Further results about L ∞ /L 1 duality and applications to the SIR epidemiological model

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***Abstract*— The L^∞/L^1 duality in optimal control problems consists in studying how to link solutions minimizing the L^∞ norm of an output function under an upper L^1 constraint on an input function (primal problem), with solutions minimizing the L^1 norm of the input function under an upper L^∞ constraint on the output function (dual problem). In this work, we bring insights on recent results on L^∞/L^1 duality in optimal control problems. In particular, we exhibit an example for which duality does not apply, and we revisit the application to the epidemiological SIR problem.**

I. INTRODUCTION

The usual tools in optimal control theory, namely the Maximum Principle of Pontryagin and the Hamilton-Jacobi-Bellman equation, provide necessary and sufficient conditions for problems whose costs are integral, terminal or both (so-called Lagrange, Mayer or Bolza problems) [3]. However, in some applications, the maximum trajectory deviation is a more relevant criterion that better reflects transient behaviors and risky situations [8]. This is typically the case in epidemiology when one aims at reducing the *epidemic peak*. This rather corresponds to an L^∞ cost, for which there is much less results available in the literature (mainly theoretical characterizations of the value function [2]). Moreover, in several applications a budget is associated to the control strategy reducing the maximum deviation. This is again the case in epidemiology. Let us for instance briefly recall the well-known SIR model with non-pharmaceutical interventions

$$\begin{cases} \dot{s} = -\beta(1-u)si \\ \dot{i} = \beta(1-u)si - \gamma i \\ \dot{r} = \gamma i \end{cases} \tag{1}$$

The components correspond to a normalized (unitary) population divided into different compartments: susceptible, infected and recovered. The non-pharmaceutical policy u acts as a social-distancing measure reducing the effective contact rate β. The space of controls is $U = [0, \overline{u}]$, with the upper bound not exceeding 1. For this model, the problem of minimizing the epidemic peak (which has been strongly motivated by the the SARS CoV2 crisis)

$$\inf_{u(\cdot)\in\mathcal{U}_K} \sup_{t\geq 0} i(t)$$

has been recently investigated for the class of controls

$$\mathcal{U}_K := \{u(\cdot) \in L_1([0,+\infty), U);\ ||u||_1 \leq K\}$$

where K denotes the *control budget* [9]. Note that without this budget constraint, the control u identically equal to $\bar{u}$ is optimal, which is of limited interest... Another similar problem has been investigated by different authors [1], [6], [7]:

$$\inf_{u(\cdot)\in\mathcal{U}} \int_0^{+\infty} u(t)dt$$

under the ICU (*Intensive Care Unit*) constraint

$$\sup_{t\geq 0} i(t) \leq K$$

for the class of controls $\mathcal{U} = L_1([0,+\infty), U)$. This problem is more classical but one has to deal with state constraint, for which the application of the Maximum's principle with state constraint is not always easy. Clearly, this two optimal control problems present a kind of *duality*. This is exactly what has been recently investigated by the authors in [4], which defines a duality in a general framework and illustrate it on the SIR model. The purpose of the present work is to complement this former work on two aspects. First, we provide an explicit counter-example to this duality, and secondly we revisit the epidemiological problem with a more general budget constraint:

$$\int_0^{+\infty} \lambda(s(t), i(t))u(t)dt \leq K$$

where λ is some positive function, that we consider to be more realistic reflecting the fact that decision makers are prone to weight differently the intervention cost depending on the current stage of the epidemic progress (accounting for instance social pressures). This cost is still of L^1 form but rather more difficult to tackle with the simple technique developed in [10], [9], [5], because of the state dependency. This is where duality gets comes into interest: depending on the application, one of the problems might be easier to solve.

The paper is organized as follows. In Section II, we present the general framework, defining the *primal* and *dual* problems. We recall in Section III the duality results, underlying the role of the viability kernels, and provide an example for which duality does not apply. Section IV is devoted to revisiting the epidemiological problem with the SIR model under generalized budget function, using the duality that is shown to apply here.

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II. Definitions and notations

We consider a controlled system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ \dot{z}(t) = g(x(t), u(t)), \\ x(0) = x_0 \in \Omega, \ z(0) = z_0 \in \mathbb{R}_+ \end{cases} \tag{2}$$

defined on $\Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ $(n > 1)$ is of non-empty interior, and the control $u(t)$ takes values in a set U of a metric space; with an output function

$$y(t) = h(x(t)), \ t \geq 0.$$

$\mathcal{U} := \mathbb{L}^0(\mathbb{R}_+; U)$ is the set of *admissible* controls functions.

Assumption 1: i. U is compact and $f : \Omega \times U \to \mathbb{R}^n$ is continuous, Lipschitz in x uniformly in u:

$$\sup_{u \in U} \sup_{x,y \in \Omega, \ x \neq y} \frac{|f(x,u) - f(y,u)|}{|x-y|} < +\infty.$$

ii. The functions $h : \Omega \to \mathbb{R}$, $g : \Omega \times U \to \mathbb{R}_+$ are bounded, continuous and Lipschitz in x uniformly in u:

$$\sup_{u \in U} \sup_{x,y \in \Omega, \ x \neq y} \frac{|g(x,u) - g(y,u)| + |h(x) - h(y)|}{|x-y|} < +\infty.$$

iii. $\Omega \times \mathbb{R}_+$ is forward invariant by (2) for any $u \in \mathcal{U}$.

Under theses assumptions, we shall denote $(x^{x_0,u}(\cdot), z^{x_0,z_0,u}(\cdot))$ the unique absolutely continuous solution of (2) for $(x_0, z_0) \in \Omega \times \mathbb{R}_+$, $u \in \mathcal{U}$, and $y^{x_0,u}(\cdot)$ the corresponding output function.

Given $x_0 \in \Omega$, $h_0 \in \mathbb{R}$, $g_0 \in \mathbb{R}_+$, we define the sets of *viable* controls related to constraints on z and y

$$\mathcal{U}_h(x_0, h_0) := \{u \in \mathcal{U}; \ y^{x_0,u}(t) \leq h_0, \ \forall t \geq 0\},$$
$$\mathcal{U}_g(x_0, g_0) := \{u \in \mathcal{U}; \ z^{x_0,0,u}(t) \leq g_0, \ \forall t \geq 0\},$$

and the associated viability kernels.

$$\text{Viab}_h(h_0) := \{x_0 \in \Omega : \ \mathcal{U}_h(x_0, h_0) \neq \emptyset\},$$
$$\text{Viab}_g(g_0) := \{x_0 \in \Omega : \ \mathcal{U}_g(x_0, g_0) \neq \emptyset\},$$

Given $u \in \mathcal{U}$, we define the functions

$$G(x_0, u) := \int_0^{+\infty} g(x^{x_0,u}(t), u(t)) dt,$$
$$H(x_0, u) := \sup_{t \geq 0} h(x^{x_0,u}(t)).$$

and the related optimal control problems.

1) Primal problem:

$$\mathcal{P}_h(x_0; h_0) : \text{minimize } G(x_0, u) \text{ over } u \in \mathcal{U}_h(x_0, h_0).$$

2) Dual problem:

$$\mathcal{P}_g(x_0; g_0) : \text{minimize } H(x_0, u) \text{ over } u \in \mathcal{U}_g(x_0, g_0).$$

Their value functions are denoted, $V_h(x_0; h_0)$ resp. $V_g(x_0; g_0)$ (set to $+\infty$ when $x_0 \notin \text{Viab}_h(h_0)$), resp. $x_0 \notin \text{Viab}_g(g_0)$).

It will be convenient to define for any subset $L \subset \Omega$ and $(x_0, u) \in \Omega \times \mathcal{U}$ the hitting time function

$$\tau_L^{x_0,u} := \begin{cases} +\infty, & \text{if } x^{x_0,u}(t) \notin L, \ \forall t \geq 0, \\ \inf\{t; \ x^{x_0,u}(t) \in L\}, & \text{otherwise.} \end{cases}$$

We give now the main results of [4].

III. The duality results

Theorem 1: If the functions $V_h(x_0; \cdot)$, $V_g(x_0; \cdot)$ are lower semi-continuous on their domains, then V_h, V_g are generalized inverse i.e.

$$\begin{aligned} V_h(x_0; h_0) &= \inf\{g_0 : \ V_g(x_0; g_0) \leq h_0\}, \\ &\qquad h_0 \in \text{Dom}\, V_h(x_0; \cdot), \\ V_g(x_0; g_0) &= \inf\{h_0 : \ V_h(x_0; h_0) \leq g_0\}, \\ &\qquad g_0 \in \text{Dom}\, V_g(x_0; \cdot). \end{aligned} \tag{3}$$

Theorem 2: Let $(x_0, h_0) \in \Omega \times \mathbb{R}$ be such that $V_h(x_0; h_0) < +\infty$ and $V_h(x_0; \cdot)$ is lower semi-continuous. Posit

$$\underline{h}_0 := \inf\{h_0' : \ V_h(x_0; h_0') = V_h(x_0; h_0)\},$$
$$g_0 := V_h(x_0; h_0) = V_h(x_0; \underline{h}_0).$$

If u^* is optimal for $\mathcal{P}_h(x_0; \underline{h}_0)$, then u^* is optimal for $\mathcal{P}_g(x_0; g_0)$. In particular, if u^* is unique, then one has

$$g_0 := V_h\left(x_0; \sup_{t \geq 0} h\left(x^{x_0,u^*}(t)\right)\right).$$

An analogous statement is obtained for V_g and problem $\mathcal{P}_h$.

In all generality, one equally notes that the viable controls satisfy a monotonicity property, i.e., $\mathcal{U}_h(x_0, h_0) \subset \mathcal{U}_h(x_0, h_0')$ as son as $h_0 \leq h_0'$. The same holds true for $\mathcal{U}_g$. As a simple consequence, $V_h(x_0, \cdot)$ is non-increasing. The lower semi-continuity of $V_h(x_0; \cdot)$ is equivalent to the right-continuity. The same property holds true for $V_g(x_0; \cdot)$.

Conditions ensuring the lower semi-continuity of the value functions van be given under the additional assumption

Assumption 2: For any $x \in \Omega$, one has

$$\bigcup_{u \in U, r \geq 0} \begin{bmatrix} f(x,u) \\ g(x,u) + r \end{bmatrix} \text{ is closed and convex.}$$

Proposition 1: Let $x_0 \in \text{Viab}_h(h_0)$ for $h_0 \in \mathbb{R}$. If there exists $\varepsilon > 0$ and a compact set $L \subset \Omega$ such that

i. for any $\bar{h} \in [h_0, h_0 + \varepsilon)$, $L \cap \text{Viab}_h(\bar{h})$ is (forward) viable with a null-cost control, i.e.

$$\forall y_0 \in L \cap \text{Viab}_h(\bar{h}), \ \exists u(\cdot) \in \mathcal{U}_h(x_0, \bar{h}) \text{ s.t.}$$
$$x^{y_0,u}(t) \in L \text{ and } g(x^{y_0,u}(t), u(t)) = 0, \ \forall t > 0;$$

ii. L is finitely reached under viable controls i.e.

$$T^\star := \sup_{\bar{h} \in [h_0, h_0 + \varepsilon)} \sup_{u \in \mathcal{U}_h(x_0, \bar{h})} \tau_L^{x_0,u} < +\infty \tag{4}$$

then $V_h(x_0, \cdot)$ is bounded and lower semi-continuous on $[h_0, h_0 + \varepsilon)$.

Similar assertions hold true for $V_g(x_0; \cdot)$.

To show the importance of the lower semi-continuity property of the value functions to obtain the duality results of Theorems 1 and 2, we give here a counter example.

Consider the dynamics in $\mathbb{R}^3$

$$\begin{cases} \dot{x}_1 = u \in [-1,1] \\ \dot{x}_2 = -\min(x_3^2,1)x_2 \\ \dot{x}_3 = 0 \end{cases}$$

with the initial condition $x(0) = x_0 = (0,1,1)^\top$, and functions

$$h(x) = |x_1|, \quad g(x,u) = \min(|x_2|,1)(1-u^2).$$

Clearly, Assumptions 1 are fulfilled.

Whatever is the control $u(\cdot)$, one has

$$x_2(t) = e^{-t}, \Rightarrow g(x(t),u) = e^{-t}(1-u^2)$$

For $h_0 = 0$, the constraint $\sup_{t\geq 0} h(x(t)) \leq 0$ implies that the solution is $x(t) = 0$ for any $t \geq 0$, and thus the control $u(\cdot)$ has to satisfy $u(t) = 0$ a.e. $t > 0$. Then the cost is

$$\int_0^{+\infty} g(x(t),u(t))dt = \int_0^{+\infty} e^{-t}dt = 1$$

For $h_0 > 0$, consider the control

$$u_{h_0}(t) = \begin{cases} 1, & t \in [2kh_0,(2k+1)h_0), \\ -1 & t \in [(2k+1)h_0,(2k+2)h_0) \end{cases}$$

Then, the corresponding solution verifies $|x_1(t)| \leq h_0$ for any $t \geq 0$ and the constraint $\sup_{t\geq 0} h(x(t)) \leq h_0$ is thus satisfied, while the cost is

$$\int_0^{+\infty} g(x(t),u_{h_0}(t))dt = 0$$

and this solution is optimal. This shows that the value function of problem $\mathscr{P}_h$ verifies $V_h(x_0;h_0) = 0$ for any $h_0 > 0$ and $V_h(x_0;0) = 1$. $V_h(x_0,\cdot)$ is thus not lower semi-continuous at 0.

Consider now the dual problem $\mathscr{P}_g$ with the constraint $\int_0^{+\infty} g(x(t),u(t))dt \leq 0$. This implies that the control has to satisfy $u^2(t) = 1$ a.e. $t \geq 0$. Then, the sequence of controls $u_{1/n}(\cdot)$ provides a sequence of solutions $x_n(\cdot)$ such that $\sup_{t\geq 0} h(x_n(t)) = 1/n$ and we obtain $V_g(x_0;0) = 0$. We do have the property

$$\inf\{h_0 \geq 0;\ V_h(x_0;h_0) \leq g_0\} = V_g(x_0;g_0)$$

for $g_0 = 0$, as in Theorem 1. However, the problem $\mathscr{P}_g(x_0;0)$ does not admit an optimal solution because having $\sup_{t\geq 0} h(x(t)) = 0$ implies $\int_0^{+\infty} g(x(t),u(t))dt = 1$ and the integral constraint is thus not satisfied. In this sense, we consider that the duality fails (Theorem 2 is not satisfied). One can note that Assumption 2 is not fulfilled for this particular choice of g (although the velocity set of the x-dynamics is convex).

Remark 1: In practical problems, Theorem 2 also gives a further intuition on a way to check the equivalence between the problems $\mathscr{P}_h$ and $\mathscr{P}_g$.
Albeit presenting state constraints, the primal problem $\mathscr{P}_h$ has known further developments in the literature and, as such, it is considered easier. If the candidates to optimality $u^{*,h_0'} \in \mathcal{U}_h(x_0;h_0')$ in connection to the problem $\mathscr{P}_h$ are unique and known for constraint profiles h_0' in a right-neighborhood of some h_0, then, in order to check lower semi-continuity of $V_h(x_0;\cdot)$ at h_0, one studies the right-continuity of the value functions $G(x_0,u^{*,h_0'})$.
This argument applies, mutatis mutandis, to $\mathscr{P}_g$.

IV. Revisiting the SIR model

We come back to model (1) with control in U/ The Euclidean state space has the dimension $n = 2$ with state variable $(s,i) \in \Omega$ where

$$\Omega := \left\{(s,i) \in \mathbb{R}^2;\ s > 0,\ i > 0,\ s+i \leq 1\right\}.$$

Indeed, due to the conservation condition $s+i+r = 1$, it suffices to argue on the first two components, the third one being naturally determined. We are concerned with *prevalence* peak constraints and the running cost is multiplicative in the control with a state-depending coefficient, i.e.,

$$h(s,i) := i, \quad g(s,i,u) = \lambda(s,i)u,$$

where λ is a smooth function. Furthermore, we will make the following assumption on the upper bound $\bar{u}$.

$$\overline{u} < 1 - \frac{\gamma}{\beta}. \tag{5}$$

Remark 2: 1) The proportion $\frac{\gamma}{\beta}$ corresponds to the endemic equilibrium without social distancing. Theoretically, with the maximal action $u = \bar{u}$, the resulting system has yet another equilibrium $\bar{s} = \frac{\gamma}{\beta(1-\overline{u})}$. If (5) holds true, $\bar{s} < 1$ and is relevant in analysis. Otherwise, this point is no longer relevant and the analysis is somewhat simplified.

2) Furthermore, this condition implies the existence of initial configurations under which the *strict confinement* should be decreed for a certain period of time.
3) Small $\overline{u}$ corresponds to a limited capacity of action, where the non-pharmaceutical measures have to be carefully considered. Finally, with the data in France on COVID-19, one has
   - $\gamma^{-1} = 14$ (time of recovery in days);
   - $R_0 = \frac{\beta}{\gamma} \approx 5.8148148$ (expressed in (individuals$\times$days)$^{-1}$, as a weighted average on $R_0 \in \{4,7,11\}$);

   As a result, $\beta \approx 0,41534$ (individuals^{-1}), and $\bar{u} \leq 0.828$ (meaning that the essential economy concerns at least 17.2% of the population, which is well bellow the reality).

A. The viability kernels and their geometric decomposition

As we have already hinted at before, we aim to illustrate the duality result by assuming that the problem $\mathscr{P}_h$ is easier than the problem $\mathscr{P}_g$. In other words, we will primarily focus on a thorough description of the viability kernel $Viab_h$. Let

$i^* \in [0,1]$ be fixed. Then, according to [1, Theorem 2.3], and provided that $i^* + \frac{\gamma}{\beta(1-\overline{u})} \leq 1$, we get the following description of the viability kernel.

$$
\begin{aligned}
&Viab_h(i^*) = \\
&\Big\{(s_0,i_0)\in\Omega\ :\ s_0 \leq \frac{\gamma}{\beta(1-\overline{u})},\, i_0 \leq i^* \text{ or} \\
&s_0 > \frac{\gamma}{\beta(1-\overline{u})}, \quad \text{and} \\
&i_0 \leq \frac{\gamma}{\beta(1-\overline{u})}\left(1+\log\left(\frac{\beta(1-\overline{u})s_0}{\gamma}\right)\right) - s_0 + i^*\Big\}
\end{aligned}
\tag{6}
$$

Let use note that the *minimal action* taken on the active boundary of $Viab_h(i^*)$ (i.e., outside the axis) is divided into three parts

1) if $s_0 \leq \frac{\gamma}{\beta}$, any control keeps $i \leq i^*$ and the minimal action is 0;
2) if $s_0 \in \left(\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\overline{u})}\right)$, the minimal action keeps $i = i^*$ and is obtained for the feed-back control $1-\frac{\gamma}{\beta s}$;
3) if $s_0 \geq \frac{\gamma}{\beta(1-\overline{u})}$, the only control maintaining $i \leq i^*$ is $\overline{u}$.

We are, therefore, concerned with this minimal action control in closed feed-back form

$$
\begin{aligned}
u^{*,i^*}(s,i) = &\max\left\{0, 1-\frac{\gamma}{\beta s}\right\}\mathbf{1}_{s\in\left(\frac{\gamma}{\beta},\frac{\gamma}{\beta(1-\overline{u})}\right),\ i=i^*} \\
&+\overline{u}\mathbf{1}_{i^*\geq i=\frac{\gamma}{\beta(1-\overline{u})}\left(1+\log\left(\frac{\beta(1-\overline{u})s}{\gamma}\right)\right)-s+i^*}
\end{aligned}
\tag{7}
$$

For further developments, we also introduce the invariance kernel associated to i^*

$$
\begin{aligned}
&Inv_h(i^\star) := \\
&\left\{(s_0,i_0)\in\Omega:\ \forall u\in\mathbb{L}^0(\mathbb{R}_+;U);\ i^{(s_0,i_0),u}(t)\leq i^\star,\ \forall t\geq 0\right\}
\end{aligned}
$$

and similar to $Viab_h(i^*)$, one has

$$
\begin{aligned}
&(s_0,i_0)\in Inv_h(i^*) \ \Leftrightarrow \\
&\left\{s_0 \leq \tfrac{\gamma}{\beta},\, i_0\leq i^*\right\} \text{ or} \\
&\left\{s_0 > \tfrac{\gamma}{\beta},\, i_0 \leq \tfrac{\gamma}{\beta}\left[1+\log\left(\tfrac{\beta s_0}{\gamma}\right)\right] - s_0 + i^*\right\}.
\end{aligned}
\tag{8}
$$

This set is the maximal one on which the trajectory driven associated with u^{*,i^*} is actually only computed with the null control. Second, we define the set of initial data for which the control u^{*,i^*} is almost surely strictly inferior to $\overline{b}$. We get the following explicit description.

$$
\begin{aligned}
&(s_0,i_0)\in B(i^*) \ \Leftrightarrow \\
&\left\{s_0 \leq \tfrac{\gamma}{\beta(1-\overline{u})},\, i_0\leq i^*\right\} \text{ or} \\
&\Big\{s_0 > \tfrac{\gamma}{\beta(1-\overline{u})}, \\
&\quad i_0 \leq \tfrac{\gamma}{\beta(1-\overline{u})}\left[1+\log\left(\tfrac{\beta(1-\overline{u})s_0}{\gamma}\right)\right] - s_0 + i^*\Big\}.
\end{aligned}
\tag{9}
$$

Concerning the main assumptions, the reader will note that we deal with a control-affine structure here such that

1) the sets $Viab_h(i^*) \subset Viab_h(1)$ are compact;
2) the Assumption 2 (convexity of the extended velocity set) is always satisfied.

We have the following simple results.

Proposition 2: For every $i^*1, i_2^* \in \left(0, 1-\frac{\gamma}{\beta(1-\overline{u})}\right)$, the following holds true.

1) $Inv_h(i_1^*) \subset B(i_1^*) \subset Viab_h(i_1^*)$;
2) If $i_1^* \leq i_2^*$, then $Inv_h(i_1^*) \subset Inv_h(i_2^*)$, $B(i_1^*) \subset B(i_2^*)$ and $Viab_h(i_1^*) \subset Viab_h(i_2^*)$.

B. The cost associated to u^*

We emphasize that the arguments hereafter work for g much more general than the multiplicative one we have here $(g(s,i,u) = \lambda(s,i)u$. The reader is referred to [6]. The reason for illustrating this example is that monotonicity is very easily obtained in this framework, without further assumptions on g and that this allows to infer a condition that echoes that on the dual approach in [9].

Proposition 3: The associated cost satisfies (see [6, Lemma 1])

$$
\begin{aligned}
&G\big(s_0,i_0,u^{*,i^*}\big) = 0,\ \text{if } (s_0,i_0)\in Inv_h(i^*); \\
&G\big(s_0,i_0,u^{*,i^*}\big) = \frac{1}{\gamma i^*}\int_{\frac{\gamma}{\beta}}^{s_1(s_0,i_0;i^*)} \lambda(l,i^*)\left(1-\frac{\gamma}{\beta l}\right)dl, \\
&\qquad\qquad \text{if } (s_0,i_0)\in B(i^*)\setminus Inv_h(i^*); \\
&G\big(s_0,i_0,u^{*,i^*}\big) = G\Big(\tfrac{\gamma}{\beta(1-\overline{u})}, i^*, u^{*,i^*}\Big) + \\
&\frac{1}{\beta(1-\overline{u})}\int_{\frac{\gamma}{\beta(1-\overline{u})}}^{s_2} \frac{\lambda\Big(s,\Big(\theta(i^*)-s+\frac{\gamma}{\beta(1-\overline{u})}\log s\Big)\Big)\overline{u}}{s\Big(\theta(i^*)-s+\frac{\gamma}{\beta(1-\overline{u})}\log s\Big)}\,ds \\
&\qquad\qquad \text{otherwise}
\end{aligned}
\tag{10}
$$

where

(a) : $s_1(s_0,i_0;i^*) > \frac{\gamma}{\beta}$ is the solution of $s_1 - s_0 - i_0 + i^* - \frac{\gamma}{\beta}\log\frac{s_1}{s_0} = 0$;

(b) : $\theta(i) := i + \frac{\gamma}{\beta(1-\overline{u})}\Big(1-\log\frac{\gamma}{\beta(1-\overline{u})}\Big)$;

(c) : $s_2 = s_2(s_0,i_0;i^*)$ is explicitly given by

$$
s_2 := \exp\left(\tfrac{\beta(1-\overline{u})}{\gamma\overline{u}}\left(s_0+i_0-\tfrac{\gamma}{\beta}\log s_0 - \theta(i^*)\right)\right).
\tag{11}
$$

The cost $G\Big(\frac{\gamma}{\beta(1-\overline{u})}, i^*, u^{*,i^*}\Big)$ used in expression (10) depends on the function λ and does not have necessarily an explicit expression, except when λ is constant as in [1], [9].

We claim that the following property is fulfilled.

Lemma 1: Fix $(s_0,i_0)\in\Omega$ and $(i^n)_n$, $n\in\mathbb{N}$, a sequence decreasing to i^*. Then, one has

$$
\lim_{n\to+\infty} G\Big(s_0,i_0,u^{*,i^n}\Big) = G\Big(s_0,i_0,u^{*,i^*}\Big).
\tag{12}
$$

Proof:

1) The reader will easily note that one has $Inv_h(i^*) = \underset{n\in N}{\cap} Inv_h(i^n)$ (decreasing limit).
2) The same assertion holds true by defining $B(i^*)$ given in (11)(a) as $B(i^*) = \underset{n\in N}{\cap} B(i^n)$, where $\left(B(i^n)\right)_n$ is a non-increasing sequence.
3) If $(s_0,i_0)\in Inv_h(i^*)$, then the equality in (12) follows easily from the inclusion $Inv_h(i^*) \subset Inv_h(i^n)$ for

every $n \in \mathbb{N}$ and by recalling that the value function is null at such points.

4) If $(s_0, i_0) \in B(i^*) \setminus Inv_h(i^*)$, then $(s_0, i_0) \in B(i^n)$, for all $n \in \mathbb{N}$. If there existed a subsequence $(\phi(n))_n$ such that $(s_0, i_0) \in Inv_h\big(i^{\phi(n)}\big)$ for any $n \in \mathbb{N}$, then, we would have $(s_0, i_0) \in Inv_h(i^*)$ which is not the case.

   It follows that, from some $n_0 > 0$ large enough and every $n \geq n_0$, one has $(s_0, i_0) \in B(i^n) \setminus Inv_h(i^n)$. One easily see that the function $i \mapsto s_1(s_0, i_0; i)$ is right-continuous for $i > 0$, and we get equality (12) for this framework.

The same arguments can be applied in order to prove (12) on $Viab_h(i^*) \setminus B(i^*)$ due to the continuity of the functions θ and s_2. ■

These considerations yield the following regularity result.

Proposition 4: Let $(s_0, i_0) \in \Omega$. Then, the value function $i^* \mapsto G\big(s, i, u^{*,i^*}\big)$ is right-continuous at every point $i^* < 1 - \frac{\gamma}{\beta(1-\overline{u})}$, such that $V_h(s_0, i_0; i^*) < \infty$.

C. Concluding using Theorem 2

In connection the the optimality of u^{*,i^*} for the problem $\mathscr{P}_h$, we recall the following result, cf. [6, Theorem 2] (see also Example 2).

Theorem 3: Let $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be non-negative, uniformly continuous such that

1) $\frac{\lambda(s_1(s_0,i_0),i^*)}{i^*} \leq \frac{\lambda(s_0,i_0)}{i_0}$, if $(s_0, i_0) \in B(i^*) \setminus Inv_h(i^*)$;
2) $\frac{\lambda\left(s_2,\theta(i^*)-s_2+\frac{\gamma}{\beta(1-\overline{u})}\log s_2\right)}{\theta(i^*)-s_2+\frac{\gamma}{\beta(1-\overline{u})}\log s_2} \leq \frac{\lambda(s_0,i_0)}{i_0}$, if $(s_0, i_0) \in Viab_h(i^*) \setminus B(i^*)$. Here, $s_2 = s_2(s_0, i_0; i^*)$ as in Proposition 3.

Then, u^{*,i^*} is the optimal control for $\mathscr{P}_h$ at any point $(s_0, i_0) \in Viab_h(i^*)$.

As a consequence, we have the following theoretical result

Proposition 5: Let $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be non-negative, uniformly continuous such that

1) $\frac{\lambda(s_1(s_0,i_0;i^*),i^*)}{i^*} \leq \frac{\lambda(s_0,i_0)}{i_0}$, if $(s_0, i_0) \in B(i^*) \setminus Inv_h(i^*)$;
2) $\frac{\lambda\left(s_2,\theta(i^*)-s_2+\frac{\gamma}{\beta(1-\overline{u})}\log s_2\right)}{\theta(i^*)-s_2+\frac{\gamma}{\beta(1-\overline{u})}\log s_2} \leq \frac{\lambda(s_0,i_0)}{i_0}$, if $(s_0, i_0) \in Viab_h(i^*) \setminus B(i^*)$. Here, $s_2 = s_2(s_0, i_0; i^*)$ as in Proposition 3.

Then, u^{*,i^*} is also the optimal control for $\mathscr{P}_g$.

Proof: Theorem 3 guarantees the optimality for $\mathscr{P}_h$. On the other hand, by Proposition 4, at (s_0, i_0), $V_h(s_0, i_0; \cdot)$ is right-continuous, hence lower semi-continuous. The conclusion follows owing to Theorem 2. ■

Let us further note the following. Geometrically speaking, the point $(s_1 = s_1(s_0, i_0; i^*), i^*)$ is the point at which the trajectory issued from (s_0, i_0) and uncontrolled $(u \equiv 0)$ hits the boundary $\partial \mathrm{Viab}_h(i^*)$ in the case $(s_0, i_0) \in B(i^*) \setminus Inv_h(i^*)$. Similarly, the point $\left(s_2, \theta(i^*) - s_2 + \frac{\gamma}{\beta(1-\overline{u})} \log s_2\right)$ is the point at which the trajectory issued from (s_0, i_0) and uncontrolled $(u \equiv 0)$ hits the boundary $\partial \mathrm{Viab}_h(i^*)$ in the remaining case.

As such, both conditions in the proposition are satisfied if one guarantees that $\partial_t \frac{\lambda\left(s^{s_0,i_0,0}(t),i^{s_0,i_0,0}(t)\right)}{i^{s_0,i_0,0}(t)}$ is non-increasing. This leads to the sufficient condition

$$\partial_s Q(s,i) - \partial_i P(s,i) \leq 0. \tag{13}$$

where

$$P(s,i) = \frac{\lambda(s,i)\left(\frac{\gamma}{\beta s} - 1\right)}{\gamma i} \text{ and } Q(s,i) = -\frac{\lambda(s,i)}{\gamma i}.$$

The reader will easily note that the control is determined by

$$udt = \left(\frac{\gamma}{\beta s} - 1\right)\frac{ds}{\gamma i} - \frac{di}{\gamma i} = P(s,i)ds + Q(s,i)di.$$

In other words, assuming some regularity (differentiability conditions on λ, under the sufficient condition (13), the optimality results in [6] imply those in [9], via our main result Theorem 2. The converse is also true under this assumption.

V. Conclusion

In this work, we have first shown the importance of the semi-continuity of the value functions with respect to the constraint level for the validity of the L^∞/L^1 duality. Let us underline that in the literature semi-continuity of the value functions in optimal controls is mainly studied with respect to the initial condition only.

In the application on the SIR model, we have also shown the role of the viability kernels in the optimal control synthesis, and thus the importance of their determination prior to the optimality analysis.

Future works could consider particular classes of dynamics for which we could refine our sufficient conditions to obtain a L^∞/L^1 duality.

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