Optimal control under action duration constraint for non-convex dynamics

Dan Goreac, Alain Rapaport

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***Abstract*— The motivation of this paper stems from a family of optimal control problems wherein the control's active duration is constrained within a predefined limit. The duration constraint can be perceived as an additional variable in the dynamics, and the relaxation of the naturally associated control problem is equivalent to a an $\mathbb{L}^1$-constraint. The paper provides a generalization of a preliminary work by the authors to encompass scenarios with non-convex dynamics. The relaxed problems are formulated through Linear Programming techniques and their qualitative properties, alongside the interrelations between different formulations, are investigated.**

I. Introduction

We consider a controlled dynamics

$$\begin{cases} \dot{x} = f(x,u), \\ x(0) = x_0, \end{cases} \tag{1}$$

under usual regularity assumptions on f, see Assumption 1. The control functions belong to

$$\mathcal{U} := \{u : [0,T] \longrightarrow U := [0,1] \text{ Borel measurable}\},$$

where $T > 0$ is a finite time horizon.
In connection to these dynamics, one can naturally consider the following Mayer problem, with cost $\Phi \in C^1$

$$(\mathscr{P}_1) \ : \ \inf_{u(\cdot)\in\mathcal{U}_\tau} \Phi(x(T)),$$

where the admissible controls take into account a *constraint on the active duration*, i.e.,

$$\mathcal{U}_\tau := \{u(\cdot) \in \mathcal{U} \ : \ meas(\{t \in [0,T] : u(t) > 0\}) \leq \tau\}.$$

Such problems have gained increasing interest in connection with epidemics. In [13], an infection peak criterion is investigated when the mitigating interventions $u(\cdot)$ verify a duration constraint of a somewhat different nature. Similar formulations appear [1], [2] seeking to optimize the final size.

Since we are interested in constraint durations, and following [10], we further introduce

$$\begin{cases} \dot{x} = f(x,u), \ x(0) = x_0, \\ \dot{z} = v, \ z(0) = z_0 \in \mathbb{R}, \end{cases} \tag{2}$$

and the control

$$w := (u,v) \in W := \Big\{(u,v) \in [0,1]^2 \ : \ uv = 0\Big\}.$$

In connection to these dynamics, a more classical control problem can be formulated

$$(\mathscr{P}_2) \ : \ \inf_{w(\cdot)\in\mathscr{W}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau,$$

with a target condition and formulated over

$$\mathscr{W} := \{w : [0,T] \longrightarrow W \text{ Borel measurable}\}.$$

Furthermore, due to the absence of convexity of the control set W, or, to be more precise, of the set $\{(f(x,u),v) \ : \ (u,v) \in W\}$, the following convexified problem is naturally considered

$$\left(\overline{\mathscr{P}_2}\right) \ : \ \inf_{w(\cdot)\in\overline{\mathscr{W}}} \Phi(x(T)) \text{ s.t. } z(T) \geq T - \tau,$$

where $\overline{\mathscr{W}} := \{w : [0,T] \longrightarrow \overline{co}\ W \text{ Borel measurable}\}$. A dual problem with $\mathbb{L}^1$-constraint on the control was formulated in [10].

$$(\mathscr{P}_3) \ : \ \inf_{u(\cdot)\in\mathcal{U}^1_\tau} \Phi(x(T)),$$

where

$$\mathcal{U}^1_\tau := \{u(\cdot) \in \mathcal{U} \ : \ \|u\|_1 \leq \tau\}, \text{ where } \|u\|_1 := \int_0^T u(t)dt.$$

Under convexity assumptions on the dynamics, it is shown in [10] that the problems $\left(\overline{\mathscr{P}_2}\right)$ and $(\mathscr{P}_3)$ are equivalent. Let us emphasize that $(\mathscr{P}_3)$ relates to the optimization of the final state of the epidemics under budgetary constraints, cf., [3].

A. A short overview of the control affine case

When the dynamics are control affine i.e.

$$f(x,u) = f_1(x) + g_1(x)u,$$

it has been proven in [10].

Theorem 1: [10, Lemmas 2.1, 2.2 and Prop. 2.1]

1) The problems $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$ are equivalent.
2) The problems $\left(\overline{\mathscr{P}_2}\right)$ and $(\mathscr{P}_3)$ are equivalent.
3) The problem $(\mathscr{P}_3)$ is equivalent to $(\mathscr{P}_1)$ formulated with relaxed controls.

A further no-gap result is obtained by a slight *time-randomization* in $(\mathscr{P}_1)$; for further details, please refer to Section II-B.

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Let us emphasize that, in particular, affine dynamics guarantee that the set-valued function

$$F(x,z) := \{(f_1(x)+g_1(x)u, v) \ : \ (u,v)\in \overline{co}\ W\}$$

is Lipschitz-continuous with convex and compact images and the results in [10] apply under such assumptions.

B. Objectives and main contributions

Our aim is to explore extensions when the dynamics f are no longer affine in control. In particular, while the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ do not need any convexity, the primary focus shifts to convexification when considering the problems $(\overline{\mathcal{P}_2})$ and $(\mathcal{P}_3)$. Additionally, as a secondary outcome, for convex dynamics, these latter problems admit optimal controls, and become more amenable to classical optimization approaches. When f is not affine in control, it may not be sufficient to consider the closed convex hull $\overline{co}\ W$, but one should, instead, consider an *occupation measure* formulation of the trajectories. Such relaxation methods have been successfully applied to classical deterministic systems, cf., [12], [5], singularly perturbed dynamics, cf., [4], or control problems with discontinuities or state constraints, cf., [6], see see also [7] for extensions to stochastic settings, or [9] for extensions to mixed controls.

The main idea is to associate, to every control policy, a normalized measure of the time/space/control occupation of the associated solution. As usual in optimal control, the objects of interest are the differential formula estimated at a test function of the trajectory, and, if available, bounds on the solutions translating into bounds of moments. For continuous costs, optimizing over such measures or, equivalently, over controls is the same as optimizing over the weak-* closed convex hull of such measures, thus transforming the problem into a Linear Programming (LP) one on an infinite-dimensional space of measures enjoying convexity and closedness, or even compactness properties.

With this intuition in mind, we will show how the problems $(\overline{\mathcal{P}_2})$ and $(\mathcal{P}_3)$ can be relaxed into LP problems in which the target constraint, and the $\mathbb{L}^1$ respectively, are taken into account via a linear restriction on spaces of measures. The different relations between the newly introduced problems, as well as the optimality issues are the main novelties in Section II-C. Further insight on time-randomization of the original problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are presented in Section II-B. Finally the overall comparison of different problems are presented in Section II-D.

II. NON-CONVEX DYNAMICS

A. Assumptions and occupation measures techniques

Throughout the paper we will enforce the following hypotheses on the data.

Assumption 1: The map f is a uniformly continuous function on $\mathbb{R}^n\times[0,1]$. Furthermore, it is Lipschitz-continuous in space uniformly with respect to the control variable $u\in U:=[0,1]$, i.e.,

$$[f]_1 := \sup_{x\neq y\in R^n,\ u\in U}\frac{|f(x,u)-f(y,u)|}{|x-y|}<\infty,$$

and with linear growth, i.e.,

$$[f]_0 := \sup_{u\in U}|f(0,u)|<\infty.$$

Under such assumptions, the equation (2) admits a unique solution, for every $(u,v)\in\overline{\mathcal{W}}$. To emphasize the initial data and the dependency on the control, as well as the decoupling in the dynamics, the solution is denoted by $(x^{x_0,u}, z^{z_0,v})$. In addition to considering the convex set $\overline{co}\ W$, we elude convexity assumptions on $\{(f(x,u),v) \ : \ (u,v)\in\overline{co}\ W\}$ by interpreting the trajectories associated to (2) as probability measures.

More precisely, given $(u,v)(\cdot)\in\overline{\mathcal{W}}$, we construct

$$\begin{cases} \gamma_1^{x_0,z_0,u,v}(S,A,V) := \\ \frac{1}{T}\int_0^T \mathbf{1}_{S\times A\times V}(t, x^{x_0,u}(t), z^{z_0,v}(t), (u,v)(t))\ dt, \\ \qquad S\subset[0,T],\ A\subset\mathbb{R}^{n+1},\ V\subset\overline{co}\ W; \\ \gamma_2^{x_0,z_0,u,v} := \delta_{x^{x_0,u}(T), z^{z_0,v}(T)}, \end{cases}$$

where δ is the Dirac mass. This provides an element $\gamma^{x_0,z_0,u,v}=(\gamma_1^{x_0,z_0,u,v},\gamma_2^{x_0,z_0,u,v})$ of the probability space $\mathcal{P}([0,T]\times\mathbb{R}^{n+1}\times\overline{co}\ W)\times\mathcal{P}(\mathbb{R}^{n+1})$. The set of all such $\gamma^{x_0,z_0,u,v}$ as $(u,v)(\cdot)\in\overline{\mathcal{W}}$ describes the family of *occupation measures* denoted by $\Gamma(x_0,z_0)$.

In view of the linear growth requirement on the coefficient f, standard estimates exhibit the existence of a T-dependent constant C such that $|x(t)|^p\le C(1+|x_0|^p)$, for all $t\in[0,T]$ and for some $p>2$. This can be added to the definition of $\Gamma(x_0,z_0)$ by asking

$$\begin{aligned}\int_{\mathbb{R}^{n+1}}(|y|^p+|z|^p)(\gamma_1([0,T],dy,dz,\overline{co}\ W)+\gamma_2(dy,dz))\\ \le C(1+|x_0|^p+|z_0|^p).\end{aligned} \tag{3}$$

In particular, we have chosen to work with $p>2$ since these uniform bounds on moments imply uniform integrability conditions in $\mathbb{L}^2$ that further translate into relative compactness in the 2-Wasserstein metric. Higher-order regularity can be imposed if necessary. Classical results, see, for instance, [5] for infinite-time horizon control, and [6, Corollary 2], see also [7], yield the following.

Proposition 1: Under Assumption 1, the weak-* closed convex hull of $\Gamma(x_0,z_0)$ is given by

$$\begin{aligned}&\Theta(x_0,z_0) := \\ &\Big\{(\gamma_1,\gamma_2)\in\mathcal{P}([0,T]\times\mathbb{R}^{n+1}\times\overline{co}\ W)\times\mathcal{P}\big(\mathbb{R}^{n+1}\big) \ : \\ &\forall\phi\in C_2^1\big(\mathbb{R}^{n+4};\mathbb{R}\big) \\ &\int_{\mathbb{R}^{n+1}}\phi(T,y,z)\gamma_2(dy,dz)=\phi(0,x_0,z_0)+ \\ &T\int_{[0,T]\times\mathbb{R}^n\times\overline{co}\ W}\mathcal{L}^{u,v}\phi(t,y,z)\gamma_1(dt,dy,dz,du,dv)\Big\}.\end{aligned} \tag{4}$$

Here, $\mathcal{P}$ stands for the set of probability measures on the Borel sigma-field on the respective metric spaces, and C_2^1 are differentiable functions that are, together with their gradients, continuous and have at most quadratic growth. Furthermore,

$$\begin{aligned}\mathcal{L}^{u,v}\phi(t,y,z) :=&\langle f(y,u), \partial_y \phi(t,y,z)\rangle \\ &+ \partial_t \phi(t,y,z) + \partial_z \phi(t,y,z) v.\end{aligned} \tag{5}$$

Remark 1: As a consequence, of (3), we actually get that the set of constraints $\Theta(x_0, z_0)$ becomes a compact subset of the Wasserstein-2 space of probability measures with finite second order moment $\mathcal{P}_2([0,T]\times\mathbb{R}^n\times U)\times\mathcal{P}_2(\mathbb{R}^n)$. For further details on Wasserstein spaces, the reader is referred to [14].

B. Time-randomization of $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$

The reader will easily note that the dynamics in (2) are obtained for usual time weighs dt. A slight generalization is obtained if $meas(E(u))$ describing the action constraints in $(\mathcal{P}_1)$ and corresponding to $\int_0^T (1-\mathbf{1}_{\{0\}})(u(t))dt$ is replaced with $\int_0^T (1-\eta(u(t)))dt$, for convenient η. T this purpose, we consider

$$\begin{aligned}H := \Big\{ &\eta : [0,1] \longrightarrow [0,1] \;\; : \\ &\eta \text{ is } u.s.c., \;\eta(u) \leq 1-u, \forall u \in [0,1] \Big\},\end{aligned} \tag{6}$$

u.s.c. standing for upper semi-continuous. Having fixed $\eta \in H$, one can naturally consider the following Mayer problem, with cost $\Phi \in C^1$

$$(\mathcal{P}_{1,\eta}) \;\;:\;\; \inf_{u(\cdot)\in\mathcal{U}_{\tau,\eta}} \Phi(x(T)),$$

where the admissible controls take into account a constraint on the action duration, i.e.,

$$\mathcal{U}_{\tau,\eta} := \left\{ u(\cdot)\in\mathcal{U} \;\;:\;\; \int_0^T (1-\eta(u(t)))dt \leq \tau \right\}.$$

The second definition extending $(\mathcal{P}_2)$ reads

$$(\mathcal{P}_{2,\eta}) \;\;:\;\; \inf_{w(\cdot)\in\mathcal{W}_\eta} \Phi(x(T)) \text{ s.t. } z(T) \geq T-\tau,$$

presenting a target condition and formulated over

$$\begin{cases}\mathcal{W}_\eta := \{w : [0,T] \longrightarrow W_\eta \text{ Borel measurable}\}, \\ W_\eta := \Big\{ w := (u,v) \in [0,1]^2 \;\;:\;\; v \leq \eta(u) \Big\}.\end{cases}$$

In the non-convex setting (i.e. without further assumptions on $\{f(x,u) \;:\; u \in U\}$ being convex), we can establish the following result.

Proposition 2: Under the assumption 1, the problems $(\mathcal{P}_{1,\eta})$ and $(\mathcal{P}_{2,\eta})$ are equivalent for every $\eta \in H$.

Proof: For $\varepsilon > 0$ we consider an admissible control $u_\varepsilon(\cdot)\in\mathcal{U}$ which is ε-optimal for problem $(\mathcal{P}_{1,\eta})$, i.e.,

$$\Phi(x_\varepsilon(T)) < \inf_{u(\cdot)\in\mathcal{U}_{\tau,\eta}} \Phi(x(T)) + \varepsilon,$$

where x_ε is the solution of (1) controlled with u_ε. By defining $v_\varepsilon := \eta(u_\varepsilon)$, the extended control $w_\varepsilon := (u_\varepsilon, v_\varepsilon)$ belongs to $\mathcal{W}_\eta$ and its associated solution of (2) satisfies

$$z_\varepsilon(T) = \int_0^T \eta(u_\varepsilon(t))dt \geq T-\tau.$$

From the arbitrariness of $\varepsilon > 0$ it follows that the value of $(\mathcal{P}_{2,\eta})$ cannot exceed the value in $(\mathcal{P}_{1,\eta})$.

For the converse, again with $\varepsilon > 0$, one considers $w_\varepsilon \in \mathcal{W}_\eta$ which is ε-optimal for $(\mathcal{P}_{2,\eta})$, i.e., such that the solution $(x_\varepsilon, z_\varepsilon)$ of (2) and associated to w_ε satisfies

$$\Phi(x_\varepsilon(T)) - \varepsilon < \left(\inf_{w(\cdot)\in\mathcal{W}_\eta} \Phi(x(T)) \text{ s.t. } z(T) \geq T-\tau \right).$$

By definition of z_ε, the qualification condition $z_\varepsilon(T) \geq T-\tau$ implies

$$T-t \leq \int_0^T v_\varepsilon(t)dt \leq \int_0^T \eta(u_\varepsilon(t))dt.$$

As a consequence, $u_\varepsilon \in \mathcal{U}_{\tau,\eta}$. Again invoking the arbitrariness of $\varepsilon > 0$, it follows that the value of $(\mathcal{P}_{1,\eta})$ cannot exceed the value in $(\mathcal{P}_{2,\eta})$, which completes our proof. ■

Remark 2: As we have already pointed out, the equivalence between $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ is obtained for $\eta = \mathbf{1}_{\{0\}}$.

Finally, we recall the following result in the convex case (cf. [10]).

Proposition 3: [10, Prop. 3.1] If

$$\{(f(x,u), v) \;\;:\;\; (u,v) \in \overline{co}\; W\} \text{ is convex } \forall x \in \mathbb{R}^n,$$

then

$$\inf_{\eta\in H} \inf_{u(\cdot)\in\mathcal{U}_{\tau,\eta}} \Phi(x(T)) = \inf_{u(\cdot)\in\mathcal{U}_\tau^1} \Phi(x(T)).$$

C. The relaxed problems

As we have already explained, we wish to extend $(\overline{\mathcal{P}_2})$ and $(\mathcal{P}_3)$ in such a way that their equivalence and the structural properties (existence of an optimum) hold true independently of convexity assumptions on f. This is achieved by naturally translating the aforementioned problems into LP formulations over the sets $\Theta(x_0, z_0)$. To achieve this program, we introduce

$$\left(\overline{\mathcal{P}_2'}\right) : \inf_{\gamma=(\gamma_1,\gamma_2)\in\Theta(x_0,0)} \int_{\mathbb{R}^n} \Phi(y)\gamma_2(dy,\mathbb{R}) \; s.t. \\ \int_{\mathbb{R}} z\gamma_2(\mathbb{R}^n, dz) \geq T-\tau.$$

The reader will easily note that this reduces to $(\overline{\mathcal{P}_2})$ when $\gamma_2(\cdot,\mathbb{R})$ is a Dirac mass.

We further introduce a problem similar to $(\mathcal{P}_3)$, in which the $\mathbb{L}^1$ constraint on the control u is no longer computed with respect to the Lebesgue measure on $[0,T]$ but dictated by the marginal of γ_1.

$$(\mathcal{P}_3') \;\;:\;\; \inf_{\gamma\in\Theta(x_0,0) \;\; s.t. \;(7)} \int_{\mathbb{R}^n} \Phi(y)\gamma_2(dy,\mathbb{R}),$$

where the restriction is given as follows

$$T\int_{[0,T]\times\overline{co}\;W} u\gamma_1(ds, \mathbb{R}^{n+1}, du, dv) \leq \tau. \tag{7}$$

1) Connection between $(\mathscr{P}_3)$ and $(\mathscr{P}_3')$:

Proposition 4: If $x_0 \in \mathbb{R}^n$ is such that the value of $(\mathscr{P}_3)$ can be approximated with the classical penalization

$$\lim_{M\to\infty} \inf_{u\in\mathcal{U}} \left[\Phi(X(T)) + M\left(\tau - \int_0^T u(t)dt\right)^+\right],$$

then $(\mathscr{P}_3)$ and $(\mathscr{P}_3')$ are equivalent at x_0.

Proof: For x_0 fixed let us denote by $V^+(x_0) < \infty$ the value function of $(\mathscr{P}_3)$ and by $V_-(x_0) \le V^+(x_0)$ the value function of $(\mathscr{P}_3')$. The reader is invited to note that

$$\Psi^M(\gamma) := \int_{\mathbb{R}^n} \Phi(y) d\gamma_2(dy, \mathbb{R}) + M\left(\tau - T\int_{[0,T]\times\mathbb{R}^n\times\overline{co}\ W} u d\gamma_1(dt, dy, \mathbb{R}, du, dv\right)^+$$

is a continuous and convex functional on the space of probability measures $\mathscr{P}_2([0,T] \times \mathbb{R}^{n+1} \times \overline{co}\ W \times \mathbb{R}^{n+1})$ if $M > 0$. Then

$$\inf_{u\in\mathcal{U}} \left[\Phi(X(T)) + M\left(\tau - \int_0^T u(t)dt\right)^+\right] = \inf_{u\in\mathcal{U}} \Psi^M\left(\gamma^{x_0,0,u,1-u}\right) = \inf_{\gamma\in\Theta(x_0,0)} \Psi^M(\gamma) \le V_-(x_0).$$

By compactness of $\Theta(x_0, 0)$, for every M there exists γ^M such

$$\Psi^M(\gamma^M) = \inf_{\gamma\in\Theta(x_0,0)} \Psi^M(\gamma).$$

Furthermore, along some sub-sequence, still indexed by M for simplicity, $\gamma^M \to \gamma^* \in \Theta(x_0, 0)$. It follows that

$$T\int_{[0,T]\times\mathbb{R}^n\times\overline{co}\ W} u d\gamma_1^*(dt, dy, \mathbb{R}, du, dv) \le \tau.$$

As a consequence, γ^* is optimal for $(\mathscr{P}_3')$ and

$$\lim_{M\to\infty} \inf_{u\in\mathcal{U}} \left[\Phi(X(T)) + M\left(\tau - \int_0^T u(t)dt\right)^+\right] = V_-(x_0, 0).$$

The proof is complete under our standing assumption. ■

Remark 3: The assumption in the previous proposition is classically satisfied if the dynamics $f(x, U)$ are convex for every $x \in \mathbb{R}^n$.

2) Equivalence between $\left(\overline{\mathscr{P}_2'}\right)$ and $(\mathscr{P}_3')$: Without any convexity assumptions, we prove the following equivalence result.

Proposition 5: Problems $\left(\overline{\mathscr{P}_2'}\right)$ and $(\mathscr{P}_3')$ are equivalent.

Proof: Let $\gamma \in \Theta(x_0, 0)$ such that

$$\int_{\mathbb{R}} z\gamma_2(\mathbb{R}^n, dz) \ge T - \tau. \tag{8}$$

On the other hand, writing the equality constraint in $\Theta(x_0, 0)$ with $\phi(z) = z$ leads to

$$\int_{\mathbb{R}} z\gamma_2(\mathbb{R}^n, dz) = T\int_{\overline{co}\ W} v\gamma_1\left([0,T], \mathbb{R}^{n+1}, du, dv\right) \ge T-\tau.$$

Finally, the construction of $\overline{co}\ W$ yields $u \le 1 - v$, which implies

$$T\int_{[0,T]\times\overline{co}\ W} u\gamma_1(ds, \mathbb{R}^{n+1}, du, dv) \le T\int_{[0,T]\times\overline{co}\ W} (1-v)\gamma_1(ds, \mathbb{R}^{n+1}, du, dv) = T - T\int_{[0,T]\times\overline{co}\ W} v\gamma_1(ds, \mathbb{R}^{n+1}, du, dv),$$

thus providing an admissible control for $(\mathscr{P}_3')$. As a consequence, the optimal value of problem $(\mathscr{P}_3')$ does not exceed the one in $\left(\overline{\mathscr{P}_2'}\right)$.

Conversely, let $\gamma \in \Theta(x_0, 0)$ such that (7) is satisfied. Then, there exists a family of controls $(u_i^m, v_i^m)_{1\le i\le k_m}$ and of coefficients $(\alpha_i^m)_{1\le i\le k_m}$ such that the combination of the associated occupation measures $\sum_{1\le i\le k_m} \alpha_i^m \gamma^{x_0,0,u_i^m,v_i^m}$ converges to γ. Then, one considers $\sum_{1\le i\le k_m} \alpha_i^m \gamma^{x_0,0,u_i^m,1-u_i^m}$ also converging (along some subsequence) to some $\eta \in \Theta(x_0, 0)$. Furthermore, since going from v_i^m to $1 - u_i^m$ does not affect the (ds, du) marginals,

$$T\int_{\overline{co}\ W} u \sum_{1\le i\le k_m} \alpha_i^m \gamma_1^{x_0,0,u_i^m,1-u_i^m}([0,T], \mathbb{R}^{n+1}, du, dv) = \sum_{1\le i\le k_m} \alpha_i^m T\int_{\overline{co}\ W} u\gamma_1^{x_0,0,u_i^m,v_i^m}([0,T], \mathbb{R}^{n+1}, du, dv).$$

Passing to the limit as $m \to \infty$, one gets that η satisfies (7) as well. Furthermore,

$$\int |1 - u - v| d\left[\sum_{1\le i\le k_m} \alpha_i^m \gamma_1^{x_0,0,u_i^m,1-u_i^m}\right] = 0,$$

which implies that $v = 1 - u,\ d\eta_1$ a.s. Arguing as in the first part, it follows that

$$\int_{\mathbb{R}} z\eta_2(\mathbb{R}^n, dz, \overline{co}\ W) = T\int_{\overline{co}\ W} v\gamma_1\left([0,T], \mathbb{R}^{n+1}, du, dv\right) = T\int_{\overline{co}\ W} (1-u)\gamma_1\left([0,T], \mathbb{R}^{n+1}, du, dv\right) \ge T - \tau.$$

The conclusion follows. ■

3) Optimality results for the relaxed problems: To complete the study of these problems, let us note that the following result on optimality holds true.

Theorem 2:

1) The problem $\left(\overline{\mathscr{P}_2'}\right)$ admits an optimal solution provided that Φ is continuous and has at most quadratic growth.
2) In this case, an optimal solution $\gamma \in \Theta(x_0, 0)$ for the problem $\left(\overline{\mathscr{P}_2'}\right)$ can be selected such that $v = 1-u,\ \gamma_1-a.s.$.
3) The problem $\left(\overline{\mathscr{P}_3'}\right)$ admits an optimal solution provided that Φ is continuous and has at most quadratic growth.

Proof: Indeed, let us denote by $V(x_0, 0)$ the optimal value for $\left(\overline{\mathscr{P}_2'}\right)$ at $(x_0, z_0 = 0)$. For a sequence of k^{-1}-

optimal solutions, i.e. $\gamma^k \in \Theta(x_0, 0)$ such that

$$\begin{cases} \int_{\mathbb{R}^n} \Phi(y)\gamma_2^k(dy, \mathbb{R}) \leq V(x_0, 0) + k^{-1}; \\ \int_{\mathbb{R}} z\gamma_2^k(\mathbb{R}^n, dz) \geq T - \tau, \end{cases}$$

due to the compactness of $\Theta(x_0, 0)$, there exists a sub-sequence (still indexed by k) W_2-converging to an element $\gamma^* \in \Theta(x_0, 0)$. Since Φ is continuous with quadratic growth implying (along the aforementioned sub-sequence)

$$\int_{\mathbb{R}^n} \Phi(y)\gamma_2^*(dy, \mathbb{R}) = \lim_{k\to\infty} \int_{\mathbb{R}^n} \Phi(y)\gamma_2^k(dy, \mathbb{R}) \leq V(x_0, 0),$$

and, since z is actually bounded on the support of measures (actually $0 \leq z \leq T$ when $z_0 = 0$), it follows that

$$\int_{\mathbb{R}} z\gamma_2^*(\mathbb{R}^n, dz) = \lim_{k\to\infty} \int_{\mathbb{R}} z\gamma_2^k(\mathbb{R}^n, dz) \geq T - \tau.$$

The conclusion follows from the definition of the value function in $\left(\overline{\mathcal{P}_2'}\right)$.
The proof of the third assertion is similar.
Let us now concentrate on the second assertion. By the first assertion, there exists $\gamma \in \Theta(x_0, 0)$ such that $\int_{\mathbb{R}} z\gamma_2(\mathbb{R}^n, dz) \geq T - t$ and $V(x_0, 0) = \int_{\mathbb{R}^n} \Phi(y)\gamma_2(dy, \mathbb{R})$, where $V(x_0, 0)$ denotes the optimal value. By the properties of $\Theta(x_0, 0)$, there exists a family of controls $(u_i^m, v_i^m)_{1\leq i\leq k_m}$ and non-negative coefficients $(\alpha_i^m)_{1\leq i\leq k_m}$ such that $\sum_{i=1}^{k_m} \alpha_i^m = 1$ and such that

$$\gamma = \lim_{m\to\infty} \sum_{i=1}^{k_m} \alpha_i^m \gamma^{x_0,0,u_i^m,v_i^m}.$$

As we have already proceeded before, we modify the aforementioned sequence to obtain $\sum_{i=1}^{k_m} \alpha_i^m \gamma^{x_0,0,u_i^m,1-u_i^m}$ and, by compactness of $\Theta(x_0, 0)$, this family converges (in a usual and in W_2 sense) to some $\gamma' \in \Theta(x_0, 0)$. Since $\gamma_1^{x_0,0,u_i^m,1-u_i^m}$ has its support on $v = 1 - u$, the same holds true for the limit γ_1'.
The linear constraint in the description of $\Theta(x_0, 0)$ with $\phi(t, y, z) = z$ yields, for $\tilde{\gamma} \in \Theta(x_0, 0)$,

$$\int_{\mathbb{R}} z\tilde{\gamma}_2(\mathbb{R}^n, dz) = T \int_{[0,T]\times\mathbb{R}^{n+1}\times\overline{co}W} v\tilde{\gamma}_1(dt, dy, dz, du, dv).$$

As a consequence, one has

$$\begin{aligned} \int_{\mathbb{R}} z\gamma_2'(\mathbb{R}^n, dz) &= \lim_{m\to\infty} \sum_{i=1}^{k_m} \alpha_i^m \int_0^T (1 - u_i^m(t))dt \\ &\geq \lim_{m\to\infty} \sum_{i=1}^{k_m} \alpha_i^m \int_0^T v_i^m(t)dt \\ &= \int_{\mathbb{R}^{n+1}} z\gamma_2(dy, dz) \geq T - \tau. \end{aligned}$$

Furthermore, if Φ is of class C^1 with at most quadratic growth and whose derivatives have at most linear growth, then, by the linear restriction and (5), it follows that $\int_{\mathbb{R}^n} \Phi(y)\gamma_2(dy, \mathbb{R})$ does not depend on the v-marginal of γ. As such, γ' is also optimal and

$$\int_{\mathbb{R}^n} \Phi(y)\gamma_2'(dy, \mathbb{R}) = V(x_0, 0).$$

■

D. Completing the circle

Finally, let us emphasize the relation between the various problems introduced.

Proposition 6: Let us enforce Assumption 1 on the controlled velocity. Furthermore, let us assume Φ to be a continuous function with at most quadratic growth. Then,

$$\begin{aligned} &\inf_{\eta\in H} \inf_{u\in\mathcal{U}_{\tau,h}} \Phi(x(T)) \\ &= \inf\Big\{\Phi(x(T)) \ : \ \eta \in H, \ w(\cdot) \in \mathcal{W}_\eta \\ &\qquad \text{s.t. } z(T) \geq T - \tau\Big\} \\ &\geq \inf\Big\{ \int_{\mathbb{R}^n} \Phi(y)\gamma_2(dy, \mathbb{R}) \ : \ \gamma \in \Theta(x_0, 0) \text{ s.t.} \\ &\qquad \int_{\mathbb{R}} z\gamma_2(\mathbb{R}^n, dz) \geq T - \tau\Big\}. \end{aligned} \tag{9}$$

Furthermore, if the dynamics are such that $\{(f(x, u), v) \ : \ w : (u, v) \in \overline{co}\ W\}$ is convex, then (9) provides an equality.

Remark 4:
1) By Theorem 2, we learn that the optimal measures in $\left(\overline{\mathcal{P}_2'}\right)$ are supported by $v = 1 - u$ and, as such, it is morally natural that the optimum in $(\mathcal{P}_{2,\eta})$ be of similar nature. But, this is only possible in such a control is admissible, which amounts to η being an extremal element

$$\eta_{extr}(u) = 1 - u,$$

and, as a consequence, this optimum should also be selected in connection to $(\mathcal{P}_{2,\eta_{extr}})$.
2) The reader is invited to note that $W_{\eta_{extr}} = \overline{co}\ W$ and, as such,

$$\mathcal{W}_{\eta_{extr}} = \overline{\mathcal{W}}.$$

3) A deeper look into the proof shows that convexity is not needed and it can be replaced with a requirement of optimality of a measure associated to a classical control for the relaxed problems $\left(\overline{\mathcal{P}_2'}\right)$ or $(\mathcal{P}_3')$.

Proof: In order to prove the first assertion, and in view of Proposition 5, it suffices to show that the inequality in (9) holds true. Let us consider $\eta \in H$ and $w(\cdot) \in \mathcal{W}_\eta$ s.t. $z(T) \geq T - \tau$. By definition, $w = (u, v)$, where $u(\cdot) \in \mathcal{U}$ satisfies $v(t) \leq \eta(u(t)) \leq 1 - u(t)$. As a consequence, $(u(t), v(t)) \in \overline{co}\ W$ and the occupation measure $\gamma^{x_0,0,u,v} \in \Theta(x_0, 0)$ further satisfies $\int_{\mathbb{R}} z\gamma_2^{x_0,0,u,v}(\mathbb{R}^n, dz) = z^{x_0,0,u,v}(T) \geq T - t$. It follows that the second term in (9) cannot exceed the third one.
In the convex case, see also Remark 3, the last term also provides the value of $(\mathcal{P}_3)$. One than applies Proposition 3 to conclude. ■

Conclusions

This paper offers an extension of the techniques proposed in [10] to non-convex dynamics. In particular, we investigate the relation between the problem of optimizing a criterion under action duration constraints $(\mathcal{P}_1)$ and problems with similar objective formulated in a Linear Programming (LP) manner ($\mathscr{P}_3'$) and taking into account $\mathbb{L}^1$ (or budgetary) restrictions.

The main advantage of providing LP formulations is that they open promising numerical perspectives through Sum-of-Squares and LMI-techniques (see, for instance, [11] or [12]for classical formulations or [8] for mixed controls) allowing to tackle problems with somewhat more unconventional constraints. Yet another advantage is that these LP formulations are naturally related, through duality, with the viscosity solutions of Hamilton-Jacobi equations (cf., [5], [6]). The exploitation of such relations is left for future work.

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