



HAL
open science

Bioeconomic Viability and Resilience of Savanna

Ivric Valaire Yatat–djeumen, Luc Doyen, Jean Jules Tewa, Bapan Ghosh

► **To cite this version:**

Ivric Valaire Yatat–djeumen, Luc Doyen, Jean Jules Tewa, Bapan Ghosh. Bioeconomic Viability and Resilience of Savanna. *Environmental Modeling & Assessment*, 2025, 30, pp.71-85. <10.1007/s10666-024-09989-3>. <hal-04691675>

HAL Id: hal-04691675

<https://hal.inrae.fr/hal-04691675v1>

Submitted on 9 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Bioeconomic Viability and Resilience of Savanna

I.V. Yatat-Djeumen^{1,2,3*}, L. Doyen⁴, J.J. Tewa^{1,3}, B. Ghosh⁵

¹University of Yaoundé I, National Advanced School of Engineering of Yaoundé,
Department of Mathematics and Physics, Yaoundé, Cameroon

²AMAP, Univ Montpellier, CIRAD, CNRS, INRA, IRD, Montpellier, France

³UMI 209 IRD/UPMC UMMISCO, Bondy, France

⁴CNRS, Center for Environmental Economics in Montpellier (CEE-M), France

⁵ Differential Equations, Modeling, and Simulation Group,
Department of Mathematics, Indian Institute of Technology Indore,
Khandwa Road, Simrol 453552, India

September 9, 2024

Abstract

Our paper investigates the bioeconomic sustainability and resilience of savanna social-ecological systems (SES). A stylized dynamics of an exploited grass-tree systems is thus considered accounting both for the competition between trees and grass along with logging and grass harvesting activities. Regarding sustainability, we rely on bioeconomic viability goals including consumption security for grass, profitability of logging and coexistence of tree-grass states. A first analytical result relates to the elicitation of sufficient sustainability conditions through the non-emptiness of the so-called viability kernel. Such sufficient conditions rely on coupled MSY (maximum sustainable yield) - MEY (maximum economic yield) reference states-controls. A larger viable set including these MSY-MEY equilibria is also identified. The resilience of such viability states-controls for savanna SES facing shocks such as fire is then put forward from both recovery through stability analysis and resistance viewpoints. These analytical results provide managers with different upper and lower bioeconomic thresholds on the savanna SESs to promote their sustainability and resilience. Simulations inspired from savanna systems in Cameroon exemplify the analytical findings.

Keywords: Savanna, Tree, Grass, Bioeconomics, Cutting, Harvesting, Dynamic model, Control, Viability, Stability, Recovery, Resistance, Cameroon.

1 Introduction

Savannas are roughly defined as ecosystems or biomes where tree and grass coexist [Scholes and Archer, 1997, Scholes, 2003, Sankaran et al., 2005] under warm climate. The largest areas of savanna are found in Africa, South America, Australia, India, the Myanmar (Burma)–Thailand region in Asia, and Madagascar. Precisely, it is estimated that they occupy about 20% of the Earth land surface, cover more than 50% of the African continent, and are observed in a large range of Mean Annual Precipitation (MAP). In Africa, they particularly occur between 100 mm and 1500 mm (and sometimes more) of total mean annual precipitation [Lehmann et al., 2011, Baudena and Rietkerk, 2013, Yatat-Djeumen et al., 2021], that is along a precipitation gradient leading from dense tropical forest to desert. They play key roles in terms of biodiversity and provisioning ecosystem services such as timber, food and drugs (Kaur, 2006). In particular, savannas provide habitats for a wide array of animals [Sankaran et al., 2005, Scholes and Walker, 1993], some of which foster the vegetation through grazing, browsing, pollinating, nutrient cycling, or seed dispersal [Scholes and Walker, 1993, Staver et al., 2021, Huntley, 2023]. Many areas of savanna are managed today to maintain large grazing mammals, such as the native fauna of Africa or the cattle used for commercial production in large areas of Australia and South and Central America. Savannas are under pressure worldwide because of global changes including climate change and demographic pressure. Savannas are affected by the overuse of woody plants for fuel.

*Corresponding author: valaire.yatat@univ-yaounde1.cm

Together with grazing and cultivation, this depletes both the grassy and woody components of the vegetative cover. Often a subsequent acceleration of soil erosion occurs. Such processes are associated, in densely settled savanna areas such as Africa north of the Equator, with the type of land degradation called desertification. Fire constitutes another important ingredient and pressure of savanna ecosystems in all regions. Fires are started naturally by lightning strikes, but in most regions humans are now the greatest cause of savanna burning. In line with these concerns, our paper examines the bioeconomic sustainability and resilience of savanna considered as a social-ecological system (SES)[Ostrom, 2009].

In the face of global changes, pressures, and uncertainties, the popularity of the concept of resilience is rising. For instance, it is included in several Sustainable Development Goals (SDGs)[FAO, 2017]: 1 (No Poverty); 2 (Zero Hunger); 13 (Climate Action); and 14 (Life Below Water). This popularity of resilience contrasts, however, with a lack of clarity over the concept across the scientific disciplines and how to operationalize it in terms of decision making and public policy [Derissen et al., 2011, Downes et al., 2013, Quinlan et al., 2016, Grafton and Little, 2017, Béné and Doyen, 2018]. Recently, Grafton et al. [2019] and Cuilleret et al. [2022] have made significant progress in the definitions, objectives and quantification of resilience-based management in particular for environmental issues and social-ecological systems (SES)[Ostrom, 2009]. In particular Grafton et al. [2019] put forward three ingredients and metrics (the 3Rs) of resilience, namely recovery, resistance and robustness (or reliability). Here we will focus on recovery and resistance to examine the bio-economic resilience of savanna SES. Recovery refers to the time to withstand shocks and uncertainties in the sense of bouncing back to viable situations [De Lara and Doyen, 2008, Oubraham and Zaccour, 2018, Cuilleret et al., 2022, Doyen et al., 2019], while resistance is the magnitude of shocks (say for state) that can be withstood [Harrison, 1979, Béné and Doyen, 2018]. The underlying viability goals we here consider for savanna systems relate to different bio-economic thresholds in line with the SDGs [FAO, 2017] and the triple bottom line of sustainable development since we deal with both security of provisioning ecosystem services (typically food or drugs), economic viability and biodiversity conservation [Baumgärtner and Quaas, 2009, Hardy et al., 2013, Cissé et al., 2013, Schuhbauer and Sumaila, 2016]. Recovery is strongly tied to stability issues [Holling, 1973, Pimm, 1984] and minimal time problems [Martin, 2004, Martinet et al., 2007].

To address the sustainability and resilience of savanna SESs, we here develop and analyse a stylized bioeconomic dynamic model of an exploited grass-tree systems in the vein of Van Langevelde et al. [2003], Yatat-Djeumen et al. [2018, 2021]. The model accounts both for the competition between trees and grass along with logging and grass harvesting activities. To deal with sustainability, bioeconomic viability goals and thresholds are considered including consumption security for grass, profitability of logging and coexistence of tree-grass states. A first analytical result consists in the identification of sufficient sustainability conditions through the non-emptiness of the so-called viability kernel. Such sufficient conditions are derived from coupled MSY (maximum sustainable yield) - MEY (maximum economic yield) reference points [Schaefer, 1954, Clark, 1973, Grafton et al., 2012, Tromeur and Doyen, 2019] articulating bioeconomic parameters of both grass and trees. The resilience of the such viability states-controls for savanna SES is then examined by first adopting a stability viewpoint. Doing so, we question the role of shocks on savanna sustainability, such as those induced by fires. We complete the resilience study by providing results in terms of resistance through the distance to the non viable zone as in Karacaoglu and Krawczyk [2021]. Finally, numerical simulations inspired from savanna systems in Cameroon exemplify the analytical findings. The calibration of the model in Cameroon stems from previous works of Yatat-Djeumen et al. [2018, 2021].

The rest of the paper is organized as follows. In Section 2, we describe the bioeconomic model of savanna with tree-grass interaction dynamics together with ecological and economic viability constraints. In Section 3 entitled ‘Results’, we provide both the generic analytical findings articulating viability - resilience viewpoints. Section 4 details the numerical simulations related to the case study. A last Section discusses the results and concludes.

2 Bio-economic model

2.1 Stylized dynamics of an exploited savanna

Our stylized model inspired by Yatat-Djeumen et al. [2018, 2021] features two coupled ordinary differential equations representing the joint dynamics of tree and grass vegetation states in savanna systems. These vegetation states potentially relate to biomasses or abundances or densities. Each equation accounts for the logistic growth of both vegetations along with their natural competition and anthropogenic activities arising from both tree cutting and grass harvesting. Thus our model reads as follows:

$$\begin{cases} \frac{dG(t)}{dt} = r_G G(t) \left(1 - \frac{G(t)}{K_G}\right) - \eta_{TG} T(t) G(t) - H_G(G(t), e_G(t)), \\ \frac{dT(t)}{dt} = r_T T(t) \left(1 - \frac{T(t)}{K_T}\right) - H_T(T(t), e_T(t)), \\ G(0) = G_0, T(0) = T_0, \end{cases} \quad (1)$$

where the variables $G(t)$ and $T(t)$ stand for grass and tree states¹ at time t , respectively. Variables $e_G(t)$ and $e_T(t)$ represent grass and tree harvest rates (efforts) at time t which can potentially play the role of controls of the system. Parameters $r_G(\geq 0)$ and $r_T(\geq 0)$ express, respectively, the intrinsic rates of grass and tree growth, while $K_G(> 0)$ and $K_T(> 0)$ correspond to, respectively, the carrying capacity of grass and tree states. Competition between grass and tree is captured by the parameters $\eta_{TG}(> 0)$ ². Anthropogenic pressure is represented by the harvest functions $H_G(G, e_G)$ and $H_T(T, e_T)$ which denote the state losses resulting from harvesting actions captured by efforts $e_G(t)$ and $e_T(t)$ (labor or capital intensities) on grass and woody vegetations, respectively.

We specify below the functional forms of harvests H_G of grass and cutting H_T of trees relating to the anthropogenic activities and pressures introduced in dynamics (1). For the sake of simplicity, we assume that the harvests of both vegetation types occur through a Schaefer production function [Schaefer, 1954, Clark, 1973] as follows:

$$H_G(G(t), e_G(t)) = q_G e_G(t) G(t), \quad H_T(T(t), e_T(t)) = q_T e_T(t) T(t), \quad (2)$$

where q_G and q_T correspond to impact rates (the so-called catchability in fisheries) on grass and tree states respectively. Hereafter, $e_i(t)$, $i \in \{G, T\}$, play the role of controls of the dynamic SES described in the system (1).

2.2 Sustainability objectives

To address the sustainability purpose, we rely on both the viable control approach and bio-economic criteria as in [Béné et al., 2001, De Lara and Doyen, 2008, Doyen et al., 2019, Schuhbauer and Sumaila, 2016, Oubraham and Zaccour, 2018]. Such viable control framework is based on thresholds and constraints to fulfill across time. By bio-economic criteria is meant the account of both ecological, economic, and social constraints.

To characterize the economic sustainability and viability, we first impose the economic profit (revenue minus variable cost) of the forestry activity to be guaranteed in the following sense:

$$\pi_T(t) = p_T H_T(T(t), e_T(t)) - c_T e_T(t) \geq \pi_T^{\text{lim}}, \quad (3)$$

where $p_T (> 0)$ is the market unit price of timber while $c_T(\geq 0)$ represents the unit cost of variable forestry effort $e_T(t)$ and $\pi_T^{\text{lim}}(> 0)$ a desired profit, potentially relating to fixed costs of the forestry. Given the Schaefer harvest formulation of $H_T(T(t), e_T(t))$ and the positivity of efforts, we can note that such a profit constraint entails the following state constraint for tree vegetation:

$$T(t) \geq \frac{c_T}{p_T q_T} = T^{\text{OA}}. \quad (4)$$

Such a state threshold T^{OA} is known in the bioeconomic literature as the open-access level [Schaefer, 1954, Clark, 1973, De Lara and Doyen, 2008].

Regarding social sustainability and viability, we focus on the provisioning ecosystem services (health, food, ...) derived from grass harvesting in the following sense:

$$H_G(G(t), e_G(t)) \geq H_G^{\text{lim}}, \quad (5)$$

where viability threshold $H_G^{\text{lim}} > 0$ can relate to current value of grass harvesting (potentially depending on human population levels).

¹In the Cameroonian example of Section 4 below, states are biomass densities, evaluated in tons per hectare ($t.ha^{-1}$)

²We here assume an asymmetric competition in the sense that $\eta_{GT} = 0$. Indeed, our model mimics tree-grass interactions in humid savannas without fire. Trees have a direct depressive effect on grasses through competition for light (i.e. tree canopy-induced shadow effect) and competition for nutrients while the effect of grass on trees is indirect and fire-mediated, see also Yatat-Djeumen et al. [2018, 2021] and references therein.

As regard biodiversity conservation, we require the co-existence of tree and grass covers. Observe that the two constraints (3) and (5) induce such a coexistence as soon as H_G^{lim} and π_T^{lim} are strictly positive.

To assess the sustainability or viability of the savanna SES, we examine the compatibility of the bio-economic constraints (3), (5), with the dynamics (1) and (2) of the SES. To do so, we rely on the mathematical concepts of invariance and viability kernels [Aubin and Frankowska, 2008, Béné et al., 2001, De Lara et al., 2007, Oubraham and Zaccour, 2018]. We start with the following invariance set:

$$\mathbb{I}nvar(e_G, e_T) = \{(G_0, T_0) \text{ s.t. constraints (1), (2), (3), (5) hold true for any time } t \geq 0\}. \quad (6)$$

Such invariance kernels $\mathbb{I}nvar(e)$ are in fact called viability niches in [Aubin, 1990] because the controls are supposed to be fixed. By contrast, the viability kernel recasts the invariance concept within the control theory framework as follows:

$$Viab = \{(G_0, T_0) \mid \exists \text{ controls } e_G(\cdot), e_T(\cdot) \text{ and states } G(\cdot), T(\cdot) \text{ s.t. constraints (1), (2), (3), (5) hold } \forall t \geq 0\}. \quad (7)$$

Of course, the following inclusion holds true

$$\bigcup_e \mathbb{I}nvar(e) \subset Viab. \quad (8)$$

However, this previous inclusion is generally strict since, within the viability kernel $Viab$, the controls $e(\cdot)$ can fluctuate with time and state (typically through feedback controls) while, in invariance kernels $\mathbb{I}nvar(e)$, the controls are supposed to remain steady over time.

3 Analytical results

3.1 Non emptiness of the viability kernel

We first focus on the non-emptiness of the viability kernel. The idea is to provide sufficient conditions on the bioeconomic parameters of the system for the existence of a state belonging to the viability kernel $Viab$ defined in (7). Our sufficient conditions here relies on well-known referenced points related to MSY³ (maximum sustainable yield) and MEY⁴ (maximum economic yield) [Schaefer, 1954, Clark, 1973, De Lara and Doyen, 2008, Grafton et al., 2012]. We first remind the characterization of MEY equilibrium (effort, state, profit) for trees obtained using first order optimality conditions:

$$\begin{cases} e_T^{MEY} = \frac{r_T}{2q_T} \left(1 - \frac{c_T}{p_T q_T K_T}\right) = \frac{r_T}{2q_T} \left(1 - \frac{T^{\text{OA}}}{K_T}\right), \\ T^{MEY} = \frac{K_T}{2} \left(1 + \frac{T^{\text{OA}}}{K_T}\right), \\ \pi_T^{MEY} = \pi_T(e_T^{MEY}, T^{MEY}) = \frac{p_T r_T K_T}{4} \left(1 - \frac{T^{\text{OA}}}{K_T}\right)^2. \end{cases} \quad (10)$$

For grass, we first need to adapt the growth rate and the carrying capacity of grass by accounting for MEY tree level T^{MEY} as follows:

$$\widetilde{r}_G = r_G - \eta_{TG} T^{MEY}, \quad \widetilde{K}_G = K_G \left(1 - \frac{\eta_{TG} T^{MEY}}{r_G}\right).$$

³We here remind the very definition of MSY for a single stock [Schaefer, 1954] governed by the following ODE

$$\dot{x}(t) = rx(t) \left(1 - \frac{x(t)}{K}\right) - qe(t)x(t). \quad (9)$$

The MSY approach aims at maximizing the catches $h(t) = qe(t)x(t)$ at equilibrium in the following sense:

$$\max_{e \text{ such that } \dot{x}(t)=0} qex.$$

⁴We here remind the very definition of MEY for a single stock [Schaefer, 1954] governed by the same ODE (9). The MEY approach aims at maximizing the profit $\pi(t)$ at equilibrium in the following sense:

$$\max_{e \text{ such that } \dot{x}(t)=0} pqex - ce.$$

Of course, MEY and MSY states, efforts and harvests coincide whenever effort costs c vanish.

We then consider the following (adapted) MSY indicators

$$\begin{cases} e_G^{MSY} = \frac{\widetilde{r}_G}{2q_G}, \\ G^{MSY} = \frac{\widetilde{K}_G}{2}, \\ H_G^{MSY} = H_G(e_G^{MSY}, G^{MSY}) = \frac{\widetilde{r}_G \widetilde{K}_G}{4}. \end{cases} \quad (11)$$

Interestingly the positivity of \widetilde{K}_G entails an upper bound on the maximum economic yield for trees T^{MEY} . Such an upper bound depends on grass parameters namely the rate $\frac{r_G}{\eta_{TG}}$. More specifically we need the condition

$$\frac{r_G}{\eta_{TG}} > \frac{1}{2}(K_T + T^{\text{OA}}). \quad (12)$$

In particular, it implies that the growth of grass r_G is high enough when compared to carrying capacity K_T of trees. Symmetrically, a low level of competition η_{TG} favors such condition (12).

Then we obtain the following sufficient conditions for the non-emptiness of the viability kernel.

Proposition 1. *Assume condition (12) and that viability thresholds $H_G^{\text{lim}} > 0$ and $\pi_T^{\text{lim}} > 0$ are such that*

$$\pi_T^{\text{lim}} \leq \pi_T^{MEY}, \quad H_G^{\text{lim}} \leq H_G^{MSY}, \quad (13)$$

then there exists harvest rates (e_G, e_T) such that the invariance kernel $\text{Invar}(e_G, e_T)$ is nonempty. Consequently the viability kernel Viab is also nonempty.

Proof. See Appendix A for the details. The skeleton of the proof consists in proving that states (G^{MSY}, T^{MEY}) along with efforts (e_G^{MSY}, e_T^{MEY}) defined in (11) and (10) comply with both the state-control constraints and the dynamics. They satisfy the viability constraints from conditions (13). As they are also equilibria, they satisfy the constraints throughout time. \square

Sufficient conditions (13) indicate that the bioeconomic viability requirements need to be not too demanding to guarantee some viability of the savanna SES. More specifically, grass harvest H_G^{lim} and tree profit π_T^{lim} thresholds need to be bounded from above with respect to the maximal bio-economic sustainability (MSY-MEY) of the two renewable resources (grass-trees). In other words, Proposition 1 informs managers and stakeholders that the field of possibilities is limited in terms of bioeconomic sustainability since some viability thresholds H_G^{lim} and π_T^{lim} are too stringent to be sustained. A sensitivity analysis of π_T^{MEY} and H_G^{MSY} values with respect to the parameters of the system gives additional insights regarding the interest of Proposition 1. Thus, the right-hand side of (13) indicates that large values of both grass carrying capacity K_G and intrinsic growth rate r_G promote the sustainability of savanna SES since, from (11), harvest H_G^{MSY} increases linearly with both r_G and K_G . In contrast, the competition parameter η_{TG} has a negative contribution on viability as it impacts negatively H_G^{MSY} through \widetilde{K}_G . Similarly, the left-hand side of (13) shows that large values of intrinsic growth rate r_T foster the sustainability of savanna SES since, from (10), profit π_T^{MEY} increases with r_T . In contrast, the role of marginal costs c_T of effort for cutting trees is negative as it alters rent π_T^{MEY} . Such intuitive results are here clearly quantified. At this stage, it can be noted that the role of the tree carrying capacity K_T is more ambiguous as, on the one hand, it enhances π_T^{MEY} and thus relax requirement (13) (left hand side) while, on the other hand, it alters T^{MEY} and thus e_G^{MSY} through \widetilde{r}_G . This ambiguity will be somehow confirmed by the next Proposition 2 with the identification of an upper viability bound on woody vegetation T .

Furthermore, from the viable control theory viewpoint, it is of interest to emphasize that the conditions (13) underpinning Proposition 1 extend the analytical viability conditions for a single renewable stock as put forward⁵ in Béné et al. [2001].

⁵In Béné et al. [2001], the sufficient and necessary conditions for the non vacuity of the viability kernel Viab are similar to those for tree vegetation in the sense that

$$0 < \pi_{\text{lim}} \leq \pi_{MEY}. \quad (14)$$

3.2 A viable rectangle

Under strict inequalities underlying (13) in Proposition 3, we can prove that the viability kernel contains not only the equilibrium state (G^{MSY}, T^{MEY}) but also a set (a rectangle) with a nonempty interior. To do so, we need to introduce different notations. First we introduce the viable grass threshold $G^{\text{lim}}(e_G)$ as well as the viable tree threshold $T^{\text{lim}}(e_T)$:

$$G^{\text{lim}}(e_G) = \frac{H_G^{\text{lim}}}{q_G e_G}, \quad T^{\text{lim}}(e_T) = \frac{\pi_T^{\text{lim}} + c_T e_T}{p_T q_T e_T} = T^{\text{OA}} + \frac{\pi_T^{\text{lim}}}{p_T q_T e_T}. \quad (15)$$

In particular, we introduce the states G^{PA} and T^{PA} as follows

$$G^{\text{PA}} = G^{\text{lim}}(e_G^{MSY}), \quad T^{\text{PA}} = T^{\text{lim}}(e_T^{MEY}). \quad (16)$$

The notation PA is inspired by the precautionary approach and thresholds of ICES for fisheries management. G^{PA} (resp. T^{PA}) could be interpreted as the minimal amount of grass (resp. tree) state that ensures viability and sustainability of harvested grass (resp. tree) using MSY-MEY efforts. We also need to consider the equilibrium function for grass dynamics⁶:

$$T^*(G, e_G) = \frac{1}{\eta_{TG}} \left(r_G \left(1 - \frac{G}{K_G} \right) - q_G e_G \right). \quad (17)$$

In particular, we introduce the particular equilibrium state $T^\#$ as follows:

$$T^\# = T^*(G^{\text{PA}}, e_G^{MSY}). \quad (18)$$

We then define the rectangle denoted by \mathcal{A} consisting in grass states larger than G^{PA} along with tree states within the interval $[T^{\text{PA}}, T^\#]$ as follows

$$\mathcal{A} = [G^{\text{PA}}, +\infty[\times [T^{\text{PA}}, T^\#]. \quad (19)$$

The following proposition captures the properties of invariance (with respect to MSY-MEY controls) and viability of such a rectangle \mathcal{A} . Such a result is illustrated by Figure 2.

Proposition 2. *Assume again conditions (12) on dynamics parameters and (13) on viability thresholds. Then the invariance kernel $\mathbb{Invar}(e_G^{MSY}, e_T^{MEY})$ and thus the viability kernel \mathbb{Viab} contains the rectangle \mathcal{A} defined in (19) in the sense that*

$$\mathcal{A} = [G^{\text{PA}}, +\infty[\times [T^{\text{PA}}, T^\#] \subset \mathbb{Invar}(e_G^{MSY}, e_T^{MEY}) \subset \mathbb{Viab}. \quad (20)$$

Moreover, if at least one the inequalities in (13) is strict, the viable rectangle \mathcal{A} has a non empty interior and is thus a real rectangle. Otherwise the viable set \mathcal{A} is reduced to the half-line $[G^{\text{PA}}, +\infty[\times \{T^{MEY}\}$.

Proof. See Appendix B for the details. The sketch of the proof consists in proving that the whole rectangle \mathcal{A} satisfies the bioeconomic constraints together with tangential (inward) viability conditions. \square

The interest of Proposition 2 is to point out that the viability kernel is not limited to an equilibrium as in Proposition 1 but contains at least a set with a non empty interior whenever one of the assumptions on viable thresholds is strict. Such a finding is exemplified in Figure 2 where the red lines stand for the boundary of the rectangle \mathcal{A} . In particular an interval $[T^{\text{PA}}, T^\#]$ emerges as a key ingredient for viability. This confirms that tree vegetation T has to be managed with cautious to remain within this specific corridor of viability. The upper bound $T^\#$ on tree vegetation for viability arises from the competition between trees and grass through parameter $\eta_{TG} > 0$. The underlying mechanism is that too many woody vegetation would put at stake the viability of grass vegetation. By contrast, grass state $G(t)$ is less demanding in terms of sustainable management since only a lower bound G^{PA} has to be guaranteed.

The room for manoeuvre underlying rectangle \mathcal{A} in terms of viable states is also of interest in terms of viable management as it also expands the set of possible viable controls beyond the equilibrium strategy (e_G^{MSY}, e_T^{MEY}) . From Aubin and Frankowska [2008], we indeed know that the set of viable controls at any

⁶In the sense that $\dot{G} = 0$.

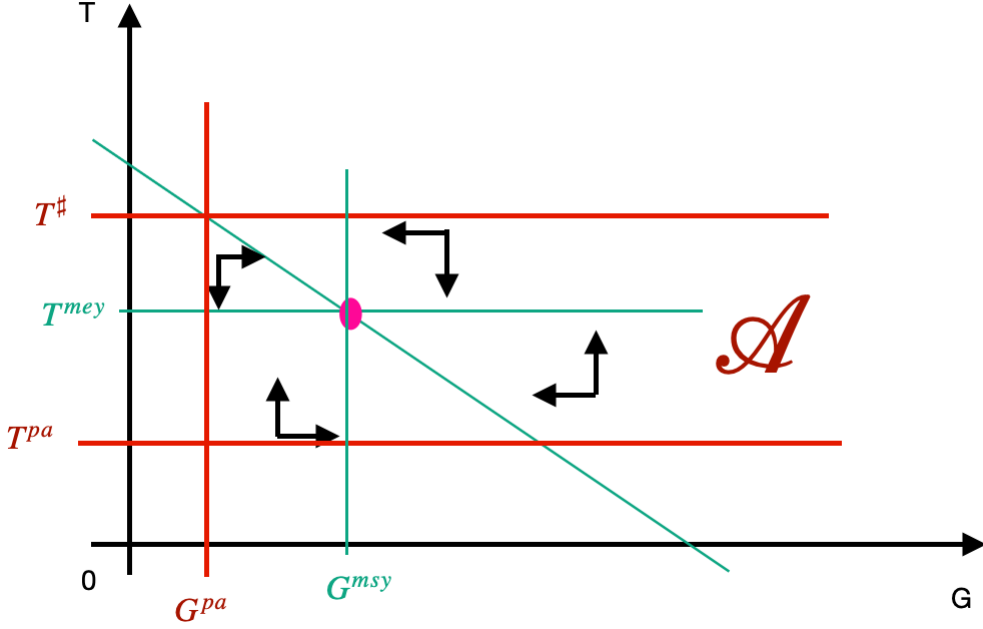


Figure 1: The viable equilibrium (G^{MSY}, T^{MEY}) and the viable rectangle \mathcal{A} . The magenta bullet denotes the coexistence equilibrium. The solid red lines delineate the viability rectangle \mathcal{A} while the solid green lines represent the equilibrium line $T^*(G, e_G) = \frac{1}{\eta r_G} \left(r_G \left(1 - \frac{G}{K_G} \right) - q_G e_G \right)$ and $T = T^{MEY}$. The arrows represent the velocities for the strategy (e_G^{MSY}, e_T^{MEY}) .

viable state $(G, T) \in \mathcal{A}$ is characterized by tangential (inward) conditions with respect to the rectangle \mathcal{A} . On the three lines of the boundary⁷ of the rectangle (see also Figure 1), these tangential conditions read:

$$\left\{ \begin{array}{ll} G = G^{PA}, T > T^{PA} & : \quad e_G \leq \frac{r_G}{q_G} \left(1 - \frac{G}{K_G} \right) - \frac{\eta r_G}{q_G} T \quad \left(\frac{\partial G}{\partial t} \geq 0 \right), \\ G > G^{PA}, T = T^{PA} & : \quad e_T \leq \frac{r_T}{q_T} \left(1 - \frac{T}{K_T} \right) \quad \left(\frac{\partial T}{\partial t} \geq 0 \right), \\ G > G^{PA}, T = T^\# & : \quad e_T \geq \frac{r_T}{q_T} \left(1 - \frac{T}{K_T} \right) \quad \left(\frac{\partial T}{\partial t} \leq 0 \right). \end{array} \right. \quad (21)$$

For the corners of the rectangle, it is enough to combine the previous tangential requirements. The new bounds on harvesting efforts e_G and e_T obtained in (21) emerging from the tangential conditions constitute additional viability requirements and inequalities. At this stage, we first stress that some of these inequalities (in particular the first two) entail upper bounds for efforts as opposed to lower bounds for efforts stemming from initial viability goals (5) and (3). Thus viability corridors again emerge, regarding here tree - grass harvesting controls and management instead of tree - grass state. We also point out that these (tangential) inequalities constitute an extension of equilibrium MSY-MEY conditions (in particular equalities $\frac{\partial G}{\partial t} = \frac{\partial T}{\partial t} = 0$) and, as such, provide more flexibility for the viable management and harvesting controls (e_T, e_G) . Therefore, globally, the tangential conditions entail more flexibility and adaptiveness in terms of management when compared to steady strategies MSY-MEY although, again, with specific viability bounds and a limited field of possibilities.

3.3 Bioeconomic resilience through stability and recovery

We now give insights into the resilience of savanna SES by providing a stability analysis of the viable states-controls MSY - MEY. In other words, we will focus on viable states that are locally and globally asymptotically stable (LAS - GAS) [Khalil and Shaw, 1999]. By LAS is here meant that there exists a neighborhood of the viable state such that every trajectory starting from this neighborhood will converge to a viable state. By GAS here meant that whenever a trajectory is initiated in the phase space, it will converge

⁷In the interior of the rectangle, there is no additional constraint.

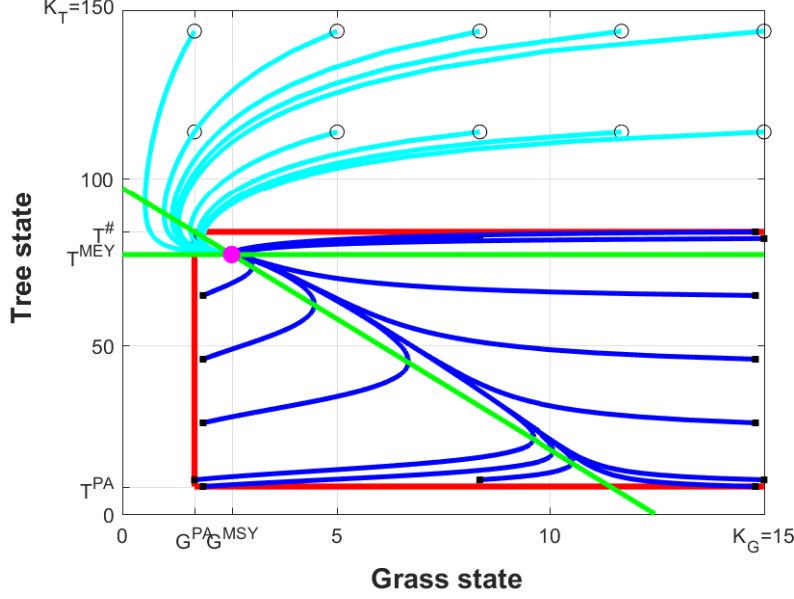


Figure 2: Phase portrait of system (1) with various initial conditions $(G_0, T_0) \in \mathcal{A}$ (black bullets) and $(G_0, T_0) \notin \mathcal{A}$ (black circles). The magenta bullet denotes the stable coexistence equilibrium $(G^{\text{MSY}}, T^{\text{MEY}})$. The solid red lines stand for the viability rectangle while the solid green lines represent the equilibrium line $T^*(G, e_G) = \frac{1}{\eta_{TG}} \left(r_G \left(1 - \frac{G}{K_G} \right) - q_G e_G \right)$ and $T = T^{\text{MEY}}$. Solid blue lines stand for viable trajectories while solid cyan lines denote non-viable trajectories.

toward the viable state. Thus such state can cope with shocks and recover in the sense of coming back to a viable situation in finite time. Such a viewpoint makes it possible to take into account shocks on savanna, such as those induced by fires. The characterization of state resilience in terms of stability is consistent with one of the 3Rs of resilience [Grafton et al., 2019, Cuilleret et al., 2022] namely recovery. Recovery is a matter of recovery time [Holling, 1973, Pimm, 1984, Martinet et al., 2007, Hardy et al., 2016]. We can prove the following Proposition 3 illustrated by Figure 2 where we also observe that the viability rectangle \mathcal{A} lies inside the basin of attraction of the viable equilibrium $(G^{\text{MSY}}, T^{\text{MEY}})$.

Proposition 3. *Assume conditions (12) on dynamics parameters and (13) on viability thresholds H_G^{lim} and π_T^{lim} . Then the state vector $X = (G^{\text{MSY}}, T^{\text{MEY}})$ together with harvest rates $(e_G, e_T) = (e_G^{\text{MSY}}, e_T^{\text{MEY}})$ is viable and globally asymptotically stable.*

Proof. The proof consists in proving first that the viable equilibrium $(G^{\text{MSY}}, T^{\text{MEY}})$ is locally asymptotically stable and, using standard arguments, also globally asymptotically stable. See Appendix C for the details. \square

Proposition 3 informs managers that resilience-based management for savanna exists in the sense that there exists harvesting efforts, at least $(e_G^{\text{MSY}}, e_T^{\text{MEY}})$, bouncing back the ecosystem and the SES toward viable states of the rectangle \mathcal{A} despite adverse events or shocks forcing the system to leave the viable rectangle for a while. Such a result is exemplified in the numerical simulations of Section 4 with wildfire shocks.

3.4 Bioeconomic resilience through resistance

We now give insights into the resilience of savanna SES by using a resistance score, another ‘R’ of the ‘3Rs’ of resilience [Grafton et al., 2019, Cuilleret et al., 2022]. Resistance is the magnitude of shocks (say for state) that can be withstood. An idea put forward in [B  n   and Doyen, 2018, Karacaoglu and Krawczyk, 2021, Cuilleret et al., 2022] to assess resistance from viability consists in considering the distance of any state X to the non viable zone namely the complementary to the viability kernel:

$$\text{Resistance}(X) = d_{\mathbb{R}^2 \setminus \text{Viab}}(X) = \min_{Y \notin \text{Viab}} \|X - Y\|. \quad (22)$$

For instance if the state lies outside or on the boundary of the viability kernel, its resistance vanishes because even very small shocks alter its viability. In terms of management, a strategy already proposed in [Cuilleret et al. \[2022\]](#) consists in promoting the resistance by maximizing the distance above (22) to the non viable area. Here we adapt such a definition of resistance by considering a lower bound with the distance to rectangle \mathcal{A} identified as a viable set in Proposition 2:

$$Resistance(G, T) \geq d_{\mathbb{R}^2 \setminus \mathcal{A}}(G, T).$$

For any state (G, T) within the rectangle \mathcal{A} , we can explicitly compute this distance (in the sense of L^1 norm) to the rectangle using its 2 corners and critical values G^{PA} , T^{PA} and T^\sharp as follows

$$d_{\mathbb{R}^2 \setminus \mathcal{A}}(G, T) = \min(G - G^{PA}, T - T^{PA}, T^\sharp - T).$$

We use this approximation of resistance in the following section to characterize the magnitude of wildfire events that can be withstood in terms of viability for a savanna SES in Cameroon.

4 Numerical example

In this section, simulations inspired from savanna systems in Cameroon exemplify the analytical findings of previous section. We follow [\[Yatat-Djeumen et al., 2018, 2021\]](#) to consider tree-grass interaction parameters for the context of a humid savanna, like in the South of Cameroon. In particular, [Yatat-Djeumen et al. \[2021\]](#) proposed, along the rainfall gradient, a parameterization of dynamic system (1) for tree-grass biomasses (t/ha) based on previous works or by re-interpreting empirical results. The parameter values are summarized in Table 1.

Table 1: Tree-grass interaction parameter values.

Symbol	K_T	K_G	r_T	r_G	η_{TG}
Value	150 t/ha	15 t/ha	1 yr^{-1}	3.5 yr^{-1}	0.03 $(t/ha)^{-2} yr^{-1}$

The economic and productive parameter values used for the simulations are summarized in Table 2. For this paper, they are rather illustrative than derived from real data. This is particularly the case of viability thresholds H_G^{lim} and π_T^{lim} that are fixed arbitrarily to guarantee strictly positive profit and harvest. Future research is thus needed to refine and calibrate these economic values.

Table 2: Economic parameter values.

Symbol	q_T	q_G	p_T	c_T	π_T^{lim}	H_G^{lim}
Value	0.5 yr^{-1}	0.5 yr^{-1}	0.5 $yr(ha/t)$	1	1	1

Based on parameter values of Tables 1-2 and characterizations (11), (10), (16), (18), key reference and tipping points of the model are deduced. They are summarized in Table 3.

Table 3: Key reference and tipping points of the model.

Symbol	e_T^{MEY}	e_G^{MSY}	T^{MEY}	G^{MSY}	π_T^{MEY}	H_G^{MSY}	G^{PA}	T^{PA}	\widetilde{r}_G	\widetilde{K}_G	T^\sharp
Value	0.9733	1.19	77	2.55	17.7633	4.4625	1.6807	8.1096	1.19	5.1	83.7614

It can be noted that the viability conditions (12), page 5 and (13), page 5 hold true for these parameters. Thus Proposition 2 about the viable rectangle \mathcal{A} applies. This viability rectangle \mathcal{A} is depicted in Figure 2, page 8 with the red solid lines. It clearly appears from Figure 2 that any trajectory of system (1) starting in the viability rectangle \mathcal{A} with steady controls (e^{MSY}, e^{MEY}) remains within \mathcal{A} throughout time. This confirms Proposition 2 regarding the viability of the rectangle \mathcal{A} . In particular, the emergence of the upper bound T^\sharp for woody state put forward in Proposition 2 to promote the sustainability of savanna SESs is confirmed in the context of many humid savannas from Southern Cameroon. Several authors have indeed emphasized that woody vegetation would be likely to reach a closed canopy situation without grasses in the absence of recurrent biomass losses induced by fires, browsers or logging that delay, block or reduce woody

biomass [Yatat-Djeumen et al., 2018, Bond et al., 2005, Bond, 2008, Jeffery et al., 2014]. For instance, if in system (1) the tree harvested effort decreases, this will enhance woody biomass. And, as the woody biomass increases above the threshold T^\sharp , its depressive impact on the grass biomass will also increase leading to a grass collapse through the competition ($-\eta_{TG}TG$). Therefore, managers need to maintain tree biomass below a ceiling level (here captured by level T^\sharp) to prevent grass exclusion and sustain the bioeconomic viability of savannas.

Expanding the sustainability viewpoint underlying the viability rectangle \mathcal{A} , we now adopt a resilience viewpoint for the case study in particular in the face of wildfires. To do so, we rely on both resistance and recovery analysis. Along with herbivory, fire is considered as a main disturbance of the dynamics of tree and grass biomasses in humid savannas. In the sequel, we consider fire as shock for tree and grass dynamics. Precisely, in humid savannas, fire does not happen continuously as several months and even years can last between two successive fire events. We also assume that they occurred suddenly and do not last more than a couple of days; hence compared to the time-scale of tree and grass biomasses dynamics, fire event is considered as impulsive (or a shock) [Yatat-Djeumen et al., 2017]. Furthermore, following [Yatat-Djeumen et al., 2017, 2018, Yatat-Djeumen and Dumont, 2018, Yatat-Djeumen et al., 2021], we assume that at a given time $\tau > 0$ when fire occurs, it induces an additional mortality in both tree and grass biomasses. Thus, denoting by τ^+ the next moment after a fire occurrence, we write:

$$\begin{cases} G(\tau^+) &= G(\tau) - \lambda_G G(\tau), \\ T(\tau^+) &= T(\tau) - \vartheta(T(\tau))\omega(G(\tau))T(\tau), \end{cases} \quad (23)$$

where $\lambda_G \in [0, 1)$ is the fire-induced mortality rate of grass biomass while $\vartheta(T(\tau))\omega(G(\tau))$ stands for fire-induced mortality rate of tree biomass. We detail and explain below the functional forms of $\vartheta(T)$ together with $\omega(G)$. The rate $\omega(G)$ takes into account that grass biomass is the main factor controlling both fire intensity and spreading capacity in savannas. Following [Yatat-Djeumen et al., 2021], we combined these two properties of fire in a single, increasing function of grass-biomass

$$\omega(G) = \frac{G^2}{G^2 + g_0^2}, \quad (24)$$

meaning that, whence average herbaceous biomass is in its highest range, fires both display the highest intensity and affect all the landscape. Conversely, low grass biomass due to aridity, grazing or tree competition, will make fires of low intensity and/or unable to reach all locations in a given year thereby decreasing the actual average frequency. Parameter g_0 is the value of grass biomass at which fire intensity, $\omega(G)$, reaches its half-saturation since $\omega(g_0) = 50\%$. Moreover, for a given level of $\omega(G)$, fire-induced tree/shrub mortality, denoted by $\vartheta(T)$ is assumed to be a decreasing, non-linear function of tree biomass [Yatat-Djeumen et al., 2021]. Indeed, fires affect differently large and small trees since fires with high intensity (flame length $> ca.$ 2m) cause greater mortality of shrubs and top-kill of trees while fires of lower intensity (flame length $< ca.$ 2m) top-kill only shrubs and sub-shrubs (Yatat-Djeumen et al. [2018] and references therein). Therefore, ϑ reads as follows [Tchuinté Tamen et al., 2017, Yatat-Djeumen et al., 2021]:

$$\vartheta(T) = \lambda_T^{min} + (\lambda_T^{max} - \lambda_T^{min})e^{-\kappa T}, \quad (25)$$

where λ_T^{min} is minimal lost portion of tree biomass due to fire in configurations with a very large tree biomass, λ_T^{max} is maximal loss of tree/shrub biomass due to fire in open vegetation while κ is proportional to the inverse of biomass suffering an intermediate level of mortality.

Facing this wildfire shocks, we now investigate the resistance issues applying ideas of Subsection 3.4. Thus, considering a state $(G(\tau), T(\tau)) \in \mathcal{A}$ before a shock (wildfire), it will be resistant if despite the shock represented by equations (23), its forward orbit stays within the viability rectangle \mathcal{A} . This means equivalently that:

$$\begin{cases} G(\tau^+) &\geq G^{PA}, \\ T(\tau^+) &\geq T^{PA}, \end{cases} \quad (26)$$

where G^{PA} and T^{PA} are given in (16), page 6. From equations (23) and (26), a necessary condition for the state $(G(\tau), T(\tau)) \in \mathcal{A}$ to be resistant is to have a low enough rate λ_G in the sense that:

$$\lambda_G \leq 1 - \frac{G^{PA}}{G(\tau)}. \quad (27)$$

In particular we set the fire-induced mortality of grass biomass to $\lambda_G = 0.25$ to guarantee resistance. Identifying resistant parameters for fire-induced tree/shrub mortality $\vartheta(T)$ is more complicated because of

the non-linearities. Following [Yatat-Djeumen et al., 2021], fire-related parameter values inducing resistance are considered in Table 4.

Table 4: Fire-related parameter values.

Symbol	λ_T^{min}	λ_T^{max}	κ	g_0
Value	0.05	0.25	0.02	2

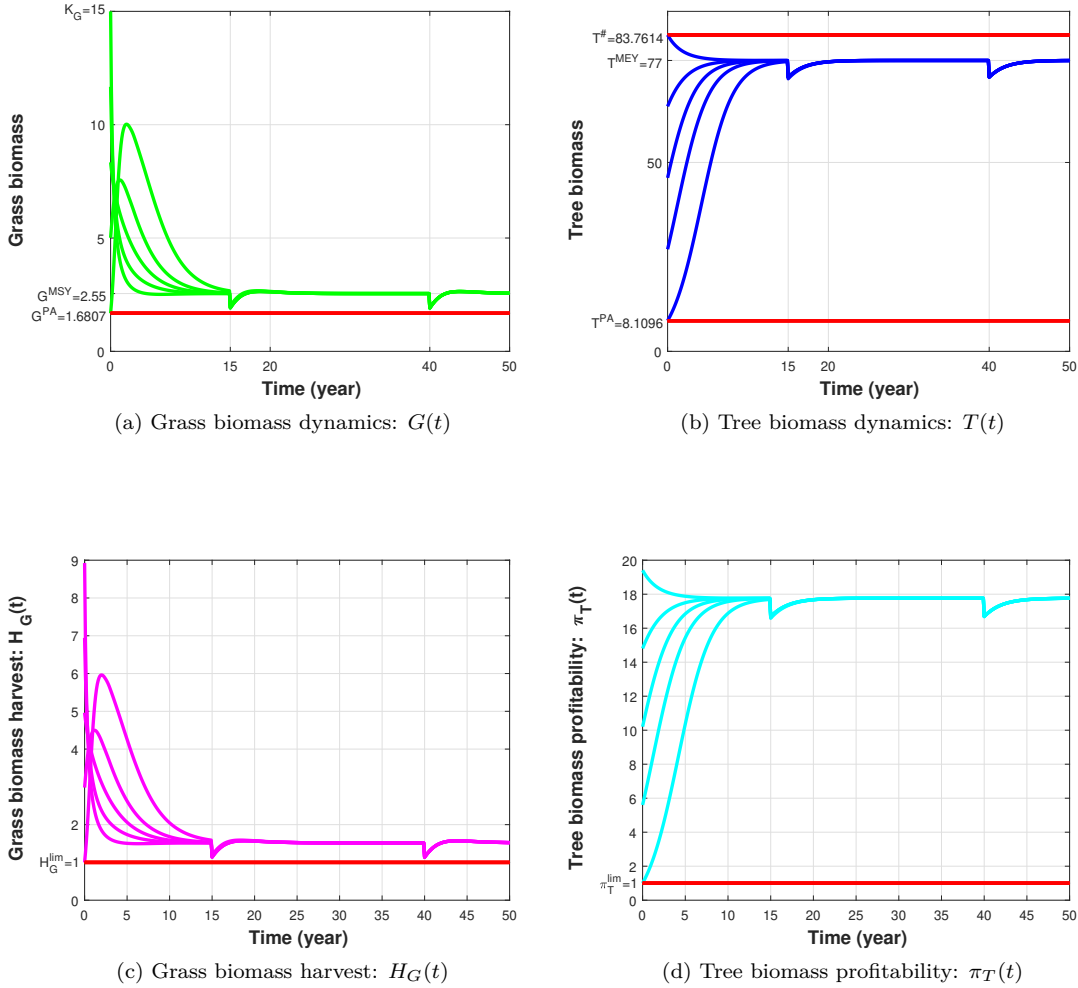


Figure 3: Dynamics of tree (panel (a)) and grass (panel (b)) biomasses showing resistance when fire events, considered as shocks, occur at $\tau_1 = 15$ years and $\tau_2 = 40$ years. The fire-induced mortality of grass biomass is $\lambda_G = 0.25$.

We now assume that fire events occur at $\tau_1 = 15$ years and $\tau_2 = 40$ years. Figure 3 depicts the dynamics of both tree and grass biomasses promoting resistance despite the occurrence of shocks, considered as fire events. More specifically, the magnitude of the shocks is such that the requirement (26) still holds true. Therefore, the states $(G(\tau_i), T(\tau_i))$, $i = 1, 2$ are resistant in the sense that they can cope with the fire shocks by still complying with the profit and harvest constraints underpinning the red thresholds in Figure 3 (c) and (d). Subfigures 3 (a) and (b) confirm that such resistance is associated with the joint viability of states $(G(t), T(t))$ within the viable rectangle $\mathcal{A} = [G^{PA}, +\infty[\times [T^{PA}, T^\#]$ as stressed in subsection 3.4.

Going further, Figure 4 illustrates the recovery property of the system underlying Proposition 3. We here increase the fire-induced mortality of grass biomass with $\lambda_G = 0.5$. Indeed, in that case, the magnitude of the shocks is such that the requirement (26) and resistance now fails as illustrated by 4 (c) where the harvest

constraint $H_G(t) \geq H_G^{\text{lim}}$ is violated for a while. However we can observe that after a transient period, the system recovers thanks to the stability of the state $(G^{\text{MSY}}, T^{\text{MEY}})$.

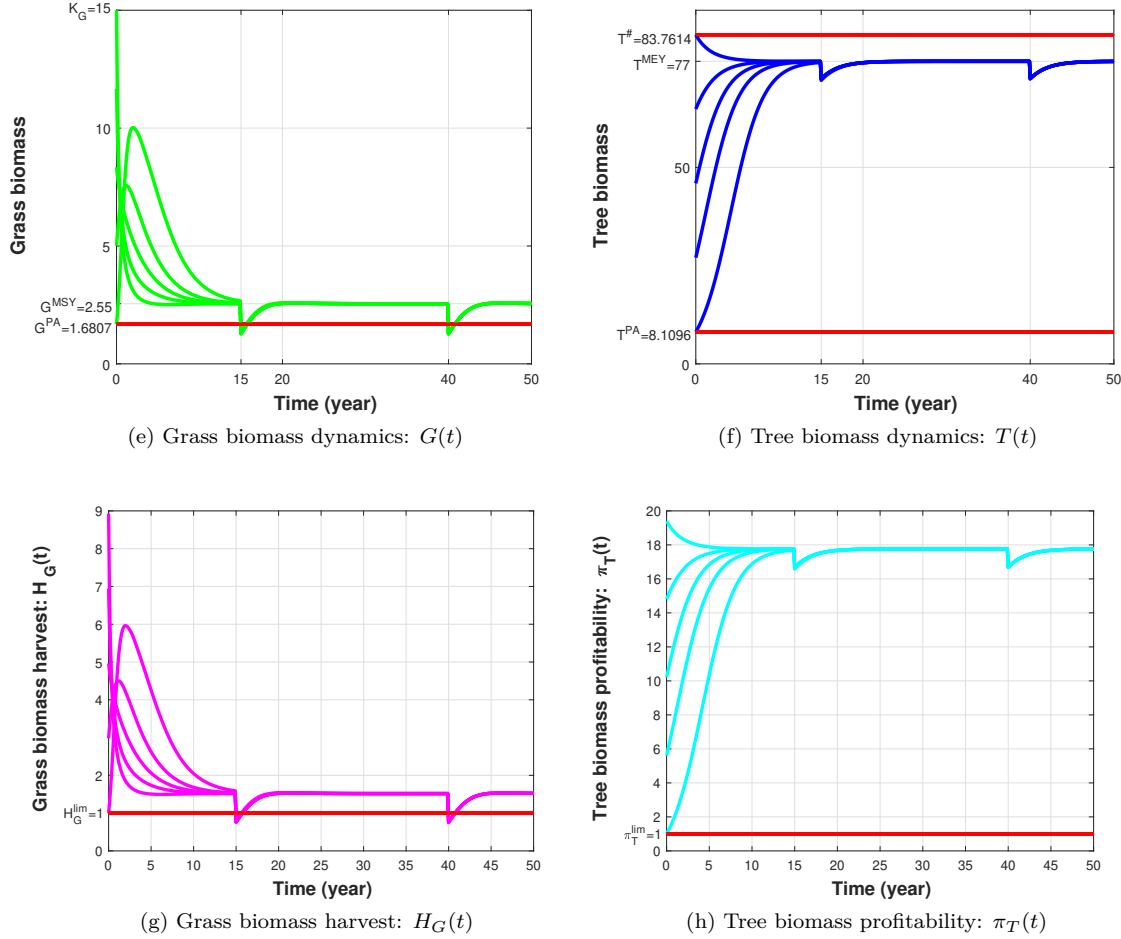


Figure 4: Dynamics of tree and grass biomasses showing recovery after fire events, considered as shocks, that occur at $\tau_1 = 15$ years and $\tau_2 = 40$ years. The fire-induced mortality of grass biomass is $\lambda_G = 0.5$.

5 Discussion and perspectives

Our paper deals with the bioeconomic sustainability and resilience-based management of savanna social-ecological systems (SES). The analysis draws on a stylized dynamics of an exploited grass-tree systems integrating both for the competition between trees and grass along with logging and grass harvesting activities which play the role of control of the system. Sustainability and resilience issues are handled through bioeconomic viability goals and thresholds including consumption security for grass, profitability of logging and coexistence of tree-grass states. Such viability goals are aligned with the triple bottom-line of sustainable development and many of the SDGs [FAO, 2017]. A first analytical result relates to the identification of sufficient sustainability conditions through the non-emptiness of the so-called viability kernel. Such sufficient conditions involving the whole set of bio-economic parameters of the SES arise from coupled MSY (maximum sustainable yield) - MEY (maximum economic yield) reference states-controls. Although MSY - MEY states and controls play a pivotal role, we prove that sustainability conditions are not limited to these equilibria as a viable rectangle including these equilibria is also elucidated. The resilience of the such viability states-controls for savanna SES is then put forward both from a stability (recovery) and viability (resistance) analysis. Simulations inspired from savanna systems in Cameroon exemplify the analytical findings.

The original contribution of our paper is threefold. The first major contribution is to pave the road for a resilience-based management of savanna SES accounting for shocks in particular those induced by fires which plays pivotal roles in the dynamics of these systems. These analytical results inform managers and

stakeholders on different state and control thresholds to sustain the bioeconomic viability and resilience of savanna SES. Of interest is that these upper and lower bounds including MSY-MEY reference levels, the corridor underpinning the viable rectangle and tangential viability conditions for tree-grass harvesting efforts provide a safe operating space [Rockström et al., 2009], room for manoeuvre, flexibility and different options to prevent undesirable bioeconomic trajectories, crisis and collapses. Such flexibility can be very useful for governance issues as compared to equilibrium or optimal strategies. Such resilience based-management should go further the stability and resistance analysis proposed here and directly copes with wildfire shocks through stochastic dynamics and control as in [Grafton et al., 2019] and Cuilleret et al. [2022]. The identification of the whole viability kernel is another future important methodological challenge. In that regard, monotonicity properties De Lara et al. [2007] underlying dynamic systems (1) should help to advance the viability for tree-grass and savanna dynamics [Accatino et al., 2014]. The application of viability and resilience-based management to savanna systems in Cameroon will be of great interest to assist local stakeholders and regulating agencies facing global changes including climate change and demographic pressure. More globally, as savanna are also found in many countries of Africa, South America or Asia, as well as in Australia or India, where they also play key roles in terms of biodiversity and provisioning ecosystem services [Scholes and Walker, 1993, Staver et al., 2021, Huntley, 2023], our sustainability and resilience insights should interest scientists, stakeholders and regulating agencies worldwide.

The second major contribution is to propose a model of intermediate complexity [Plaganyi et al., 2014, Doyen, 2018, Gomes et al., 2020] to deal with savanna social-ecological systems (SES). By intermediate complexity is meant a model with a limited complexity but capturing the major ingredients of the management issues at play, namely savanna management through cutting and harvesting. The complexity here stems from the combination of non-linear coupled dynamics of tree and grass biomasses together with ecological, social and economic constraints in line with the SDGs. Accounting for herbivory processes through grazing and browsing mechanisms constitutes a key extension of our work towards models of intermediate complexity and an ecosystem-based approach of savanna SES [Van Langevelde et al., 2003].

The third major contribution is to extend analytical viability results for bioeconomic systems initiated in Béné et al. [2001]. The sufficient conditions of Proposition 1 for the non-emptiness involving coupled MSY-MEY referenced points for both tree and grass indeed expand the sufficient conditions for a single renewable stock stressed in Béné et al. [2001] involving only the MEY of this single stock. Such result paves thus the road for sufficient viability conditions in ecosystems with larger complexity and dimensionality as many of those relating to biodiversity management. The viable rectangle underpinning Proposition 2 is another important step to expand viability conditions beyond equilibrium conditions. This rectangle stresses the need to manage stocks within safety corridors including generalized MSY-MEY states but not limited to them.

Statements and Declarations

Acknowledgements

The authors thank the anonymous reviewers and the editor for their helpful comments that greatly improved this work.

Funding

This study was funded by the CNRS through the research project IEA (International Emerging Action) entitled ERICA (Ecological-ecoNomic resilience of TROPICAL Ecosystem in Africa) in collaboration with the Ecole Nationale Supérieure Polytechnique de Yaoundé (ENSPY) and the University of Yaoundé 1. The role of the International Research Network (IRN) named QARESS and supported by CNRS has also to be pointed out. I.V.Y.D. acknowledges the support of the AFRICAM project funded by the French Agency for Development.

Competing Interests

L.D. is an Editor of the journal *Environmental Modeling & Assessment*.

Author Contributions

I.V.Y.D. and L.D. wrote the main manuscript text; I.V.Y.D. prepared all figures and wrote the revised text. All authors reviewed the manuscript.

Ethical Approval

Not applicable.

Availability of data and materials

Available upon request.

References

- F. Accatino, R. Sabatier, C. De Michele, D. Ward, K. Wiegand, and K. Meyer. Robustness and management adaptability in tropical rangelands: a viability-based assessment under the non-equilibrium paradigm. *Animal*, 8(8):1272–1281, 2014. ISSN 1751-7311. doi: <https://doi.org/10.1017/S1751731114000913>. URL <https://www.sciencedirect.com/science/article/pii/S1751731114000913>.
- J.-P. Aubin. A survey of viability theory. *SIAM J. Control Optim.*, 28(4):749–788, 1990.
- J.-P. Aubin and H. Frankowska. *Set-Valued Analysis*. Modern Birkhäuser Classics. Birkhauser Boston, Secaucus, NJ, 1 edition, Nov. 2008.
- M. Baudena and M. Rietkerk. Complexity and coexistence in a simple spatial model for arid savanna ecosystems. *Theor. Ecol.*, 6(2):131–141, 2013. URL <https://doi.org/10.1007/s12080-012-0165-1>.
- S. Baumgärtner and M. Quaas. Ecological-economic viability as a criterion of strong sustainability under uncertainty. *Ecological Economics*, 68(7):2008–2020, 2009. ISSN 0921-8009. URL https://econpapers.repec.org/article/eeeecolec/v_3a68_3ay_3a2009_3ai_3a7_3ap_3a2008-2020.htm. Publisher: Elsevier.
- C. Béné and L. Doyen. From Resistance to Transformation: A Generic Metric of Resilience Through Viability. *Earth's Future*, 6(7):979–996, July 2018. ISSN 23284277. doi: 10.1002/2017EF000660. URL <http://doi.wiley.com/10.1002/2017EF000660>.
- C. Béné, L. Doyen, and D. Gabay. A viability analysis for a bio-economic model. *Ecological economics*, 36(3):385–396, 2001. Publisher: Elsevier.
- W. J. Bond. What limits trees in c4 grasslands and savannas? *Annu. Rev. Ecol. Evol. Syst.*, 39:641–659, 2008.
- W. J. Bond, F. I. Woodward, and G. F. Midgley. The global distribution of ecosystems in a world without fire. *New Phytol.*, 165(2):525–538, 2005. URL <https://doi.org/10.1111/j.1469-8137.2004.01252.x>.
- A. Cissé, S. Gourguet, L. Doyen, F. Blanchard, and J.-C. Péreau. A bio-economic model for the ecosystem-based management of the coastal fishery in French Guiana. *Environment and Development Economics*, 18(03):245–269, 2013. Publisher: Cambridge Univ Press.
- C. W. Clark. The economics of overexploitation. *Science*, 181(4100):630–634, 1973.
- M. Cuilleret, L. Doyen, H. Gomes, and F. Blanchard. Resilience management for coastal fisheries facing with global changes and uncertainties. *Economic Analysis and Policy*, 74:634–656, June 2022. ISSN 0313-5926. doi: 10.1016/j.eap.2022.03.016. URL <https://www.sciencedirect.com/science/article/pii/S0313592622000455>.
- M. De Lara and L. Doyen. *Sustainable management of natural resources: mathematical models and methods*. Springer Science & Business Media, 2008.

- M. De Lara, L. Doyen, T. Guilbaud, and M.-J. Rochet. Monotonicity properties for the viable control of discrete-time systems. *Systems & Control Letters*, 56(4):296–302, 2007. ISSN 0167-6911. doi: <https://doi.org/10.1016/j.sysconle.2006.10.007>. URL <https://www.sciencedirect.com/science/article/pii/S0167691106001782>.
- S. Derissen, M. F. Quaas, and S. Baumgärtner. The relationship between resilience and sustainability of ecological-economic systems. *Ecological Economics*, 70(6):1121–1128, Apr. 2011. ISSN 09218009. doi: 10.1016/j.ecolecon.2011.01.003. URL <https://linkinghub.elsevier.com/retrieve/pii/S0921800911000103>.
- B. J. Downes, F. Miller, J. Barnett, A. Glaister, and H. Ellemor. How do we know about resilience? An analysis of empirical research on resilience, and implications for interdisciplinary praxis. *Environmental Research Letters*, 8(1):014041, Mar. 2013. ISSN 1748-9326. doi: 10.1088/1748-9326/8/1/014041. URL <https://iopscience.iop.org/article/10.1088/1748-9326/8/1/014041>.
- L. Doyen. Mathematics for Scenarios of Biodiversity and Ecosystem Services. *Environmental Modeling & Assessment*, 23(6):729–742, 2018. Publisher: Springer.
- L. Doyen, C. Armstrong, S. Baumgärtner, C. Béné, F. Blanchard, A. Cissé, R. Cooper, L. Dutra, A. Eide, D. Freitas, and others. From no whinge scenarios to viability tree. *Ecological Economics*, 163:183–188, 2019. Publisher: Elsevier.
- FAO. FAO and the SDGs. Indicators: Measuring Up to the 2030 Agenda for Sustainable Development [Policy Support and Governance] Food and Agriculture Organization of the United Nations, 2017. URL <https://www.fao.org/policy-support/tools-and-publications/resources-details/fr/c/854006/>.
- H. Gomes, C. Kersulec, L. Doyen, F. Blanchard, A. Cisse, and N. Sanz. Climate warming vs ecological competition for marine tropical biodiversity and fisheries. *Bordeaux Economics Working Papers*, 2020-13 (BxWP2020 -1), 2020.
- Q. R. Grafton, T. Kompas, T. N. Che, L. Chu, and R. Hilborn. BMEY as a fisheries management target. *Fish and Fisheries*, 13(3):303–312, 2012. Publisher: Wiley Online Library.
- R. Q. Grafton and L. R. Little. Risks, resilience, and natural resource management: Lessons from selected findings. *Natural Resource Modeling*, 30(1):91–111, Feb. 2017. ISSN 08908575. doi: 10.1111/nrm.12104. URL <http://doi.wiley.com/10.1111/nrm.12104>.
- R. Q. Grafton, L. Doyen, C. Béné, E. Borgomeo, K. Brooks, L. Chu, G. S. Cumming, J. Dixon, S. Dovers, D. Garrick, A. Helfgott, Q. Jiang, P. Katic, T. Kompas, L. R. Little, N. Matthews, C. Ringler, D. Squires, S. I. Steinshamn, S. Villasante, S. Wheeler, J. Williams, and P. R. Wyrwoll. Realizing resilience for decision-making. *Nature Sustainability*, 2(10):907–913, Oct. 2019. ISSN 2398-9629. doi: 10.1038/s41893-019-0376-1. URL <http://www.nature.com/articles/s41893-019-0376-1>.
- P.-Y. Hardy, C. Béné, L. Doyen, and A.-M. Schwarz. Food security versus environment conservation: A case study of Solomon Islands’ small-scale fisheries. *Environmental Development*, 8:38–56, 2013. Publisher: Elsevier.
- P.-Y. Hardy, C. Béné, L. Doyen, J. Perea, and D. Mills. Viability and resilience of small-scale fisheries through cooperative arrangements. *Environment and Development Economics*, 21(6):713–741, Dec. 2016. ISSN 1355-770X, 1469-4395. doi: 10.1017/S1355770X16000152. URL https://www.cambridge.org/core/product/identifidier/S1355770X16000152/type/journal_article.
- G. W. Harrison. Stability under Environmental Stress: Resistance, Resilience, Persistence, and Variability. *The American Naturalist*, 113(5):659–669, May 1979. ISSN 0003-0147, 1537-5323. doi: 10.1086/283424. URL <https://www.journals.uchicago.edu/doi/10.1086/283424>.
- C. S. Holling. Resilience and stability of ecological systems. *Annual review of ecology and systematics*, 4(1): 1–23, 1973. Publisher: Annual Reviews 4139 El Camino Way, PO Box 10139, Palo Alto, CA 94303-0139, USA.
- B. J. Huntley. *Herbivory: Mammalian Grazers and Browsers*, pages 167–182. Springer International Publishing, Cham, 2023. ISBN 978-3-031-18923-4. doi: 10.1007/978-3-031-18923-4_8. URL https://doi.org/10.1007/978-3-031-18923-4_8.

- K. J. Jeffery, L. Korte, F. Palla, G. M. Walters, L. White, and K. Abernethy. *Fire management in a changing landscape: a case study from Lopé National Park, Gabon*, volume 20.1. International Union for Conservation of Nature and Natural Resources, 2014. URL <https://doi.org/10.2305/iucn.ch.2014.parks-20-1.kjj.en>.
- G. Karacaoglu and J. B. Krawczyk. Public policy, systemic resilience and viability theory. *Metroeconomica*, 72(4):826–848, 2021. doi: <https://doi.org/10.1111/meca.12349>. URL <https://onlinelibrary.wiley.com/doi/abs/10.1111/meca.12349>.
- H. K. Khalil and S. W. Shaw. *Stability Theory, Nonlinear*. John Wiley & Sons, Ltd, 1999. ISBN 9780471346081. doi: <https://doi.org/10.1002/047134608X.W1042>. URL <https://onlinelibrary.wiley.com/doi/abs/10.1002/047134608X.W1042>.
- C. E. R. Lehmann, S. A. Archibald, W. A. Hoffmann, and W. J. Bond. Deciphering the distribution of the savanna biome. *New Phytol.*, 191(1):197–209, 2011. URL <https://doi.org/10.1111/j.1469-8137.2011.03689.x>.
- S. Martin. The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication. *Ecology and Society*, 9(2), 2004. Publisher: JSTOR.
- V. Martinet, O. Thébaud, and L. Doyen. Defining viable recovery paths toward sustainable fisheries. *Ecological Economics*, 64(2):411–422, Dec. 2007. ISSN 09218009. doi: 10.1016/j.ecolecon.2007.02.036. URL <https://linkinghub.elsevier.com/retrieve/pii/S0921800907001656>.
- E. Ostrom. A General Framework for Analyzing Sustainability of Social-Ecological Systems. *Science*, 325(5939):419–422, July 2009. doi: 10.1126/science.1172133. URL <https://www.science.org/doi/10.1126/science.1172133>. Publisher: American Association for the Advancement of Science.
- A. Oubraham and G. Zaccour. A survey of applications of viability theory to the sustainable exploitation of renewable resources. *Ecological economics*, 145:346–367, 2018. Publisher: Elsevier.
- S. L. Pimm. The complexity and stability of ecosystems. *Nature*, 307(5949):321–326, Jan. 1984. ISSN 0028-0836, 1476-4687. doi: 10.1038/307321a0. URL <http://www.nature.com/articles/307321a0>.
- É. E. Plaganyi, A. E. Punt, R. Hillary, E. B. Morello, O. Thébaud, T. Hutton, R. D. Pillans, J. T. Thorson, E. A. Fulton, A. D. Smith, and others. Multispecies fisheries management and conservation: tactical applications using models of intermediate complexity. *Fish and Fisheries*, 15(1):1–22, 2014. Publisher: Wiley Online Library.
- A. E. Quinlan, M. Berbés-Blázquez, L. J. Haider, and G. D. Peterson. Measuring and assessing resilience: broadening understanding through multiple disciplinary perspectives. *Journal of Applied Ecology*, 53(3): 677–687, June 2016. ISSN 00218901. doi: 10.1111/1365-2664.12550. URL <http://doi.wiley.com/10.1111/1365-2664.12550>.
- J. Rockström, W. Steffen, K. Noone, \. Persson, F. S. Chapin, E. F. Lambin, T. M. Lenton, M. Scheffer, C. Folke, H. J. Schellnhuber, and others. A safe operating space for humanity. *nature*, 461(7263):472–475, 2009. Publisher: Nature Publishing Group.
- M. Sankaran, N. P. Hanan, R. J. Scholes, J. Ratnam, D. J. Augustine, B. S. Cade, J. Gignoux, S. I. Higgins, X. Le Roux, F. Ludwig, J. Ardo, F. Banyikwa, A. Bronn, G. Bucini, K. K. Caylor, M. B. Coughenour, A. Diouf, W. Ekaya, C. J. Feral, E. C. February, P. G. H. Frost, P. Hiernaux, H. Hrabar, K. L. Metzger, H. H. T. Prins, S. Ringrose, W. Sea, J. Tews, W. J., and N. Zambatis. Determinants of woody cover in african savannas. *Nature*, 438(7069):846–849, 2005. URL <http://dx.doi.org/10.1038/nature04070>.
- M. B. Schaefer. Some aspects of the dynamics of populations important to the management of the commercial marine fisheries. *Inter-American Tropical Tuna Commission Bulletin*, 1(2):23–56, 1954.
- R. Scholes and S. R. Archer. Tree-grass interactions in savannas. *Annu. Rev. Ecol. Evol. Syst.*, pages 517–544, 1997. URL <http://dx.doi.org/10.1146/annurev.ecolsys.28.1.517>.
- R. Scholes and B. Walker. *An African savanna: Synthesis of the Nylsvley study*. Cambridge University Press, 1993.

- R. J. Scholes. Convex relationships in ecosystems containing mixtures of trees and grass. *Environ. Resour. Econ.*, 26(4):559–574, 2003. URL <https://doi.org/10.1023/B:EARE.0000007349.67564.b3>.
- A. Schuhbauer and U. R. Sumaila. Economic viability and small-scale fisheries-A review. *Ecological Economics*, 124:69–75, 2016. URL http://ac.els-cdn.com/S092180091630132X/1-s2.0-S092180091630132X-main.pdf?_tid=b3a751dc-b18c-11e6-9133-00000aab0f02&acdnat=1479913035_05838483be6bc12fe950da520adb8f40. Publisher: Elsevier.
- A. C. Staver, J. O. Abraham, G. P. Hempson, A. T. Karp, and J. T. Faith. The past, present, and future of herbivore impacts on savanna vegetation. *Journal of Ecology*, 109(8):2804–2822, 2021. doi: <https://doi.org/10.1111/1365-2745.13685>. URL <https://besjournals.onlinelibrary.wiley.com/doi/abs/10.1111/1365-2745.13685>.
- A. Tchuinté Tamen, Y. Dumont, J. J. Tewa, S. Bowong, and P. Couteron. A minimalistic model of tree–grass interactions using impulsive differential equations and non-linear feedback functions of grass biomass onto fire-induced tree mortality. *Math. Comput. Simulation*, 133:265–297, 2017. URL <http://dx.doi.org/10.1016/j.matcom.2016.03.008>.
- E. Tromeur and L. Doyen. Optimal Harvesting Policies Threaten Biodiversity in Mixed Fisheries. *Environmental Modeling & Assessment*, 24(4):387–403, Aug. 2019. ISSN 1420-2026, 1573-2967. doi: [10.1007/s10666-018-9618-2](https://doi.org/10.1007/s10666-018-9618-2). URL <http://link.springer.com/10.1007/s10666-018-9618-2>.
- F. Van Langevelde, C. Van De Vijver, L. Kumar, J. Van De Koppel, N. De Ridder, J. Van Andel, A. Skidmore, J. Hearne, L. Stroosnijder, W. Bond, et al. Effects of fire and herbivory on the stability of savanna ecosystems. *Ecology*, 84(2):337–350, 2003. URL <https://www.jstor.org/stable/3107889>.
- S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer-Verlag New York, 2003. URL <https://doi.org/10.1007/b97481>.
- I. Yatat-Djeumen, Y. Dumont, A. Doizy, and P. Couteron. A minimalistic model of vegetation physiognomies in the savanna biome. *Ecological Modelling*, 440:109381, 2021. ISSN 0304-3800. doi: <https://doi.org/10.1016/j.ecolmodel.2020.109381>. URL <https://www.sciencedirect.com/science/article/pii/S0304380020304452>.
- V. Yatat-Djeumen and Y. Dumont. FKPP equation with impulses on unbounded domain. In *Mathematical Methods and Models in Biosciences - BIOMATH 2017*, number 1, pages 1–21, 2018. URL <http://dx.doi.org/10.11145/texts.2017.11.157>.
- V. Yatat-Djeumen, P. Couteron, J. J. Tewa, S. Bowong, and Y. Dumont. An impulsive modelling framework of fire occurrence in a size-structured model of tree–grass interactions for savanna ecosystems. *J. Math. Biol.*, 74:1425–1482, 2017. URL <http://dx.doi.org/10.1007/s00285-016-1060-y>.
- V. Yatat-Djeumen, A. Tchuinté Tamen, Y. Dumont, and P. Couteron. A tribute to the use of minimalistic spatially-implicit models of savanna vegetation dynamics to address broad spatial scales in spite of scarce data. *BIOMATH*, 7:1812167, 2018. URL <http://dx.doi.org/10.11145/j.biomath.2018.12.167>.

A Proof of Proposition 1

Let us consider the harvest effort vector $E = (e_G^{MSY}, e_T^{MEY})$ together with state vector $X = (G^{MSY}, T^{MEY})$. Let us prove that $X \in \text{Invar}(E)$.

First it turns out that the state-control (E, X) fulfills the bio-economic constraints. From the very definition G^{MSY} and e_G^{MSY} in (11) along with the second assumption of (13), we indeed have

$$H_G(G^{MSY}, e_G^{MSY}) = H_G^{MSY} \geq H_G^{\text{lim}}.$$

Thus viability constraints (5) also holds true.

Similarly, from definitions (10) and assumptions (13), we also have

$$\pi_T(e_T^{MEY}, T^{MEY}) = \pi_T^{MEY} \geq \pi_T^{\text{lim}}.$$

Thus profit constraints (3) holds true.

Observe also that the coexistence constraint is satisfied since both grass and tree states are strictly positive namely $G^{MSY} > 0$ and $T^{MEY} > 0$.

Let us now prove that the state X is an equilibrium of dynamics (1) associated with efforts E . We first have

$$\begin{aligned} r_T \left(1 - \frac{T^{MEY}}{K_T}\right) - q_T e_T^{MEY} &= r_T \left(1 - \frac{\frac{1}{2}(K_T + T^{OA})}{K_T}\right) - q_T \frac{r_T}{2q_T} \left(1 - \frac{T^{OA}}{K_T}\right), \\ &= r_T \left(\frac{1}{2} - \frac{T^{OA}}{2K_T} - \frac{1}{2} + \frac{T^{OA}}{2K_T}\right) = 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} r_G \left(1 - \frac{G^{MSY}}{K_G}\right) - \eta_{TG} T^{MEY} - q_G e_G^{MSY}, \\ = r_G \left(1 - \frac{K_G \left(1 - \frac{2\eta_{TG} T^{MEY}}{r_G}\right)}{2K_G}\right) - \eta_{TG} T^{MEY} - \frac{r_G}{2}, \\ = r_G \left(\frac{1}{2} + \frac{\eta_{TG} T^{MEY}}{r_G}\right) - \eta_{TG} T^{MEY} - \frac{r_G}{2} = 0. \end{aligned}$$

Consequently, X is a steady state of the system (1). Therefore, $X \in \text{Invar}(E)$. Consequently, from inclusion (8), we deduce that $X \in \text{Viab}$.

B Proof of Proposition 2

Let us consider again the harvest effort vector $E = (e_G^{MSY}, e_T^{MEY})$.

First it turns out that any state-controls (X, E) where $X \in \mathcal{A}$ fulfills the bio-economic constraints.

From the very definition of G^{PA} in (15) and (16) together with the very definition of e_G^{MSY} in (11) along with the second assumption of (13), we indeed have:

$$G \geq G^{\text{PA}} \implies H_G(G, e_G^{MSY}) \geq H_G(G^{\text{PA}}, e_G^{MSY}) = q_G e_G^{MSY} \frac{H_G^{\text{lim}}}{q_G e_G^{MSY}} = H_G^{\text{lim}}.$$

Thus viability constraints (5) also holds true.

Similarly, from definitions (10) and assumptions (13), we have

$$T \geq T^{\text{PA}} \implies \pi_T(T, e_T^{MEY}) \geq \pi_T(T^{\text{PA}}, e_T^{MEY}) = p_T q_T e_T^{MEY} \frac{\pi_T^{\text{lim}} + c_T e_T^{MEY}}{p_T q_T e_T^{MEY}} - c_T e_T^{MEY} = \pi_T^{\text{lim}}.$$

It means that profit constraints (3) holds true.

Let us now prove that the set \mathcal{A} is invariant for system (1) associated with efforts E . Following tangential conditions⁸ of Aubin [1990], we need to distinguish between five cases corresponding to the 3 sides and 2 corners of rectangle \mathcal{A} :

⁸Consider any set A defined in terms of inequalities as follows:

$$A = \{x \in \mathbb{R}^n, a_j(x) \geq 0, j = 1, \dots, m\}$$

with $a_j(\cdot)$ continuously differentiable functions. Consider any dynamic system

$$\dot{x} = f(x).$$

Invariance tangential conditions are

$$\forall x, \forall j \text{ s.t. } a_j(x) = 0, \frac{\partial a_j}{\partial x}(x) \cdot f(x) \geq 0.$$

- **Side 1: (bottom horizontal side):** $T = T^{\text{PA}}$ and $G > G^{\text{PA}}$: As $T^{\text{PA}} \leq T^{\text{MEY}}$ from assumption (13) and definition (16), we have

$$\frac{\dot{T}}{T} = r_T \left(1 - \frac{T}{K_T}\right) - q_T e_T^{\text{MEY}} \geq r_T \left(1 - \frac{T^{\text{MEY}}}{K_T}\right) - q_T e_T^{\text{MEY}} = 0.$$

Thus T increases and thus remains within the rectangle \mathcal{A} .

- **Side 2 (left vertical side):** $G = G^{\text{PA}}$ and $T^{\text{PA}} < T < T^\sharp$: In that case, using the very definition of the equilibrium function $T^*(G, e_G)$ in (17), we claim that

$$\begin{aligned} \frac{\dot{G}}{G} &= r_G \left(1 - \frac{G}{K_G}\right) - \eta_{TG} T - q_G e_G^{\text{MSY}} > r_G \left(1 - \frac{G}{K_G}\right) - \eta_{TG} T^*(G, e_G^{\text{MSY}}) - q_G e_G^{\text{MSY}}, \\ &= r_G \left(1 - \frac{G}{K_G}\right) - \left(r_G \left(1 - \frac{G}{K_G}\right) - q_G e_G^{\text{MSY}}\right) - q_G e_G^{\text{MSY}} = 0. \end{aligned}$$

Thus G (strictly) increases and thus remains within the rectangle \mathcal{A} .

- **Side 3 (top horizontal side):** $T = T^\sharp$ and $G > G^{\text{PA}}$: As $G^{\text{PA}} \leq G^{\text{MSY}}$ from assumption (13) and $T^*(G)$ is a decreasing function of G , we have

$$T = T^*(G^{\lim}(e_G^{\text{MSY}}, e_G^{\text{MSY}})) \geq T^*(G^{\text{MSY}}, e_G^{\text{MSY}}) = T^{\text{MEY}}$$

and consequently

$$\frac{\dot{T}}{T} = r_T \left(1 - \frac{T}{K_T}\right) - q_T e_T^{\text{MEY}} \leq r_T \left(1 - \frac{T^{\text{MEY}}}{K_T}\right) - q_T e_T^{\text{MEY}} = 0.$$

Thus T decreases and thus remains within the rectangle \mathcal{A} .

- **Corner 1 (left bottom):** $T = T^{\text{PA}}$ and $G = G^{\text{PA}}$: Using arguments similar to previous cases, namely $G = G^{\text{PA}} \leq G^{\text{MSY}}$ and $T = T^{\text{PA}} \leq T^{\text{MEY}}$ from (13), we first can write

$$\begin{aligned} \frac{\dot{G}}{G} &= r_G \left(1 - \frac{G}{K_G}\right) - \eta_{TG} T - q_G e_G^{\text{MSY}} = r_G \left(1 - \frac{G^{\text{PA}}}{K_G}\right) - \eta_{TG} T^{\text{PA}} - q_G e_G^{\text{MSY}}, \\ &\geq r_G \left(1 - \frac{G^{\text{MSY}}}{K_G}\right) - \eta_{TG} T^{\text{MEY}} - q_G e_G^{\text{MSY}} = 0. \end{aligned}$$

since joint state $(G^{\text{MSY}}, T^{\text{MEY}})$ is at equilibrium. Similarly, we have

$$\begin{aligned} \frac{\dot{T}}{T} &= r_T \left(1 - \frac{T}{K_T}\right) - q_T e_T^{\text{MEY}} = r_T \left(1 - \frac{T^{\text{PA}}}{K_T}\right) - q_T e_T^{\text{MEY}}, \\ &\geq r_T \left(1 - \frac{T^{\text{MEY}}}{K_T}\right) - q_T e_T^{\text{MEY}} = 0, \end{aligned}$$

since T^{MEY} is at equilibrium.

Thus both states T and G increase and thus remains within the rectangle \mathcal{A} .

- **Corner 2 (left top):** $T = T^\sharp$ and $G = G^{\text{PA}}$: Using similar arguments, in that case, we have $\dot{T} = 0$ and $\dot{G} \geq 0$.

We conclude that all along the boundary of the rectangle \mathcal{A} we have the required tangential conditions for invariance and thus for viability. Consequently, \mathcal{A} is an invariant set of dynamics (1) with control E . Therefore, $\mathcal{A} \subset \mathbb{I}nvar(E)$. Furthermore, from inclusion (8), we deduce that $\mathcal{A} \subset \mathbb{V}iab$.

C Proof of Proposition 3

We consider again the harvest effort vector $E = (e_G^{MSY}, e_T^{MEY})$ together with state vector $X = (G^{MSY}, T^{MEY}) \in \mathcal{A}$. From Proposition 1, we already know that X is an equilibrium point of system (1) and that $X \in \text{Invar}(E)$.

Moreover, it turns out that X is LAS. Indeed, the Jacobian matrix for system (1) at X is

$$J(X) = J(G^{MSY}, T^{MEY}) = \begin{pmatrix} -\frac{r_G}{K_G} G^{MSY} & -\eta_{TG} G^{MSY} \\ 0 & -\frac{r_T}{K_T} T^{MEY} \end{pmatrix}.$$

$J(X)$ is a triangular matrix. Hence its eigenvalues are $\lambda_1 = -\frac{r_G}{K_G} G^{MSY}$ and $\lambda_2 = -\frac{r_T}{K_T} T^{MEY}$ which are both (strictly) negative. Hence, the equilibrium point X is LAS. Moreover, system (1) is dissipative and does not admit close orbits. Therefore, it follows from the Poincaré-Bendixson theorem [Wiggins, 2003, Theorem 9.0.6, page 120] together with the LAS property of X , that the equilibrium X is GAS. Hence, when it exists, the savanna equilibrium $(G^{MSY}, T^{MEY}) \in \mathcal{A}$ is GAS.