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ABSTRACT

The concept of regime shift generally refers to a deterministic dynamical system switching from an attraction basin to another after an isolated perturbation. However, when the dynamics is constantly submitted to random perturbations, the system can go back and forth between attraction basins and the usual concept seems inadequate. To address this issue, we consider a stochastic dynamical system and we assume that its functioning is satisfactory in a subset of its state space, called the satisfaction set. We define regimes with respect to the propensity of the system to become and to remain satisfactory. We investigate two indicators available in the literature: (i) the first-exit time from the satisfaction set and (ii) the sojourn time in the satisfaction set. Using statistics on these indicators, we define regimes of durable or resilient satisfaction. Applying the same definitions to the dissatisfaction set, we define durable or resilient dissatisfaction. Our results show the emergence of a tipping set, equivalent to tipping point in the deterministic case. We illustrate our approach using three different types of dynamics: two theoretical models based on the exploitation of natural resources and predator–prey dynamics, as well as the eutrophication of the French Lake Bourget.

1. Introduction

1.1. Assessing regime shifts in social-ecological systems

The concept of regime shift is a powerful tool for describing how dramatic changes take place in complex dynamical systems, in particular social-ecological systems (SES), such as changes in tropical forest (Hirota et al., 2011), in coral reefs (Hughes et al., 2017), in the thermohaline circulation (Vellinga and Wood, 2002), in exploited ecosystems (Mathias et al., 2020) or the collapse of Greenland ice sheet (Pritchard et al., 2009). This concept is mobilized in various research topics such as the assessment of early warnings (Dakos et al., 2008), the analysis of experimental data (Dakos et al., 2012) or related theoretical developments (Lenton et al., 2008; Scheffer et al., 2009). Generally, these researchers refer to the mathematics of dynamical systems and use the concepts of equilibrium (stable or unstable), attractor or attraction basin. A regime is classically associated with the attraction basin of a stable equilibrium state (Scheffer et al., 2001) (or a set of attraction basins) and a regime shift is classically defined by the transition from one attraction basin to another, because of a perturbation or a slight change in internal feedbacks. The system therefore encounters an unstable equilibrium, often called a tipping point (Milkoreit et al., 2018). A small perturbation at the tipping point may push the system into one attraction basin or into the other (Scheffer et al., 2009; Lenton, 2013). The perturbations that may lead to a regime shift are considered one by one, implying that the system reaches a stable state after each perturbation. Before and after the perturbation, the system evolves with a single possible trajectory, defined without uncertainty. Hence if the system crosses a tipping point because of the perturbation, it then remains in the new attraction basin and a next perturbation is generally not considered.

However, in practice, socio-ecological systems are characterized by significant uncertainties (Nuno et al., 2014) and they are subject to frequent perturbations, for instance because of weather variations. Mathematically, the uncertainty or the perturbations are modeled by adding randomness into the dynamics (Scheffer et al., 2009). At any moment, the system's trajectory has different possible directions for continuing its path, and one of them is drawn according to its probability. The dynamics are then said to be uncertain. In this case, defining attractors and attraction basins is less easy. Moreover, supposing this done, the standard definition of regime shift as a transition between two attraction basins may become misleading. Indeed, suppose for instance that, during a period of observation, a system stays in a regime except for a very short time spent in a second regime. This situation can easily occur because of the randomness. According to the standard definition, there would be two regime shifts (one from the first regime to the

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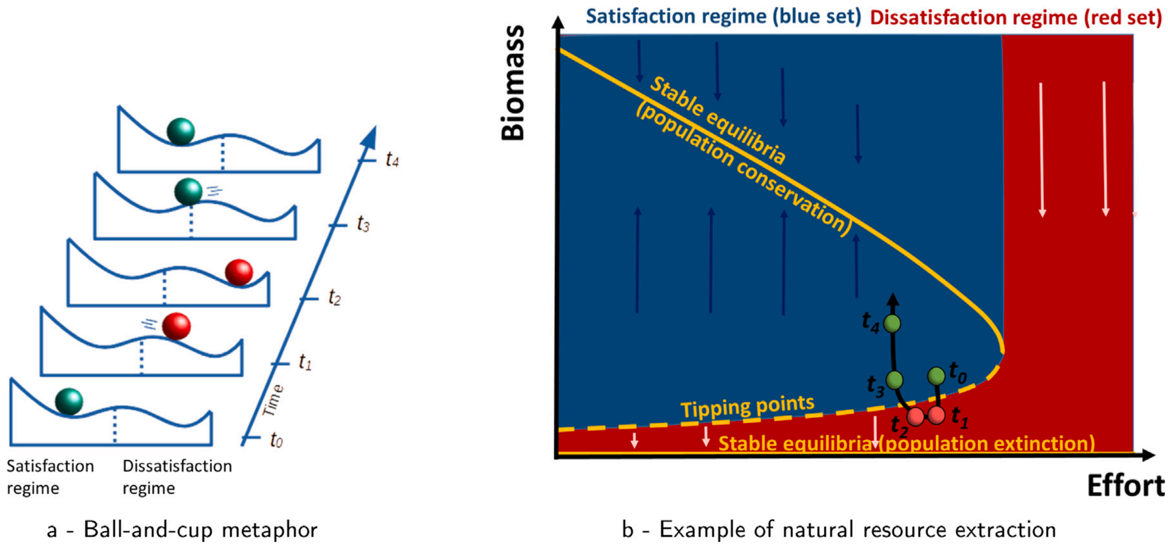

Fig. 1. Assessing regime shifts in the case of uncertain dynamics. In Fig. 1a, the ball-and-cup metaphor may be misleading in the case of a series of perturbations. A regime shift is defined as the transition from a regime to another one as depicted between time t_0 and t_1. However, if the dynamics are uncertain, the system may bounce back after a short time (here time t_3). This metaphor is represented in Fig. 1b in the case of natural resources extraction. The blue set corresponds to the basin of attraction of non-zero stable equilibria (any states within this set will converge towards a non-zero stable equilibria) whereas the red set corresponds to the basin of attraction of zero stable equilibria (any states within this set will converge towards a null biomass). In many cases, the main assumption is that stable equilibria are satisfactory whereas stakeholders want to avoid population extinction.

second one, the other from the second regime to the first one). But does this very short time spent in the second regime really matter? This question is particularly crucial for stakeholders who use or manage environmental systems. For instance, if an oligotrophic lake becomes mesotrophic during one season (due to an extreme rainfall), stakeholders would like to know if the water has high chances to remain of bad quality in the long-term or if it is likely to soon recover its good quality.

1.2. The ball-and-cup metaphor can be misleading

The ball-and-cup metaphor (Lamothe et al., 2019) provides an intuitive representation of regime shifts (see Fig. 1a). This metaphor has first been used for describing ecosystem stability (Willems, 1970), its resilience (Holling, 1973) as well as regime shifts (Scheffer and Carpenter, 2003). The ball rolling on a hilly landscape represents the state of SES changing over time. The bottom of the cup represents stable equilibriums where the SES tends to remain. SES may move about within the cup, never settling at the bottom of the cup because of uncertainties. On the other hand, SES may also go over a hill and settle in a new cup. This ball-and-cup metaphor enables us to understand and discuss the importance of perturbations in SES regime shifts (Scheffer and Carpenter, 2003). This metaphor is relevant when considering an isolated perturbation of deterministic dynamics, but it becomes questionable when random perturbations constantly occur. For instance, let us imagine two consecutive perturbations (as shown in Fig. 1a): the first one pushes the system in a new cup (at time t_1 and the second one pushes back the system into the initial cup (at time t_3). Do we call it a regime shift? In practice, let us consider the problem of lake eutrophication in the case of variations in weather (variations in temperature and rainfall for instance). In this case, hot temperature with high rainfall may cause a temporary eutrophication of the lake. How long should this episode last to be qualified as "regime shift"? A few days is too short and several years more than enough for most stakeholders, but what about one month or one year? The answer to this question is clearly partly normative. In the following, we try to generalize and formalize this example.

2. Methods

2.1. Which indicators for assessing regime shifts?

2.1.1. The starting point: defining a "satisfaction set"

In the field of socio-ecological systems, regimes are not only based on the characteristics of the dynamics (attractors, attraction basins), they are also related to a qualitative assessment of the state space. For instance, the oligotrophic states of a lake are considered positive (or desirable, or satisfactory) because in such states, the water of the lake is clear, which is favorable to tourism and biodiversity. On the contrary, the eutrophic states are considered negative (or undesirable or unsatisfactory) because the water is then invaded by algae and becomes opaque, which is bad for tourism and biodiversity. Similarly, in the case of natural resource management, the states offering a profitable activity while keeping the resource level high enough would be considered as positive. As we can see, the qualitative assessment of the states inherently encompasses normative aspects. For instance, in France, policy-makers should keep the chlorophyll-a concentration in a lake below 30 mg m^{-3}, otherwise the law imposes some restrictions on recreational activities in the lake. In the same vein, by national regulation, forest managers should keep a minimum volume of deadwood per hectare of forest in order to favor biodiversity. As shown in Fig. 2, in some cases such management thresholds are directly derived from basins of attraction of the system's dynamics, but in other cases, probably significantly more frequent, social, political or legislative processes are their strongest determinants. At the extreme, the goal can simply be to prevent the system from collapsing. Our framework, closely related to viability theory (Aubin, 1991; Delara and Doyen, 2008), focuses on keeping a system within a satisfaction set, without making specific assumptions about this set. It therefore fits all the previously mentioned usual ways to frame problems of sustainability.

Indeed, at the starting point of our approach, we assume the existence of a classification of the system states be it derived from an existing legislation, from the expressed preferences of a group of stakeholders or from the assessment of the system's basins of attraction by scientists. We say that the system is satisfactory when it is in a positive (or desirable or acceptable) state and unsatisfactory otherwise. The term "satisfactory" is general enough to include "desirable" or

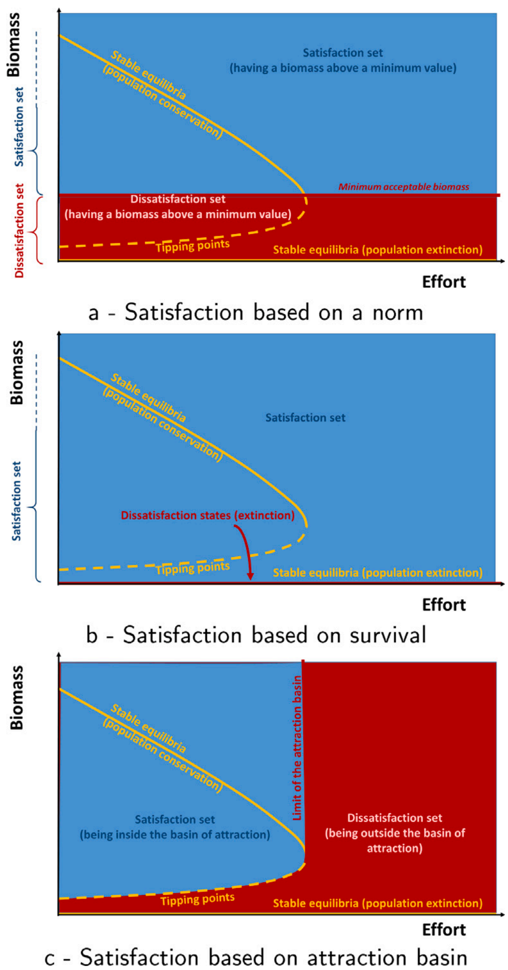

Fig. 2. Usual sustainability problems formulated as keeping the system within a satisfaction set. Each panel shows the satisfaction set (represented in blue) and the dissatisfaction set (represented in red) associated with a usual sustainability problem. Panel a illustrates the objective of norm satisfaction: stakeholders aim at keeping the biomass above a given threshold. Panel b is quite similar with an objective of avoiding collapse (maintaining survival). Finally, panel c shows the case of staying in the basin of attraction of desirable equilibria (non-zero equilibria).

"acceptable" as different levels of satisfaction. We assume that a mathematical function, classifying states as satisfactory or unsatisfactory, is available.

Importantly, we also assume that a reliable stochastic model of the system's dynamics is available. From any state of the system, this model provides the set of likely trajectories at the next time steps, with their probabilities. We are aware that this assumption is demanding but it seems reasonable to postulate ideal conditions at first in order to derive clear concepts.

Using this model of the system's dynamics, a regime is characterized by the propensity of the system to remain satisfactory (when the system is already satisfactory) or to become and remain satisfactory (when it is unsatisfactory). Building upon the literature about indicators characterizing how long a stochastic system stays in a set, such as the first-exit time (Zhang et al., 2019; Serdukova et al., 2016; Lindner and Hellmann, 2019), the first escape probability (Zhang et al., 2019), the sojourn time (Lindner and Hellmann, 2019) or stochastic basins of attraction of a set (Lindner and Hellmann, 2019; Serdukova et al., 2016), we now define durable satisfaction (or dissatisfaction) and resilient satisfaction (or dissatisfaction).

2.1.2. Durable satisfaction or dissatisfaction

If the system is in a state of the satisfaction set, within which its trajectory is likely to remain a significant time, then the system is in a state of durable satisfaction. The word durable is used for underlining that the system is likely to remain in the satisfaction set for a duration which is significant according to relevant stakeholders. For this purpose, we consider a time horizon T, which represents a specific period of interest. We characterize states of durable satisfaction by distinguishing two situations described by Fig. 3, from which we derive two possible indicators (see Annex A for all mathematical details):

- **the mean sojourn time (MST) in the satisfaction set S**. The sojourn time in the satisfaction set S within horizon T is the proportion of a trajectory of duration T located in the satisfaction set. The mean sojourn time (MST) from a given state of the satisfaction set is the average sojourn time in the satisfaction set within horizon T over all possible trajectories from the considered state. The MST measures the durability of satisfaction: a value close to 1 indicates that the system is likely to remain satisfactory most of the time horizon. On the contrary, a value close to 0 indicates that the system is likely to be unsatisfactory most of the time horizon. It is worth noting that other statistical operators, such as the median or quantiles, may also be relevant for defining this indicator.
- the **median first-exit time (MFET) from satisfaction set S**. The concept of "life expectancy" has been used as a measure of resilience and has been formalized as the mean exit time from an attraction basin (Arani et al., 2021). Exit time is a common way to characterize the stability of stochastic systems (Zhang et al., 2019; Serdukova et al., 2016; Lindner and Hellmann, 2019; Arani et al., 2021). In this study, we propose to use this concept as the basis for an indicator. Suppose that the system has just crossed the limit of the satisfaction set S and is in the dissatisfaction set (denoted $\overline{S}$ as the complementary set of the satisfaction set S). We assume that decision-makers want to know how long the system will remain in this dissatisfaction set $\overline{S}$, or in other words, what is its "life expectancy" in this set. To answer this question, we aim to determine the exit time of the system from this dissatisfaction set $\overline{S}$ and its corresponding entrance time into the satisfaction set S (time t_3 in Fig. 1). Similarly, decision-makers may also want to know how long the system remains in the satisfaction set S. In both cases, we use an indicator based on the distribution of the first-exit time described in Fig. 3. Specifically, we focus on the median first-exit time (MFET), which corresponds to the probability of 0.5 for the system to stay in the satisfaction set for at least a time equal to the MFET. We note that the mean first-exit time was not used because it can be infinite only with a single trajectory remaining infinitely in the satisfaction set, making its approximation challenging, whereas we just need to determine the cumulative exit-time distribution for 50% of the population for the MFET. Therefore, an infinite value of first-exit time for a single trajectory does not lead to an infinite value of the median: the value of the first-exit time (finite or infinite) itself does not matter in this case, what matters is its comparison with the threshold defining the median. Finally, in practice, we consider trajectories of finite duration T - the time horizon of interest - and we divide the exit-time by T, getting a normalized value between 0 and 1. A normalized MFET equal to 0.8 corresponds to a state from which half the trajectories stay in the satisfaction set during more than $0.8T$.

A particular attention has been paid to selecting these indicators, ensuring that they apply to uncertain dynamics and converge to the classical definition of deterministic regimes when the noise vanishes (refer to SI for proofs). We associate a "durability threshold" (d) to both the normalized MFET (Mean First Exit Time) and MST (Mean Sojourn Time) indicators interchangeably. For clarity, the durability threshold

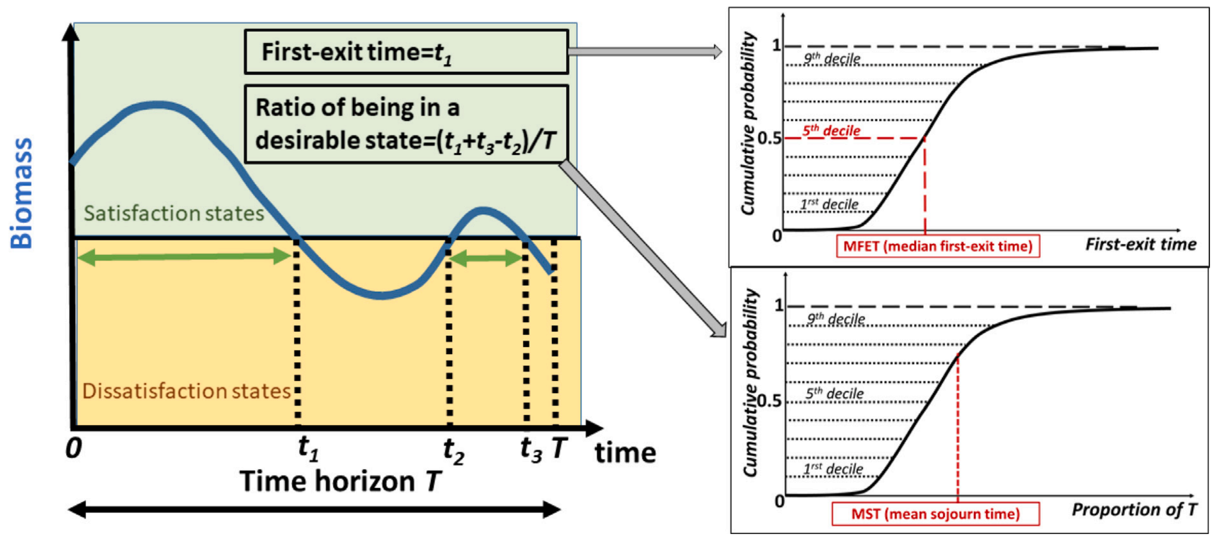

Fig. 3. Indicators used for assessing regime shifts. The satisfaction set is shown in green and the dissatisfaction set is shown in yellow. For defining durable satisfaction and durable dissatisfaction regimes, we use cumulative probability of two indicators for assessing regime shifts. We use the median first-exit time in order to avoid infinite sum in the case of mean indicator (e.g. only one collapsing trajectory will keep the system for an infinite time, involving an infinite mean). We also use the mean proportion of time being in a satisfaction state during a given time horizon T.

$d \in [0,1]$ represents the minimum value of the indicator required for the satisfaction from this state to be considered as durable. The durable satisfaction set denoted by $D(S,d)$ (and, the durable dissatisfaction set by $D(\overline{S},d)$) is thus the set of states from which the durability indicator is (MST or MFET) is higher than d.

2.1.3. Resilient satisfaction or dissatisfaction

In addition to defining states of durable satisfaction it is also important to assess if, from unsatisfactory states (i.e. located in the dissatisfaction set), the system could become durably satisfactory after a reasonable time. Indeed, from some states located in the satisfaction set and outside the durable satisfaction set, the system could also reach the durable satisfaction set after a reasonable time. This corresponds to the core idea of resilience, which is to recover a property (Holling, 1973, 1978; Martin, 2004; Deffuant and Gilbert, 2011; Mathias et al., 2018). Here the property to recover is the durable satisfaction. The “reasonable time” needs to be assessed by an indicator that measures the time required to reach the durable satisfaction set $D(S,d)$. Among possible indicators, we use the median reaching time (MRT) into the durable satisfaction set within the time horizon T (this is equivalent to the median first-exit time from the complementary of the durable satisfaction set), divided by time horizon T in order to get a value of the indicator between 0 and 1. Then, the states of resilient satisfaction are the states where the satisfaction is not durable, and the median reaching time to the durable satisfaction set is below a given threshold $r < 1$. The resilient satisfaction set $R(S,d,r)$ is the set of states located outside the durable satisfaction set $D(S,d)$, for which the median reaching time to $D(S,d)$ within horizon T is lower than rT. The resilient dissatisfaction set $R(\overline{S},d,r)$ is defined similarly by replacing S by $\overline{S}$.

2.1.4. Tipping regime and regime shifts

Overall we defined the following regimes:

- a “durable satisfaction” regime defined by the set of points in the durable satisfaction set. The system in such a regime has the indicator of satisfaction durability (mean sojourn time MST or median exit time MFET) higher than durability threshold d, therefore the system is expected to stay in the satisfaction set S longer than dT, where T is the time horizon. This set is represented by dark blue color in the figures and is denoted by $D(S,d)$ in what follows;
- a “durable dissatisfaction” regime defined by the set of points already in the durable dissatisfaction set. Similarly, the system is in this regime when its indicator of dissatisfaction durability (MST of MFET) is higher than the durability threshold d, therefore the system is expected to stay in the dissatisfaction set longer than dT. This set is represented by dark red color in the figures and is denoted by $D(\overline{S},d)$ in what follows;
- a “resilient satisfaction” regime defined by the set of initial points located outside the durable satisfaction set $D(S,d)$ that have a high probability to come back in this durable satisfaction set in less than rT where r is the resilience threshold (and the probability measure is based on the median reaching time). This set is represented by light blue color in the figures and is denoted by $R(S,d,r)$ in what follows;
- a “resilient dissatisfaction” regime defined by the set of initial points located outside the durable dissatisfaction set $D(\overline{S},d)$ that have a high probability to come back in this durable dissatisfaction set in less than rT according to the resilience indicator r (The median reaching time indicator is lower than r). This set is represented by light red color in the figures and is be denoted by $R(\overline{S},d,r)$ in what follows;

Moreover, the union of the previously defined sets generally fails to cover the whole state space, leaving a set of states where the satisfaction and the dissatisfaction are neither durable nor resilient. We called this set the “tipping set” that we associate with the “tipping regime”. In this regime, the satisfaction or dissatisfaction are expected to be unstable. This set is colored in white in the figures and is denoted by $Tip(S,d,r)$ in what follows. As we will see in the following examples, this tipping set shows similarities with tipping points in the deterministic case. Moreover, the tipping set converges to tipping points when noise vanishes. Finally, regime shift takes place when the trajectory of the system shifts from one regime to another one.

3. Results

3.1. Illustration with a generic model of exploited ecosystem: emergence of tipping sets

3.1.1. A stylized model of renewable resource exploitation

We consider a generic model of exploitation of renewable resources, such as forest or fisheries for instance. Among the existing models, we

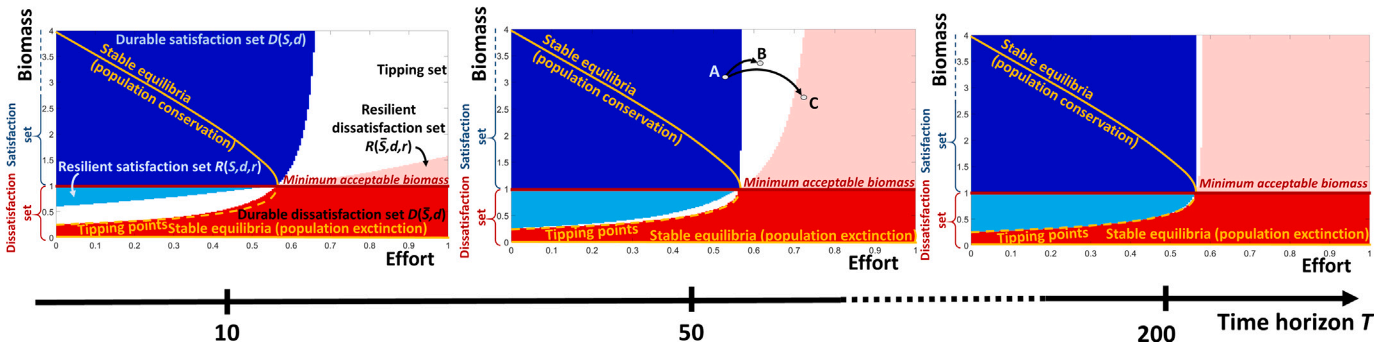

Fig. 4. Influence of time horizon T on deterministic regimes (with a durability threshold $d = 0.75$ and a resilience threshold $r = 0.1$). Regime defined by MFET and MST are similar because of the bistable properties of our system (see main text). Durable (dis-)satisfaction sets (dark blue and red sets) correspond to system states that may stay in a (dis-)satisfaction set during $d \times T$ (e.g. 37.5 time steps for $T = 50$, for $d = 0.75$). The resilient (dis-)satisfaction states sets (light blue and red sets) includes the states that may reach in a durable (dis-)satisfaction set during $r \times T$ (e.g. 5 time steps for $T = 50$ for $r = 0.1$). Considering a time horizon T leads to the appearance of tipping sets (white sets). A regime shift is defined as the shift between two regimes, for instance from point A to point B or from point A to point C (see for $T = 50$). Increasing the time horizon T (see for instance $T = 200$) leads to fit the classical definitions of regimes (represented in Fig. 1b).

choose the dynamics of a harvested resource, represented by the bioeconomic model proposed by Clark (1973) and inspired from Anderies et al. (2019) (see Annex B for more details):

$$B(t+1) = B(t) + [g(K - B(t))(B(t) - \alpha) - hB(t)]\Delta t + \sigma_B \varepsilon_{t+1} \Delta t \tag{1}$$

Biomass $B(t)$ of the ecosystem at time t depends on the level of harvest h (we set catchability to 1), the ecosystem regeneration capacity g, the maximum carrying capacity K of the ecosystem and a critical depensation α. ε are independent draws from a unit normal distribution, and σ_B is the standard deviation of the noise for the biomass $B(t)$. We reset biomass value to zero when they turn negative due to system dynamics, and then initiate a system restart through an exogenous contribution. The satisfaction is defined by the biomass being higher than a threshold ($x(t) > 1$).

3.1.2. Deterministic dynamics

In this section, we first consider the deterministic case (i.e; noises $\sigma_B = 0$). Hence, there is only one trajectory from a given initial state. Therefore, MFET and MRT are simply the first exit and reaching times and the MST is the proportion of time a trajectory of duration T spends in the (dis-)satisfaction set. In the case of bistable systems, like the bioeconomic model we consider, once the system has exited the (dis-)satisfaction set, it does not come back. Therefore MFET- and MST give the same results because the normalized first-exit time corresponds to the proportion time for a given time horizon T.

Here we consider a durability threshold $d = 0.75$ and a resilience threshold $r = 0.1$ and different values of time horizon T. For the sake of coherence, we choose a value of d (threshold on duration of the satisfaction) greater than the value of r (threshold on time to reach durable satisfaction): we privilege situations with high durability and high resilience.

As shown in Fig. 4, the durable satisfaction set $D(S, d)$ (dark blue set) is larger than the reference case (see Fig. 1b) for small time horizon (see for instance $T = 10$) because this set contains states that were previously in the set from which dissatisfaction states can be reached. In this case, these states are in the durable satisfaction set because of the time scale of interest: even if the system ultimately reaches dissatisfaction states, it reaches them beyond the time horizon.

Besides, a tipping set appears (white areas). In this set, either the (dis)satisfaction is not durable (white areas in the (dis)satisfaction set) or the medium time to reach the durable (dis)satisfaction set is too large. Then, when the time horizon increases (see $T = 200$ for instance), regimes of durable (dis)satisfaction converge to the classical definition of regimes, associated with the attraction basin of satisfactory or unsatisfactory states. If the time horizon is infinite, when starting from any state within the durable satisfaction set, the system remains in the satisfaction set indefinitely. Moreover, all states that reach the durable satisfaction set in finite time belong to the resilient satisfaction set, as illustrated by Fig. 1b.

Finally, a regime shift is defined as the shift from one regime to another. For instance (see figure for time horizon $T = 50$), we have a regime shift from point A to point C as well as another regime shift from point A to point B. Note that other definitions of regime shifts may be used as the shift from durable/resilient satisfaction regimes (blue sets) to durable/resilient dissatisfaction regimes (red sets).

3.1.3. Uncertain dynamics: ecosystem may benefit from uncertainties in collapsing case

We now consider perturbations in our system in terms of biomass by adding noises to dynamics described in Equation 1 ($\sigma_B = 0.3$). We consider the same durability and resilience thresholds d and r used in the previous section ($d = 0.75$, $r = 0.1$) in order to compare uncertain cases with deterministic cases. Figs. 5-a and -b represent the different regimes in the uncertain case.

Considering uncertainties make MFET- and MST-based results different: unlike the deterministic case, the system can exit the (dis-)satisfaction set and come back inside because of the uncertainties. For instance, a trajectory can exit very early the satisfaction set and come back durably in the satisfaction set: this case corresponds to a low value of the first-exit time and a high value of the sojourn time. Therefore, MST-based regimes are larger than MFET-based regimes, especially when the time horizon increases because the impact of early exits on MFET is greater than on MST. Compared to the deterministic case (Fig. 4), adding uncertainties to the dynamics also increases the tipping sets and changes the shapes of regimes. Finally, the effect of uncertainties on the regimes depends on the time horizon T: the higher the time horizon T, the higher the probability to exit the satisfaction set, the smaller the durable satisfaction set. Therefore, uncertainties are more likely to affect the shapes of durable/resilient (dis-)satisfaction sets when the time horizon is large (see $T = 200$ for instance) than when time horizon is small (see $T = 10$ for instance).

Because uncertainties shrink not only the durable satisfaction set but also the durable dissatisfaction set, uncertainties may actually help save the collapsing ecosystem by tipping it towards the tipping set and potentially towards a satisfaction state.

3.2. Illustration with Lotka–Volterra dynamics: regime shifts of cyclic ecosystem

3.2.1. The Lotka–Volterra model

We now consider a cyclic ecosystem with the Lotka–Volterra predator–prey model. Let us consider dynamics of prey density x and

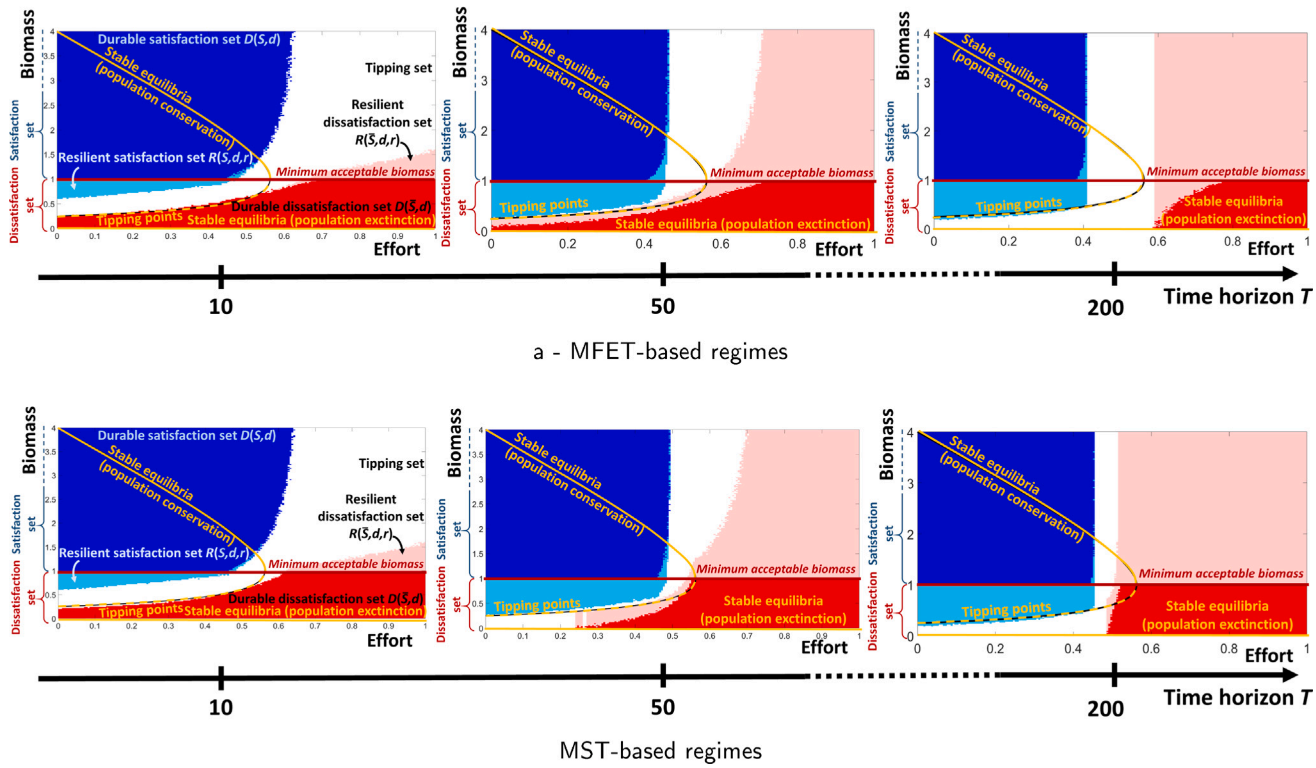

Fig. 5. **Influence of time horizon** T **on uncertain regimes (with a durability threshold** $d = 0.75$ **and a resilience threshold** $r = 0.1$). Uncertainties shrink not only the durable satisfaction set S (dark blue) but also the durable dissatisfaction set $\overline{S}$ (dark red), especially when the time horizon increases, whereas uncertainties increase the size of resilient (dis-)satisfaction sets (light blue and light red).

predator density y as follows (see Annex B for more details):

$$\begin{cases} x(t+1) = & x(t) + x(t)[a - by(t)]\Delta t + \sigma_x \epsilon_{t+1} \Delta t \\ y(t+1) = & y(t) + y(t)[cx(t) - e]\Delta t + \sigma_y \epsilon_{t+1} \Delta t \end{cases} \quad (2)$$

with $a = 1, b = 0.5, c = 0.5$ and $e = 0.5$. ϵ are independent draws from a unit normal distribution, and σ_x and σ_y are the standard deviations of the noise for the prey density $x(t)$ and for predator density $y(t)$ respectively. Note that we use the Runge–Kutta method for solving Eq. (2). The satisfaction is defined by the prey density being above a threshold ($x(t) > 1$). Deterministic and uncertain regimes are reported in Fig. 6 in the case of MFET and MST indicators.

The Lotka–Volterra model demonstrates a “center” equilibrium, around which the system trajectories form orbits. However, due to the cyclic nature of the dynamics, the system never actually reaches this equilibrium point. Instead, the orbits continue indefinitely, exhibiting periodic behavior. This equilibrium is represented by a yellow star in Fig. 6.

Unlike in the previous case, the MFET-based and MST-based regimes are now completely different because the dynamics is cyclic. On one hand, the cycles lead the system to exit the satisfaction set in a short time, yielding a low value of MFET (dark blue set in Fig. 6a). On the other hand, the time spent in dissatisfaction states is short, with the majority of cycles being in a satisfaction state. This yields a larger durable satisfaction set for the MST-based regime (dark blue set in Fig. 6b).

Furthermore, MFET-based regimes are more sensitive to the predator–prey densities because the predator–prey cycles depend on the initial conditions. For instance, the first-exit time from the satisfaction set is higher for high initial density of preys and low initial density of predators because the density changes are then faster.

Finally, the noise also modifies the shapes of the MFET or MST-based regimes differently. While it increases the size of the MFET-based tipping set (as shown in Fig. 6c), it has a relatively minor influence on MST-based regimes (as shown in Fig. 6d), since the noise does not significantly alter the mean sojourn time spent in the satisfaction set.

3.3. Illustration with Lake Bourget

3.3.1. Introduction

Lake Bourget is located in the French Alps, with a surface area of 44.5 km^2; mean depth equal to 80 m (with a maximum depth equal to 147 m). There are two main input flows into the lake (the Leysse and Sierroz rivers) which are responsible for 80% of the water inflow. Stakeholders have dealt with eutrophication issues since the 1960s (Vinçon-Leite et al., 1995). Several mitigation measures have been done in order to reduce the eutrophication of Lake Bourget (Jacquet et al., 2012) with an objective of phosphorous loading reduction to 30 tons.year^{-1} and a maximum phosphorous concentration in the lake of 10 µg L^{-1}. For instance, two mitigations are the building of an effluent treatment station in the 80 s and a polluted-water retention basin more recently to prevent the effluents from being rejected into the lake.

3.3.2. Eutrophication model with inter-annual rainfall variability

We use the model developed and calibrated in Brias et al. (2018). A full discussion about the model and the lake history can be found in Brias et al. (2018). Hereafter, we remind the main rationale. We use a linear relationship between total rainfall and total phosphorous load in catchments (Ockenden et al., 2016). The phosphorous loading into the lake at year t, is denoted $L(t)$, and can be written as a linear function of rainfall level $R(t)$ at year t:

$$L(t) = a_1 R(t) + a_0 \quad (3)$$

From the CISALB annual reports (Jacquet et al., 2012) that give annual phosphorous loading and annual rainfall from 2004 to 2016, a_0 and a_1 have been calibrated (see Brias et al. (2018) for more details). Rainfall is assumed to follow a normal distribution based on the measures from 2004 to 2016 (with a mean $\mu_R = 1154$ and a standard deviation $\sigma_R = 178.2$). We use a model of phosphorous dynamics developed by Carpenter et al. (1999) and also used in Rougé

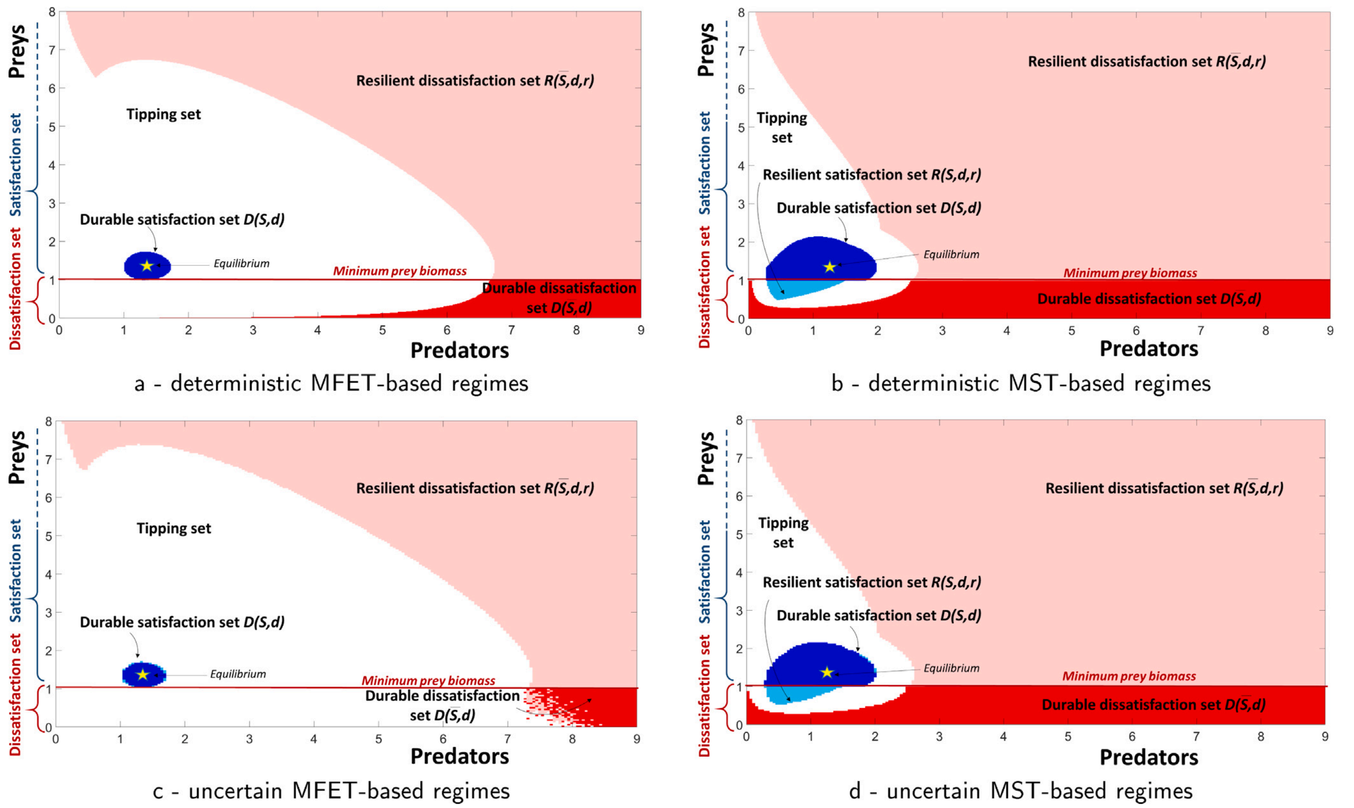

Fig. 6. MFET- and MST regimes, Lotka–Volterra case (**with a durability threshold** $d = 0.2$, **a resilience threshold** $r = 0.1$ **and a time horizon** $T = 20$). Uncertainties shrink regimes compared to the deterministic case (Figs. 6-a and -b). However, uncertainties much more impact the MFET-based regimes than the MST-regimes. Indeed, the mean sojourn time is less impacted because of the cyclic nature of the dynamics.

et al. (2013). This models represents the dynamics of the phosphorous mass in the lake as follows:

$$P(t+1) = P(t) + \left[L(t) - (s+h)P(t) + v\frac{P(t)^q}{P(t)^q + m^q} \right] \Delta t \tag{4}$$

where s is rate of sedimentation, h is rate of out-flooding $P(t)$, v is maximal mass of $P(t)$ recycled by sediments, q is the exponent of the recycling curve, depending on the type of the lake, and m is the $P(t)$ mass for which recycling reaches half of the maximal rate. There are interactions between these processes, so the differential equation has to account for the possibility of several sedimentation and recycling events during a single year. Using Eq. (3), Eq. (4) becomes:

$$P(t+1) = P(t) + \left[a_1 R(t) + a_0 - (s+h)P(t) + v\frac{P(t)^q}{P(t)^q + m^q} \right] \Delta t \tag{5}$$

It enables to express directly the phosphorous concentration according to rainfall inter-annual variability. The satisfaction is defined as the phosphorus concentration being below the mesotrophic threshold.

3.3.3. Results

Fig. 7 presents results in terms of dynamics and regimes of Lake Bourget. Firstly, we depict the "classical" regimes, which are defined by the basins of attraction of deterministic equilibriums, in Fig. 7a. Subsequently, based on this classical definition, we identify the years 2007, 2008 and 2013, 2014 as regime shifts due to the system crossing a tipping point.

Despite the apparent regime shift in 2013 according to the classical definition of regimes, it was already in a durable satisfaction regime for both MFET- and MST-based regimes (Figs. 7b and c). It is noteworthy that the shape of regimes, represented in Figs. 7b and c, is entirely different in the uncertain case compared to the classical definitions of regimes in Fig. 7a. This is due to the significance of rainfall dynamics resulting from rainfall uncertainties. It leads to two regime shifts: one in 2006 (from durable dissatisfaction set to the tipping set) and another one in 2013 (from the tipping set to durable satisfaction set).

Indeed, if we consider a deterministic constant rainfall around its annual mean (1154 mm per year), the phosphorus dynamics would converge to a satisfactory attractor of around 20 tons of phosphorus in the lake. However, this would not provide information regarding the trade-off between the dynamics of phosphorus and the influence of uncertainties coupled with this dynamics (such as the crossing of tipping points and tipping set in 2013 caused by uncertainties).

Finally, it is worth highlighting the difference between MFET- and MPT-based regimes for heavy rainfall (>1500 mm) just below the mesotrophic threshold (around 35–40 tons of phosphorous input). In this region, the lake has two key characteristics: (1) a high probability of crossing the mesotrophic threshold in the short term due to the deterministic component of the lake dynamics (i.e., basin of attraction of unsatisfactory attractor), leading to a low value of MFET; and (2) a high probability of returning below the mesotrophic threshold after leaving the satisfaction set because of the uncertain component of the dynamics (i.e., rainfall mean is in the basin of attraction of satisfactory attractor), leading to a high value of MST. Therefore, this region (i.e., heavy rainfall under the mesotrophic threshold) is included in the MST-based durable satisfaction set, but it is not in the MFET-based durable satisfaction set. This example illustrates how the choice between MFET and MST can be made in practice: if the political objective is to avoid any dissatisfaction, stakeholders should privilege MFET. Conversely, if temporary dissatisfaction is acceptable, MST is preferable.

4. Discussion and conclusion

Social-ecological systems are inherently complex, which makes many concepts, such as "regime shifts" or "safe operating space" (Rockström et al., 2009), hotly debated within the SES community. These

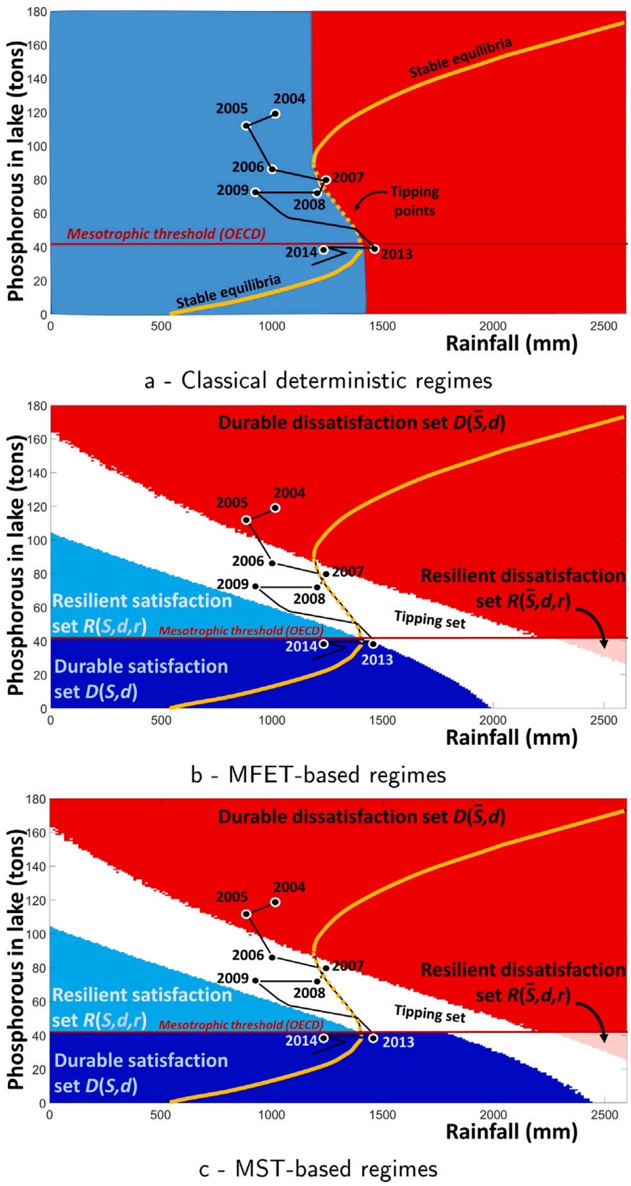

Fig. 7. Dynamics of lake Bourget and representation of regimes. We choose a time horizon T equal to 13 years – like experimental data – with an objective to stay 7 years satisfied (leading to a durability threshold $d = 0.54$) and to come back on 3 years if the lake is not oligotrophic (leading to a resilience threshold $r = 0.23$). According to classical definition of regime shifts, lake Bourget encountered regime shifts in 2007, 2008 and 2013, 2014 (see Fig. 7a) because it crossed a tipping point. According to MFET- and MST based approach, the lake was in an dissatisfaction regime until 2005. Then, in 2006, it enters the tipping set, leading to the first regime shift. A second regime shift occurs when the lake enters the durable satisfaction set in 2013.

debates often refer to concepts of regime shifts and tipping points, which are defined for deterministic systems, while the dynamics of SES is generally deeply uncertain. With the aim to clarify these debates, we propose definitions of regimes and regime shift that explicitly deal with uncertainty. It is important to stress that, at first, our approach requires defining a satisfaction set. Indeed, the regimes are directly related to the likelihood to keep or recover and keep this satisfaction. We use statistics on the exit time from the satisfaction set or the sojourn time in it as indicators of durability. These indicators clearly depend on the objectives defined by stakeholders. According to the case study, both indicators may or may not provide similar insights. For instance, in lake eutrophication, differences in sets defined by MST or MFET are small (see Figs. 5 and 7). However, using one indicator or another may yield significant differences as depicted with the Lotka–Volterra model (see Fig. 6); choosing MFET or MST significantly change the durable (dis-)satisfaction sets. The regimes are thus defined with respect to the satisfaction set and the chosen indicators. The satisfaction set generally incorporates social and economic norms but it can also be a hypothesis to be tested. Acknowledging the normative or hypothetical nature of this set is an essential clarification. Indeed, all the subsequent results depend on these initial choices hence they can be considered as normative as well. Moreover, the approach also relies on other normative choices: what is the time horizon? How long should the system remain, on average, in the satisfaction set for the satisfaction to be considered as durable? What is the maximum acceptable average reaching time for the satisfaction to be considered resilient? The relevance of these choices and of the ones related to the satisfaction set, is rooted in the general knowledge about the socio-ecological system (Anderies et al., 2019), including social and economic norms of its stakeholders. Changing norms (Nyborg et al., 2016) or information tools (Bourceret et al., 2023) can thus have a strong impact on the satisfaction set and consequently on the durable or resilient satisfaction regimes.

The practical use of our approach remains to be tested on the ground. We can expect its flexibility to be an asset. In particular, the objectives of the stakeholders may inform the choice of the indicator of durability. Indeed, if stakeholders focus on the life expectancy of the system, the exit time is certainly preferable while if they focus on the ratio of good functioning, the sojourn time is better. For some applications, these thresholds may depend on "norms": for instance keeping lake concentration above a given threshold during the operating period. In other cases, these thresholds may emerge from discussions between stakeholders. Finally, sensitivity analysis may be used in order to analyze the trade-off between durability and resilience. It may help stakeholders to better understand not only the interplay between durability and resilience but also the effect of uncertainty and of the time horizon. However, a serious limitation to the practical use of our approach is its requirement for a reliable mathematical expression of the system's dynamics, including its uncertainty, which is not often available. We may apply our approach to different types of stochastic models. However, the main issue will be the computation time. For instance, in the case of agent-based models, we will face problems associated with the heavy computation time because of the number of state variables required for describing the system. Therefore, computation of satisfaction sets seems difficult in practice for agent-based models. However, assessing the durability of (dis-)satisfaction from a given initial state of an agent-based model seems feasible by calculating the metrics for this initial state by running several replica. Assessing the resilience of this state will require to calculate the durability of all states reached from this initial state. Furthermore, identifying uncertainties in SES remains a key-challenge with limited solutions, not only from experimental data but also in scenarios and models in support of decision-making (Rounsevell et al., 2021). Therefore, new tools that can more effectively capture the uncertain dimensions of SES need to be developed from uncertainties assessment to their integration in decision-making.

CRediT authorship contribution statement

Jean-Denis Mathias: Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Conceptualization. **Guillaume Deffuant:** Writing – review & editing, Supervision, Methodology, Conceptualization. **Antoine Brias:** Writing – review & editing, Writing – original draft, Methodology, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Jean-Denis Mathias reports financial support was provided by French

National Research Agency. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

I have shared the link for the code in the paper.

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Annex A. Theoretical framework

A.1. Dynamical system

The system is described by state variable $x(t) \in \mathbf{X}$, varying over time, representing for instance the phosphorous concentration in a lake or the biomass of an exploited ecosystem. More generally, the state can involve several dimensions, like the concentration of phosphorus and the temperature of a lake. For simplicity, we assume that the time is discrete and the dynamics of the system writes:

$$x(t+1) = f(x(t), w(t+1)). \tag{6}$$

Function f determines the next state $x(t+1)$ of the system from current state $x(t)$. The uncertainty of the system is modeled by the value $w(t+1)$ which is independently drawn from a random variable $W(t+1)$ according to a given probability distribution. This random variable can represent for instance the variations of phosphorus input in a lake, because of rain intensity variations.

A.2. Satisfaction and dissatisfaction sets

The satisfaction set $S \subset \mathbf{X}$ is often defined by thresholds on the state variables. For instance in the case of a single state variable, the satisfaction set is often defined as follows:

$$S = \left\{ x \in \mathbf{X}, C^{\min} \leq x \leq C^{\max} \right\} \tag{7}$$

Where $C^{\min}$ and $C^{\max}$ are the minimum and maximum acceptable values of the state variable. In the case of a resource exploitation, $C^{\min}$ would be the minimum acceptable level of the resource from an ecological point of view and $C^{\max}$ the maximum acceptable level of the resource from economic stakeholders.

However, of course in some cases the satisfaction set is defined with more complicated mathematical functions than constants. Here, we simply assume that a mathematical function is available for computing if any state x is in the satisfaction set or not.

The dissatisfaction set $\overline{S}$ is the set of states that do not belong to the satisfaction set S.

Now, we mathematically define the indicators assessing how long the system is likely to remain in the satisfaction set. We assume that a time horizon T, which expresses the long term in the context of the system, is identified.

A.3. Durable satisfaction and dissatisfaction defined with median exit time

Assume that the system is initially i the satisfaction set at state $x(0) = x_0 \in S$ and that the components of $\omega_T = (w(1), \dots, w(T))$ are realizations of the random variables $(W(1), \dots, W(T))$. From the values x_0 and ω_T we compute a trajectory $(x(0), \dots, x(T)) = x(x_0, \omega_T)$ of the system by iterating equation (6).

The first-exit time $\tau_S(x_0, \omega_T)$ of trajectory $x(x_0, \omega_T)$ is the maximum time $t \leq T$ for which the trajectory $x(x_0, \omega_t)$ of duration t remains in the satisfaction set S:

$$\tau_S(x_0, \omega_T) = \max\{t \in \{0, 1, \dots, T\} \mid x(x_0, \omega_t) \subset S\}. \tag{8}$$

Note that, if the trajectory remains in S for the whole time horizon T, the exit time is T, which approximates infinity in this context.

The median first-exit time from $x_0 \in S$ within horizon T, $\hat{\tau}_S(x_0)$, is the median value of the first-exit times $\tau_S(x_0, \omega_T)$ over all trajectories $x(x_0, \omega_T)$ generated by all the realizations ω_T. Therefore$\tau_S(x_0, \omega_T)$ is defined by:

$$\mathbb{P}\left(\tau_S\left(x_0, \omega_T\right) < \hat{\tau}_S(x_0)\right) = 0.5. \tag{9}$$

In practice, this median value is approximated by running a large number of random trajectories of T time steps from x_0.

Then, defining the set of durable satisfaction requires to choose a minimum proportion $0 < d \leq 1$ of the time horizon. Denoted by $D_\tau(S, d)$, the set of durable satisfaction includes all states $x_0 \in S$ for which the median exit-time from S within horizon T is greater than dT:

$$D_\tau(S, d) = \left\{ x_0 \in S, \hat{\tau}_S(x_0) \geq dT \right\}. \tag{10}$$

Note that the lower the threshold d, the less demanding is the condition on the trajectories, hence the larger is the set of durable satisfaction.

Similarly, we define the set of durable dissatisfaction by replacing the satisfaction set S by the dissatisfaction set $\overline{S}$

$$D_\tau(\overline{S}, d) = \left\{ x_0 \in \overline{S}, \hat{\tau}_{\overline{S}}(x_0) \geq dT \right\}. \tag{11}$$

In the deterministic case, there is only one possible trajectory and one value of the first-exit time for a given initial state. Therefore, the median first exit time is then this first-exit time of the single trajectory.

A.4. Durable satisfaction and dissatisfaction defined with mean sojourn time

We now define $\eta_S(x_0, \omega_T)$, the sojourn time of the trajectory $x(x_0, \omega_T)$ in S within time horizon T, as the number of time steps t for which the trajectory $x(x_0, \omega_T)$ is in $\mathbf{S}$:

$$\eta_S(x_0, \omega_T) = \#\{t \in \{1, \dots, T\} \mid x(t) \in S\}, \tag{12}$$

where $\#A$ denotes the number of elements in set A. In our case, it remains to count the number of time steps during which the system stays desirable. From this sojourn time, we define the mean sojourn time from point $x_0 \in S$, $\hat{\eta}_S(x_0)$, as the mean of sojourn times over all the values of ω_T :

$$\hat{\eta}_S(x_0) = \sum_{\omega_T} \eta_S(x_0, \omega_T) \mathbb{P}(\omega_T). \tag{13}$$

In this definition, for sake of simplicity we assume that the set of possible noise vectors ω_T is discrete. Generalizing to a continuous set of noise vectors would change the sum into an integral. In practice, this indicator is approximated on a large number of random trajectories: we run 1000 simulations and we calculate the mean value of the sojourn time of these 1000 simulations.

Then, the set of durable satisfaction, $D_\eta(S, d)$, is defined like previously using a minimum proportion $0 < d \leq 1$ of the time horizon T, as the set of the points in S having a mean sojourn time in S within horizon T higher than dT:

$$D_\eta(S, d) = \left\{ x_0 \in S, \hat{\eta}_S(x_0) \geq dT \right\}. \tag{14}$$

Of course, the set of durable dissatisfaction is defined like previously. Only the indicator changes, mean sojourn time replacing median exit-time.

$$D_\eta(\overline{S}, d) = \left\{ x_0 \in \overline{S}, \hat{\eta}_{\overline{S}}(x_0) \geq dT \right\}. \tag{15}$$

A.5. Resilient satisfaction and dissatisfaction

We now aim at identifying states that are not in the durable satisfaction set, but are likely to reach this set in a relatively short time (with respect to horizon T). Following several previous works (Martin, 2004; Deffuant and Gilbert, 2011), we say that these points belong to the resilient satisfaction set.

Let the set of durable satisfaction be $D(S, d)$. We assume that the indicator used to determine this set can be exit or sojourn time and we omit the subscript designating this indicator. The set of resilient satisfaction $R(S, d, r)$ is defined as the points reaching $D(S, d)$ in a maximum proportion $0 < r < 1$ of the horizon, as follows:

$$R(S, d, r) = \left\{ x_0 \in \overline{D(S, d)}, \hat{\tau}_{\overline{D(S,d)}}(x_0) \leq rT \right\}, \tag{16}$$

where $\overline{D(S, d)}$ is the complement of the set of durable satisfaction $D(S, d)$. Indeed, the exit time from $\overline{D(S, d)}$ is the reaching time into $D(S, d)$. Note that, with deterministic dynamics, if $r = 1$ and the horizon T tends to infinity, this definition coincides with the attraction basin of $D(S, d)$.

Similarly, we define the resilient dissatisfaction set, $R(\overline{S}, d, r)$ as:

$$R(\overline{S}, d, r) = \left\{ x_0 \in \overline{D(\overline{S}, d)}, \hat{\tau}_{\overline{D(\overline{S},d)}}(x_0) \leq rT \right\}. \tag{17}$$

A.6. Tipping set

Finally, a part of the state space in which the satisfaction or the dissatisfaction are neither durable nor resilient. This set is called the tipping set:

$$Tip(d, r) = \overline{D(S, d) \cup D(\overline{S}, d) \cup R(S, d, r) \cup R(\overline{S}, d, r)}. \tag{18}$$

Note that, for deterministic dynamics, when time horizon T tends to infinity, the tipping set tends to the set of unstable equilibriums, which are the classical tipping points.

Annex B. Simulations

B.1. Introduction

For stochastic simulations, we run 1000 simulations for calculating MFET, MST and MRT indicators. We employ a discretization of 200 steps per dimension for the state spaces in the following models. The time step Δt is equal to 1. The codes are available here: https://github.com/jdmathias/tippingset

B.2. A harvested population model

The dynamics of a harvested population, in our model, are represented by the classic model proposed by Clark (1973) and is inspired from Anderies et al. (2019):

$$B(t+1) = B(t) + [g(K - B(t))(B(t) - \alpha) - hB(t) + \sigma_B \epsilon_{t+1}]\Delta t \tag{19}$$

Where the variable $B(t)$ represents the biomass of the ecosystem at time t and depends on the level of exploitation h. Overexploitation may lead to the collapse of the population and is characterized by the level of biomass. Therefore, $C_{\max}$ represents the minimum acceptable biomass level and is set at 1 according the simulations. The ecosystem parameters have been fixed according to Anderies et al. (2019): r that represent the ecosystem regeneration capacity (set at $g = 0.25$); K that represents the maximum carrying capacity (set at $K = 4$) of the ecosystem and α that represents the sigmoid predation consumption coefficient (set at $\alpha = 0.25$) of the ecosystem. ϵ are independent draws from a unit normal distribution, and σ_B is the standard deviations of the noise for the biomass $B(t)$. We have chosen the medium values of σ_B : high values would lead to recurrent collapse of the ecosystem whereas low values would lead to obvious results (close to the classical deterministic ones). We set $\sigma_B = 0.3$. Note that when biomass $x(t) < 0$, we set it to 0 but we let the possibility to increase again (exogenous regeneration). Finally, calculations have been done with the durability threshold $d = 0.75$ and the resilience threshold $r = 0.1$.

B.3. The Lotka–Volterra model

We now consider a cyclic ecosystem with the Lotka–Volterra predator–prey model. Let us consider dynamics of preys x and predators y as follows :

$$\begin{cases} x(t+1) = & x(t) + x(t)[a - by(t)]\Delta t + \sigma_x \epsilon_{t+1}\Delta t \\ y(t+1) = & y(t) + y(t)[cx(t) - e]\Delta t + \sigma_y \epsilon_{t+1}\Delta t \end{cases} \tag{20}$$

with $a = 1, b = 0.5, c = 0.5$ and $e = 0.5$. ϵ are independent draws from a unit normal distribution, and σ_x and σ_y are the standard deviations of the noise for the preys $x(t)$ and for predators $y(t)$ respectively. The sustainable constraint boils down to maintain the prey density above a minimum acceptable biomass ($x(t) > 1$) during a time horizon T. For the algorithm, we use a Runge–Kutta method (with a step h equal to 0.1 time step) as follows :

$$\begin{cases} k1x = & x(t) * (a - b * y(t)) \\ k1y = & y(t) * (c * x(t) - e) \\ k2x = & (x(t) + h/2 * k1x) * (a - b * (y(t) + h/2 * k1y)) \\ k2y = & (y(t) + h/2 * k1y) * (c * (x(t) + h/2 * k1x) - e) \\ k3x = & (x(t) + h/2 * k2x) * (a - b * (y(t) + h/2 * k2y)) \\ k3y = & (y(t) + h/2 * k2y) * (c * (x(t) + h/2 * k2x) - e) \\ k4x = & (x(t) + h * k3x) * (a - b * (y(t) + h * k3y)) \\ k4y = & (y(t) + h * k3y) * (c * (x(t) + h * k3x) - e) \\ x(t+h) = & x(t) + h/6 * (k1x + 2 * k2x + 2 * k3x + k4x) + \sigma_x^h \epsilon_{t+h} \\ y(t+h) = & y(t) + h/6 * (k1y + 2 * k2y + 2 * k3y + k4y) + \sigma_y^h \epsilon_{t+h} \end{cases} \tag{21}$$

We set $\sigma_x^h = \sigma_y^h = 0.005$ for uncertain simulations.

B.4. Lake eutrophication

A full discussion about the model and the lake history can be found in Brias et al. (2018). The phosphorous loading into the lake at year t, is denoted $L(t)$, and can be written as a linear function of rainfall level $R(t)$ at year t:

$$L(t) = a_1 R(t) + a_0 \tag{22}$$

Rainfall follows a normal distribution (Jacquet et al., 2012) based on the measures from 2004 to 2016 (with a mean $\mu_R = 1154$ and a standard deviation $\sigma_R = 178.2$). We use a model of phosphorous dynamics (Carpenter et al., 1999; Rougé et al., 2013):

$$P(t+1) = P(t) + \left[L(t) - (s+h)P(t) + v\frac{P(t)^q}{P(t)^q + m^q} \right] \Delta t \tag{23}$$

where s is rate of sedimentation ($s = 2.1476 yr^{-1}$), h is rate of outflooding $P(t)$ ($h = 0.12 yr^{-1}$), v is maximal mass of $P(t)$ recycled by sediments ($v = 367.04 tons.yr^{-1}$), q is the exponent of the recycling curve ($q = 2.2222$), depending on the type of the lake, and m is the $P(t)$ mass for which recycling reaches half of the maximal rate ($m = 96.85$). Using Eq. (22), Eq. (23) becomes:

$$P(t+1) = P(t) + \left[a_1 R(t) + a_0 - (s+h)P(t) + v\frac{P(t)^q}{P(t)^q + m^q} \right] \Delta t \tag{24}$$

References

Anderies, J.M., Mathias, J.-D., Janssen, M.A., 2019. Knowledge infrastructure and safe operating spaces in social–ecological systems. Proc. Natl. Acad. Sci. 116 (12), 5277–5284.

Arani, B.M., Carpenter, S.R., Lahti, L., Van Nes, E.H., Scheffer, M., 2021. Exit time as a measure of ecological resilience. Science 372 (6547), eaay4895.

Aubin, J., 1991. Viability Theory. In: Systems & Control, Birkhäuser.

Bourceret, A., Amblard, L., Mathias, J.D., 2023. How do farmers' environmental preferences influence the efficiency of information instruments for water quality management? Evidence from a social-ecological agent-based model. Ecol. Model. 478, 110300.

Brias, A., Mathias, J.-D., Deffuant, G., 2018. Inter-annual rainfall variability may foster lake regime shifts: An example from Lake Bourget in France. Ecol. Model. 389, 11–18.

Carpenter, S.R., Ludwig, D., Brock, W.A., 1999. Management of eutrophication for lakes subject to potentially irreversible change. Ecol. Appl. 9 (3), 751–771.

Clark, C., 1973. The economics of overexploitation. Science 181 (4100), 630–634.

Dakos, V., Carpenter, S., Brock, W., Ellison, A., Guttal, V., Ives, A., Kéfi, S., Livina, V., Seekell, D., van Nes, E., Scheffer, M., 2012. Methods for detecting early warnings of critical transitions in time series illustrated using simulated ecological data. PLoS ONE 7 (7).

Dakos, V., Scheffer, M., Van Nes, E., Brovkin, V., Petoukhov, V., Held, H., 2008. Slowing down as an early warning signal for abrupt climate change. Proc. Natl. Acad. Sci. USA 105 (38), 14308–14312.

Deffuant, G., Gilbert, N., 2011. Viability and Resilience of Complex Systems: Concepts, Methods and Case Studies from Ecology and Society. Springer,

Delara, M., Doyen, L., 2008. Sustainable Management of Natural Resources. Mathematical Models and Methods. Springer.

Hirota, M., Holmgren, M., Nes, E.H.V., Scheffer, M., 2011. Global resilience of tropical forest and savanna to critical transitions. Science 334 (6053), 232–235.

Holling, C.S., 1973. Resilience and stability of ecological systems. Annu. Rev. Ecol. Syst. 4, 1–23.

Holling, C., 1978. Adaptive Environmental Assessment and Management. John Wiley and Sons, London.

Hughes, T.P., Barnes, M.L., Bellwood, D.R., Cinner, J.E., Cumming, G.S., Jackson, J.B., Kleypas, J., Van De Leemput, I.A., Lough, J.M., Morrison, T.H., et al., 2017. Coral reefs in the Anthropocene. Nature 546 (7656), 82–90.

Jacquet, S., Barbet, D., Cachera, S., Caudron, A., Colon, M., Girel, C., Guillard, J., Hébert, A., Kerrien, F., Laine, L., et al., 2012. Suivi environnemental des eaux du lac du bourget pour l'année 2011. Article in French. INRA-CISALB.

Lamothe, K.A., Somers, K.M., Jackson, D.A., 2019. Linking the ball-and-cup analogy and ordination trajectories to describe ecosystem stability, resistance, and resilience. Ecosphere 10 (3), e02629.

Lenton, T.M., 2013. Environmental tipping points. Annu. Rev. Environ. Res. 38 (1), 1–29.

Lenton, T., Held, H., Kriegler, E., Hall, J., Lucht, W., Rahmstorf, S., Schellnhuber, H., 2008. Tipping elements in the Earth's climate system. Proc. Natl. Acad. Sci. USA 105 (6), 1786–1793.

Lindner, M., Hellmann, F., 2019. Stochastic basins of attraction and generalized committor functions. Phys. Rev. E 100 (2), 022124.

Martin, S., 2004. The cost of restoration as a way of defining resilience: A viability approach applied to a model of lake eutrophication. Ecol. Soc. 9 (2).

Mathias, J.-D., Anderies, J.M., Baggio, J., Hodbod, J., Huet, S., Janssen, M.A., Milkoreit, M., Schoon, M., 2020. Exploring non-linear transition pathways in social-ecological systems. Sci. Rep. 10 (1), 4136.

Mathias, J.-D., Clark, S.S., Onat, N., Seager, T.P., 2018. An integrated dynamical modeling perspective for infrastructure resilience. Infrastructures 3 (2), 11.

Milkoreit, M., Hodbod, J., Baggio, J., Benessaiah, K., Calderón-Contreras, R., Donges, J.F., Mathias, J.-D., Rocha, J.C., Schoon, M., Werners, S.E., 2018. Defining tipping points for social-ecological systems scholarship—an interdisciplinary literature review. Environ. Res. Lett. 13 (3), 033005.

Nuno, A., Bunnefeld, N., Milner-Gulland, E., 2014. Managing social–ecological systems under uncertainty: implementation in the real world. Ecol. Soc. 19 (2).

Nyborg, K., Anderies, J.M., Dannenberg, A., Lindahl, T., Schill, C., Schlüter, M., Adger, W.N., Arrow, K.J., Barrett, S., Carpenter, S., Chapin, F.S., Crépin, A.-S., Daily, G., Ehrlich, P., Folke, C., Jager, W., Kautsky, N., Levin, S.A., Madsen, O.J., Polasky, S., Scheffer, M., Walker, B., Weber, E.U., Wilen, J., Xepapadeas, A., de Zeeuw, A., 2016. Social norms as solutions. Science 354 (6308), 42–43.

Ockenden, M., Deasy, C., Benskin, C., Beven, K., Burke, S., Collins, A., Evans, R., Falloon, P., Forber, K., Hiscock, K., Hollaway, M., Kahana, R., Macleod, C., Reaney, S., Snell, M., Villamizar, M., Wearing, C., Withers, P., Zhou, J., Haygarth, P., 2016. Changing climate and nutrient transfers: Evidence from high temporal resolution concentration-flow dynamics in headwater catchments. Sci. Total Environ. 548–549, 325–339.

Pritchard, H., Arthern, R., Vaughan, D., Edwards, L., 2009. Extensive dynamic thinning on the margins of the Greenland and Antarctic ice sheets. Nature 461 (7266), 971–975.

Rockström, J., Steffen, W., Noone, K., Persson, Å., Chapin, F.S., Lambin, E.F., Lenton, T.M., Scheffer, M., Folke, C., Schellnhuber, H.J., et al., 2009. A safe operating space for humanity. Nature 461 (7263), 472–475.

Rougé, C., Mathias, J.D., Deffuant, G., 2013. Extending the viability theory framework of resilience to uncertain dynamics, and application to lake eutrophication. Ecol. Indic. 29, 420–433.

Rounsevell, M.D., Arneth, A., Brown, C., Cheung, W.W., Gimenez, O., Holman, I., Leadley, P., Luján, C., Mahevas, S., Maréchaux, I., Pélissier, R., Verburg, P.H., Vieilledent, G., Wintle, B.A., Shin, Y.J., 2021. Identifying uncertainties in scenarios and models of socio-ecological systems in support of decision-making. One Earth 4 (7), 967–985.

Scheffer, M., Bascompte, J., Brock, W.A., Brovkin, V., Carpenter, S.R., Dakos, V., Held, H., van Nes, E.H., Rietkerk, M., Sugihara, G., 2009. Early-warning signals for critical transitions. Nature 461 (7260), 53–59.

Scheffer, M., Carpenter, S.R., 2003. Catastrophic regime shifts in ecosystems: linking theory to observation. Trends Ecol. Evol. 18 (12), 648–656.

Scheffer, M., Carpenter, S.R., Foley, S., Folke, C., Walker, B., 2001. Catastrophic regime shifts in ecosystems. Nature 413, 591–596.

Serdukova, L., Zheng, Y., Duan, J., Kurths, J., 2016. Stochastic basins of attraction for metastable states. Chaos 26 (7), 073117.

Vellinga, M., Wood, R., 2002. Global climatic impacts of a collapse of the atlantic thermohaline circulation. Clim. Change 54 (3), 251–267.

Vinçon-Leite, B., Tassin, B., Jaquet, J.M., 1995. Contribution of mathematical modeling to lake ecosystem understanding: Lake Bourget (savoy, France). Hydrobiologia 300 (1), 433–442.

Willems, J.L., 1970. Stability Theory of Dynamical Systems, vol. 1, Wiley-Interscience.

Zhang, H., Xu, W., Lei, Y., Qiao, Y., 2019. Early warning and basin stability in a stochastic vegetation-water dynamical system. Commun. Nonlinear Sci. Numer. Simul. 77, 258–270.