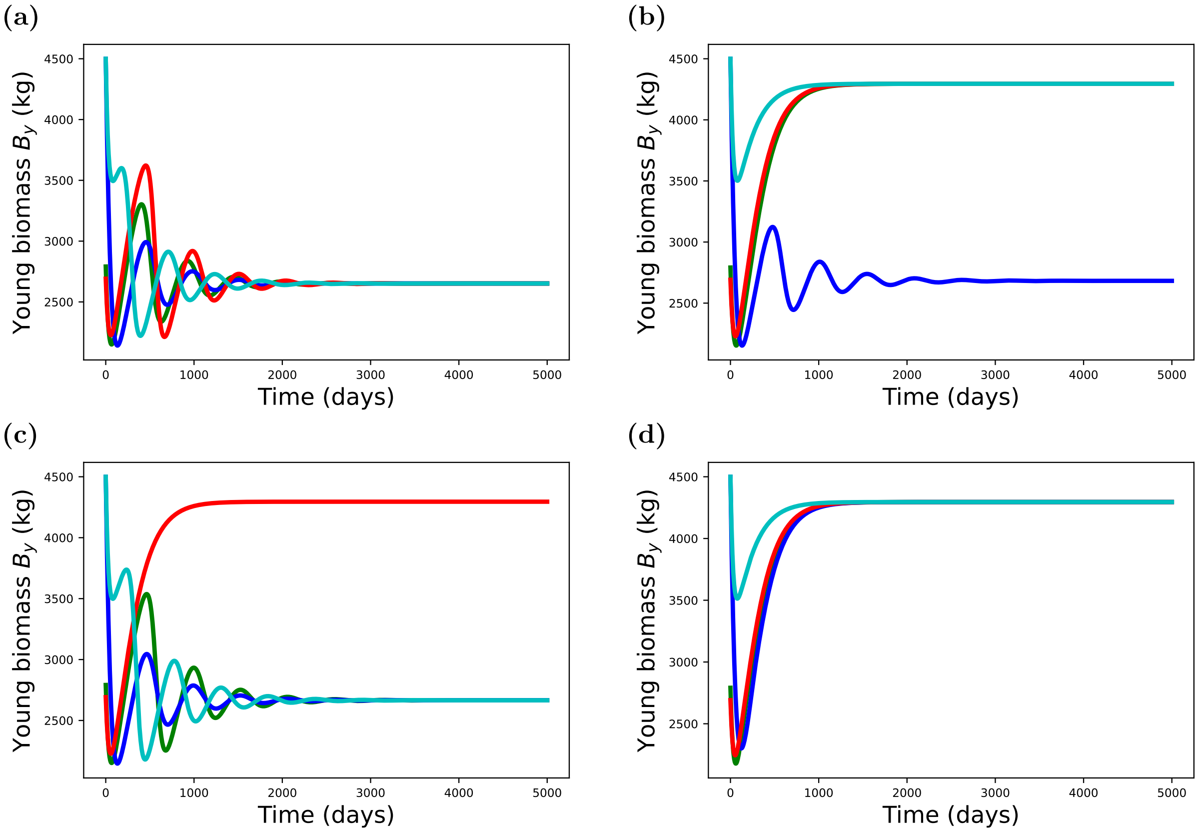

Figure 14: Dynamics of state variable B_y of model (3) with: (a) $\alpha = 7 \times 10^{-3}$ day^{-1}, (b) $\alpha = 1.35 \times 10^{-2}$ day^{-1} (same value as Fig. 11), (c) $\alpha = 0.01$ day^{-1} (same value as Fig. 13) and (d) $\alpha = 0.09$ day^{-1} (same value as Fig. 12). The initial conditions are the same as in Fig. 11. The remaining parameter values are given in Table 1 with $\delta_M = 10^{-5}$ USD^{-1}. We have $\alpha^{0'} \approx 7.34 \times 10^{-3}$ day^{-1} and $\alpha^{s'} \approx 0.022$ day^{-1}.