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# BOUNDED CONFIDENCE MODEL ON GROWING POPULATIONS

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This paper studies the bounded confidence model on growing fully-mixed populations. In this model, in addition to the usual opinion clusters, significant secondary clusters of smaller size appear systematically, while those secondary clusters appear erratically and include much fewer agents when the population is fixed. Through simulations, we derive the bifurcation diagram of the growing population model and compare it to the diagram obtained with an evolving probability density instead of agents, and with their equivalent having a fixed population. Our tests, when changing the usual bounded confidence function into a smooth bounded confidence function, suggest that these secondary clusters are mainly generated by a different mechanism when the population is growing than when it is fixed.

*Keywords*: Agent-based models; growing populations; opinion dynamics; bounded confidence models.

## 1. Introduction

Opinion dynamics models express mathematically some hypotheses about social interactions and provide means to investigate their effect on large populations of interacting agents. In particular, bounded confidence models [5, 10], assume that when an agent's opinion is too far from the one of its interlocutor, it has no influence.

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This assumption is grounded in well established results in social psychology [11]. Running this model on a population where all the agents hold the same confidence bound leads to one or more separate clusters of opinions, depending on the value of the confidence bound. Many papers are devoted to studying these models and their variants [14]. Early papers study the model in which all agents have the same confidence bound, showing the details of the bifurcation diagrams when model parameters change. Other researchers investigate populations with different confidence bounds. Several studies focus on including the so-called extremists agents, whose opinion is at the border of the opinion interval [4, 6, 17]. Others consider agents with confidences drawn in a given interval [20]. Many papers assess the effect of different types of networks of interactions [9, 15, 21], and in some of them the topology is allowed to change in terms of the opinion dynamics [2, 12]. Introducing noise in these models also significantly modifies their qualitative behavior [7, 18, 19]. Several review papers are totally or partly devoted to these models [3, 8, 14].

In this paper, we study the bounded confidence model when new agents are progressively added to the population. As far as we know, the model has not been studied in these conditions yet. The growing population can be related to online communities of agents that appear and then may grow more or less rapidly. Recent models inspired by the physics of gels address, more specifically, the dynamics of aggregation and desegregation of online groups [16]. In this paper, we focus on the simple case of a fully mixed population and on the growing phase of the group.

The main new feature appearing when the population is growing is the emergence of significant and stable secondary clusters, located approximately at one confidence bound distance of the main clusters. Secondary or minor clusters were already identified in the fixed population model, more particularly in the density version of the model [1]. However, in simulations of the agent models, the secondary clusters do not appear systematically and when they do, they generally include very few isolated agents. Hence the secondary clusters are often ignored in papers about agent models. However, when the population is growing, it is impossible to ignore them as they appear systematically and include significant numbers of agents, though remaining much smaller than the number of agents in the primary clusters.

These observations led us to question the origin of the secondary clusters. With this aim, we also run simulations of the model with a smooth influence function. Actually, as noticed by several scholars [4, 6], the bounded confidence influence function shows a strong discontinuity when the distance of the opinions are around the confidence bound, which seems difficult to justify psychologically. We propose a smooth influence function that eliminates this discontinuity and is simpler than the previous versions of smooth bounded confidence influence functions [4, 6]. The analysis using the smooth influence function helps us to uncover that the main mechanism responsible for the systematic presence of secondary clusters in the version with a growing population is the arrival of new agents in regions that are already free of attraction by the primary clusters while the secondary clusters appearing in the model with a fixed population are due to the discontinuity in the interaction function.

This paper is arranged as follows. Section 2 presents the model in all the versions considered in the paper: agent-based, density-based, growing or fixed population, and finally, the smooth influence function. Section 3 is devoted to the simulation results, where we compare the models in detail, showing their similarities and differences. Section 4 proposes a discussion of these results.



## 2. Methods

### 2.1. *Agent model with standard influence*

We consider a population of $N_t$ agents growing with time $t$, from an initial size of $N_0$. An agent $i \in \{1, \ldots, N_t\}$ is characterized by an opinion $a_i(t) \in [0, 1]$. All the agents share the same confidence bound $\epsilon$. While a time limit $T$ is not reached, the algorithm repeats choosing a couple of agents at random and performing the classical bounded confidence interaction as in [5]. Moreover, after each interaction, with probability $\frac{\delta_N}{N_t}$, $N_t$ being the current population size and $\delta_N \leq N_0$ being a parameter, a new agent is added to the population with an opinion uniformly drawn in the opinion interval $[0, 1]$.

We assume that the time needed for each agent to interact once on average is constant. Indeed, we assume that our sequential program represents interactions that take place mostly in parallel so that the time for every agent of the population to perform one interaction is constant whatever the population size. This explains why, in the algorithm, the time $t$ is incremented of $1/N_t$ at each interaction. With this definition of time, in the sequential program, one interaction takes less and less time as the population grows in order to account for a larger and larger number of parallel interactions.

Then, we assume a constant number $\delta_N$ of incoming agents over each time unit. Indeed, a new agent arrives into the population with probability $\delta_N/N_t$ after each interaction, which results in $\delta_N$ incoming agent on average every time unit. This assumption can be related to the Barabasi–Albert networks, where the number of entering agents is constant, independently from the system size.

Moreover, we assume that this constant growing flow takes place during a given number of time units $T$ and we focus on this period of constant growth.

More precisely, the algorithm is as follows:

(1) Set $T$ the maximum time of the simulation;
(2) Set $t = 0$;
(3) Initialize $N_0$ agents, with opinions uniformly drawn in $[0, 1]$;
(4) While $t < T$:

   (a) Choose two distinct agents $i$ and $j$ at random and

$$\text{If } |a_i(t) - a_j(t)| < \epsilon \text{ then} \begin{cases} a_i(t+1) = a_i(t) + \mu(a_j(t) - a_i(t)), \\ a_j(t+1) = a_j(t) + \mu(a_i(t) - a_j(t)), \end{cases} \quad (1)$$

where $\mu$ is a parameter of the model, fixed to 0.5 in our simulations;
(b) With probability $\delta_N/N_t$, add a new agent to the population with an opinion uniformly drawn in $[0, 1]$;
(c) $N_t :=$ size of population.
(d) $t := t + \frac{1}{N_t}$



### 2.2. *Density model*

Developing a density model approximating the agent version is a classical practice in agent-based modeling (see, for instance, [1, 4]). The principle is to consider the evolution of the distribution for the probability of the presence of agents instead of considering the agents themselves. The density model can be seen as an agent model with an infinite number of agents starting from a perfectly uniform distribution and synchronous interactions. Comparing agent and density models allows identifying the effects of irregularities and noise with respect to an ideal case.

In practice, to run the evolution of the density distribution numerically, it is necessary to cut the opinion axis into a large number $M$ of intervals and consider the density of agents in each of these intervals. Therefore, the system's state is a vector $d(t) = (d_1, \ldots, d_M)$ of $M$ continuous values, approximating the continuous density. The algorithm is based on the agent model rules and computes the probabilities that the density increases or decreases in each interval. This is generally done through a master equation expressing the inflow and outflow sum at each interval. The repeated action of these changes at each interval produces the evolution of the density.

Here, we use an algorithm for applying the master equation by averaging the dynamics of the model and storing all the changes of a time unit into a vector denoted by $\Delta$. Then, all the changes of the density during a time unit are performed at once by adding $\Delta$ to the current density $d(t)$. Overall, the buffer vector $\Delta$ stores the balance of inflows and outflows in each density cell, as the master equation would do.

Then, we add $\delta_N$ agents to the population. Let $\omega = \delta_N/N_0$ be the proportion of the initial population added at each time unit. As the sum of $d_i(t) + \Delta_i$, over all the intervals $i \in (1, \ldots, M)$, is 1, we add the vector $(\frac{\omega}{M(1+\omega t)}, \ldots, \frac{\omega}{M(1+\omega t)})$. This vector is divided by $1 + \omega t$ because the size of the population at $t$ is $N_0(1 + \omega t)$.

Then we normalize $d(t + 1)$. Finally, after the $T$ time units, we multiply $d(t)$ by $(1 + \omega t)/(1 + \omega T)$ for $t \in (0, \ldots, T)$, to get growing values of $d(t)$ like in the agent model.

Indeed, when computing the opinion density in the agent model at a given time step $t$, we count the number of agents in regular intervals of the opinion axis and we divide the number in each interval by the total number of agents at the end of the simulation (at time step $T$). In this way, we visually represent the progressive increase of the number of agents. This increase is visible on Figure 3, showing evolving densities with a small initial number of agents and an increase at each step that is higher in proportion than in the simulations starting with numerous agents shown on Figures 1 and 2.

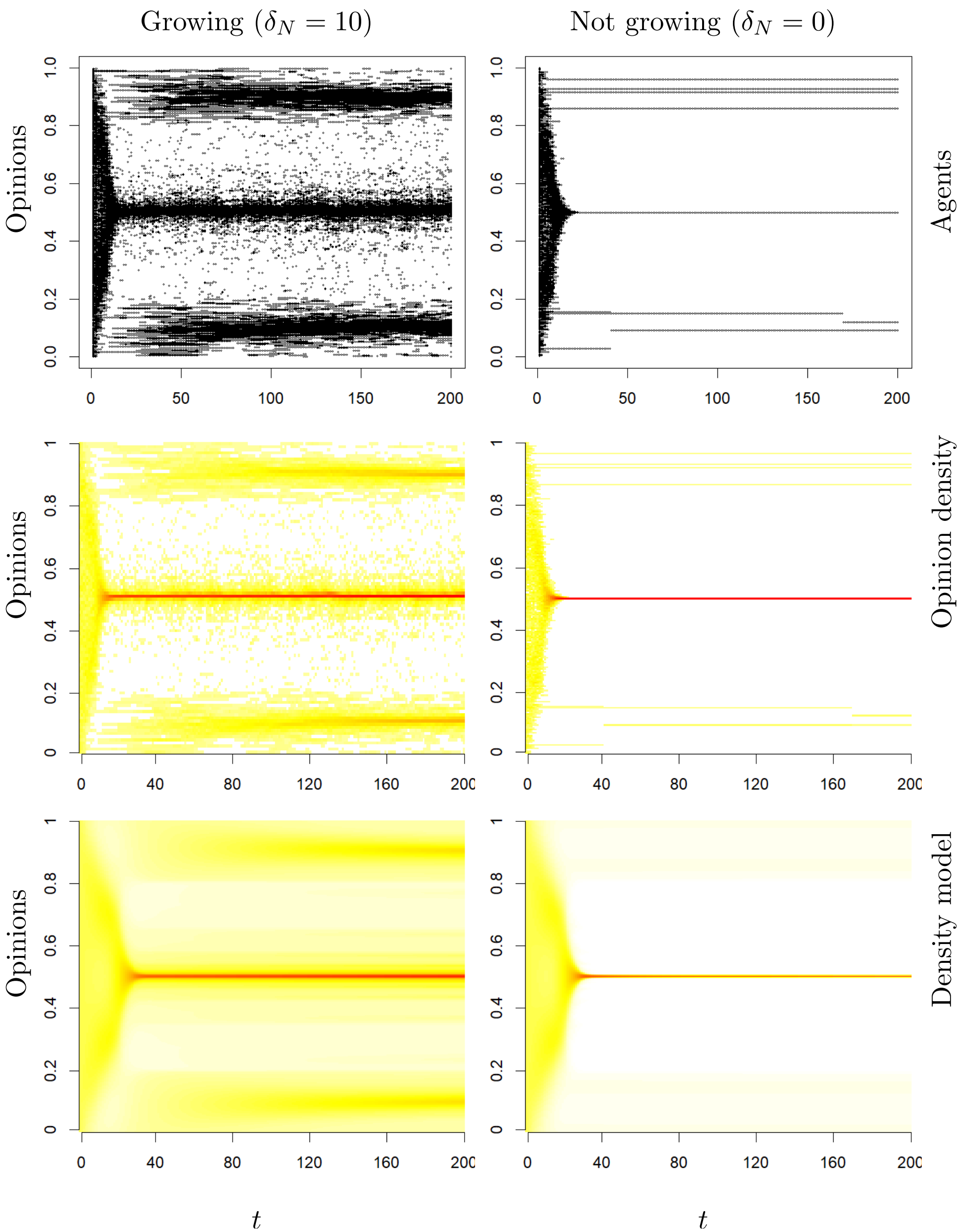


Fig. 1. Comparing growing ($\delta_N = 10$) and non-growing ($\delta_N = 0$) cases for the confidence bound $\epsilon = 0.3$, for both the agent and the density models. The upper panels show the evolution of opinions in one simulation of the agent model. The middle panels represent the evolution of the density of opinions from the same simulation of the agent model. The bottom panels show the evolution of the density model.

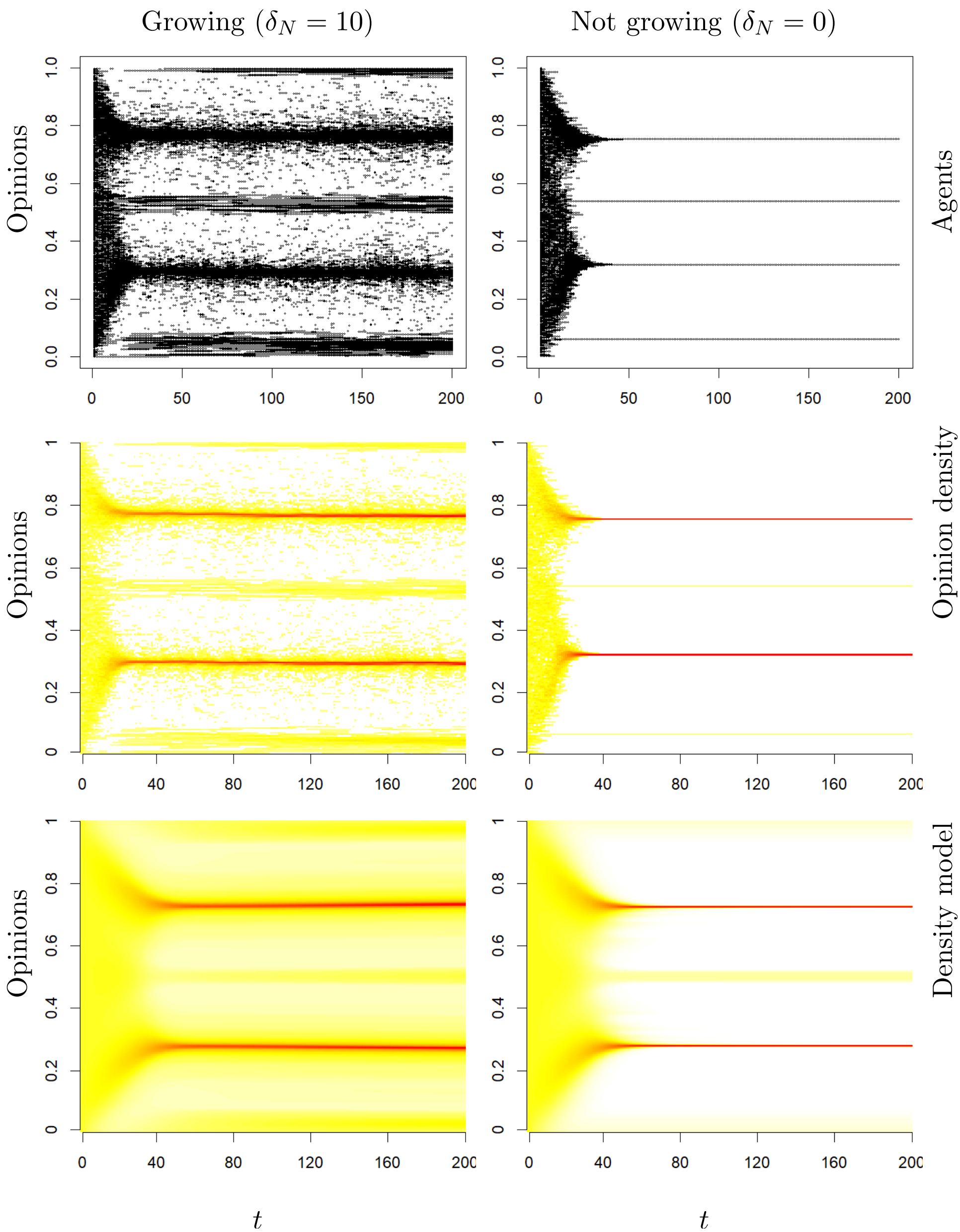


Fig. 2. Comparing growing ($\delta_N = 10$) and non-growing ($\delta_N = 0$) cases for the confidence bound $\epsilon = 0.2$, for both the agent and the density models. The upper panels show the evolution of opinions in a simulation of the agent model. The middle panels represent the evolution of the density of opinions from the agent model for the same simulation. The bottom panels show the evolution of the density model.

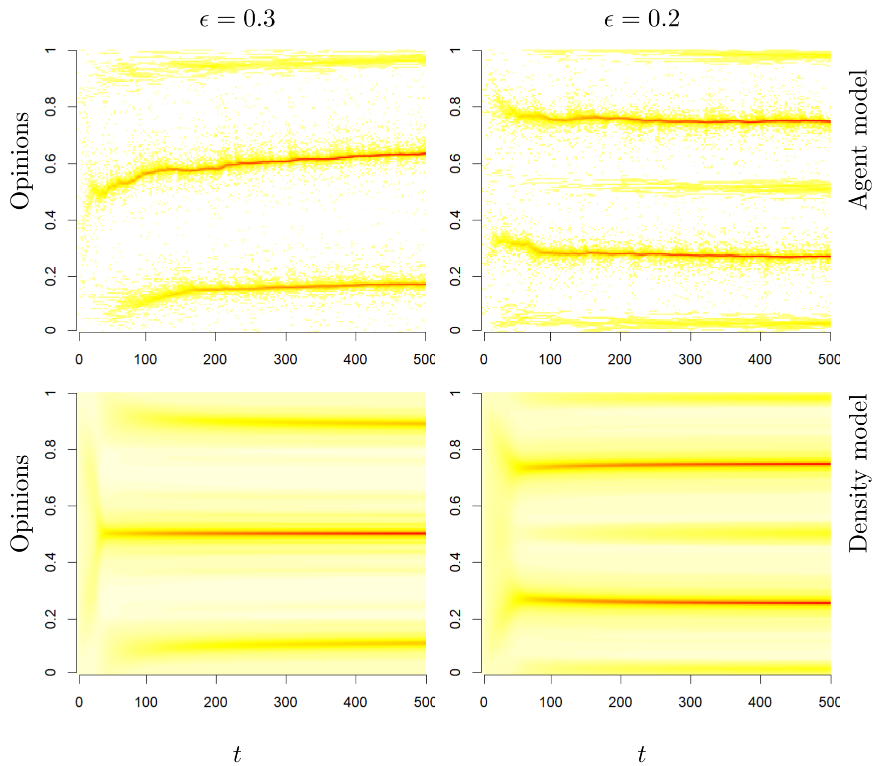


Fig. 3. Comparing the opinion density from one simulation of the agent model (upper panels) and the density model (bottom panels), for the confidence bound $\epsilon = 0.3$ in the left panels and $\epsilon = 0.2$ in the right ones. Initial number of agents is set to $N_0 = 5$, and adding $\delta_N = 5$ agents at each time unit.

The algorithm is as follows:

(1) Initialize $d(0) = (\frac{1}{M}, \ldots, \frac{1}{M})$, $\omega = \delta_N / N_0$;
(2) For $t \in (0, \ldots, T-1)$:
  (a) Initialize $\Delta = (0, \ldots, 0)$ vector of size $M$;
  (b) For $i \in (1, \ldots, M)$
    - For $j \in (1, \ldots, M)$
      – if $|i - j| < \epsilon M$,
        * $k = \text{round}(i + \mu(j - i))$;
        * $\Delta_i := \Delta_i - d_i(t) d_j(t)$;
        * $\Delta_k := \Delta_k + d_i(t) d_j(t)$; ($d_i(t) d_j(t)$ expresses the probability of interaction between agents in sites $i$ and $j$)
  (c) $\omega = \delta_N / N_0$;

(d) For $i \in (1, \ldots, M)$,

$$d_i(t+1) = d_i(t) + \Delta_i + \frac{\omega}{M(1+\omega t)};$$

(e)

$$d_i(t+1) := \frac{d_i(t+1)}{\sum_{j=1}^{M} d_j(t+1)}$$

(3) For $t \in (0, \ldots, T)$,

$$d(t) = \frac{1+\omega t}{1+\omega T} d(t)$$

### 2.3. *Agent model with smooth influence*

With the standard influence rule, the influence of agent $i$ on agent $j$ is strongly discontinuous when the distance of the opinions $|\theta_{ij}(t)| = |a_i(t) - a_j(t)|$ is around $\epsilon$. Indeed, if $|\theta_{ij}(t)|$ is slightly below $\epsilon$, the change of opinion $|a_j(t+1) - a_j(t)|$ is close to $\mu\epsilon$, which is the maximum possible change. However, if $|\theta_{ij}(t)|$ reaches $\epsilon$, the change suddenly drops to 0.

This discontinuity seems difficult to justify psychologically to the eyes of some authors who proposed some changes in the influence function that eliminate this discontinuity (see e.g., [4, 6]). In this paper, we propose the following variant, with $\theta_{ij}(t) = a_i(t) - a_j(t)$:

$$\text{If } |d_{ij}(t)| < \epsilon \text{ then} \begin{cases} a_i(t+1) = a_i(t) + \mu\theta_{ji}(t)\left(1 - \dfrac{|\theta_{ji}(t)|}{\epsilon}\right), \\ a_j(t+1) = a_j(t) + \mu\theta_{ij}(t)\left(1 - \dfrac{|\theta_{ij}(t)|}{\epsilon}\right). \end{cases} \tag{2}$$

We refer to this version as the smooth influence and to the previous version as the standard influence. Qualitatively, for the same value of parameter $\mu$, opinions change significantly slower with the smooth than with the standard influence and there is no sharp limit between influence and no influence.

The smooth influence can, of course, be introduced in both versions: the agent model and the density model. In the following, we compare the agent and density models with standard and smooth influence.

## 3. Results

### 3.1. *Simulations with the standard influence function*

#### Examples for $N_0 = 1000$ and $N_0 = 5$

Figures 1 and 2 compare the model simulations with the standard influence function, for both the agent and the density models. We also compare the case of growing population, adding $\delta_N = 10$ agents at each time unit (left) with constant population $\delta_N = 0$ (right). The upper panels show the evolution of opinions in one simulation of the agent model. Then, the panels of the middle row represent the density of opinions, from the same simulation of the agent model. The density of opinions is obtained by counting the number of opinions in 200 regular intervals of the opinion axis. This number is divided by the total number of agents at the end of the simulation. Finally, the bottom panels show the result of the density model for $\omega = \delta_N/N_0 = 0.01$. Indeed, in this density model, the initial number $N_0$ cannot be taken into account, as the initial state is a density in which individual agents cannot be distinguished nor counted. Figure 1 shows the case of $\epsilon = 0.3$, while Fig. 2 shows for $\epsilon = 0.2$. We are comparing the cases of epsilon $= 0.3$ and $0.2$ because these are the cases where one and two main clusters appear, respectively, in the model with constant population.

The main observations from these first examples are as follows:

- In the growing population model, the secondary clusters are significantly larger than in the non-growing case.
- In the growing model, there is a density of agents remaining around the major clusters, while this density is null around the major clusters in the non-growing model.
- In general, the results of the agent and density models are similar.

In the above examples, the simulations started with a high number of participants ($N_0 = 1000$), with respect to $\delta_N = 10$, in Fig. 3 we consider the case of a small initial population ($N_0 = 5$) and adding $\delta_N = 5$ agents at each time unit. The upper panels show examples of the agent model. The bottom panels show the density model for $\delta_N/N_0 = \omega = 1$.

We observe more differences between the agent and the density models in this case than when the initial population is large. Indeed, we note more variations in the positions of the clusters in the agent model than in the density model, especially at the beginning of the simulations, when the population remains small. This could be expected, as the density model should rather be considered as the average result of an infinity of simulations of the agent model, and starting with a small population introduces stronger initial variations from one simulation to the other in the agent model.

*Configurations of clusters when the confidence bound* $\boldsymbol{\epsilon}$ *varies*

Figure 4 shows the positions of the clusters for the density model when growing ($\omega = 1$) and not growing ($\omega = 0$), as a function of $1/2\epsilon$. The figure distinguishes between primary, secondary and intermediate clusters. The distinction is based on the effective weight of the cluster (the computation of effective weight is described in Appendix A). Primary clusters are defined by having an effective weight higher than 0.6, secondary clusters having an effective weight lower than 0.2, and intermediate clusters by an effective weight between 0.2 and 0.6.

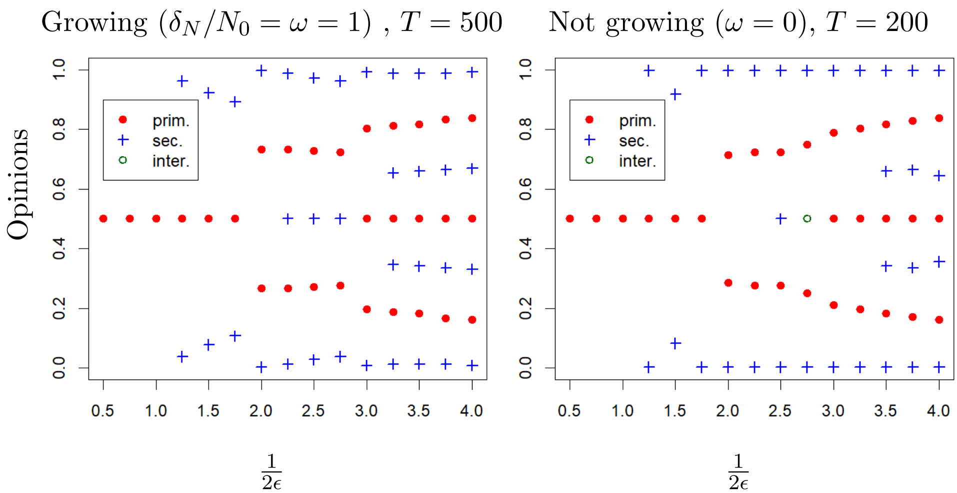


Fig. 4. Positions of clusters from the density model for both growing and non-growing cases for different values $\frac{1}{2\epsilon}$. The primary (prim. in the legend) clusters are such that their effective weight is higher than 0.6, the secondary clusters (sec. in the legend) are such that their effective weight is lower than 0.2. The intermediate clusters (inter. in the legend) are such that their effective weight is between 0.2 and 0.6. See the appendix for the definition of the effective weight of a cluster.

Overall, the growing and non-growing cases yield similar patterns of cluster positions. A few differences are, however, noticeable: for $\frac{1}{2\epsilon} = 2.25$, a central secondary cluster is detected only in the growing case. Similarly, for $\frac{1}{2\epsilon} = 3.25$, the growing case shows two secondary clusters more than the non-growing case. For $\frac{1}{2\epsilon} = 2.75$, the central cluster is secondary for the growing case, while it is intermediate for the non-growing case. Finally, the positions of the clusters are a bit different. These positions change more sharply at the transitions in the growing case than in the non-growing one.

The configurations of clusters obtained with the density model are references for the agent model. For a given value of the confidence bound $\epsilon$, depending on the other parameters ($N_0$ and $\delta_N$), the agent model can yield more or less often the same configuration as the density model.

The top panels of Fig. 5 show the average over 100 replicas of the positions of the clusters for the agent-based model in two cases. In one case (left panel), the initial number $N_0 = 1000$, the average number of agents added at each time unit is $\delta_N = 10$, and the cluster positions are measured after $T = 200$ time unit (on average 3000 agents). In the other case (right panel), the initial number $N_0 = 5$, the average number of agents added at each time unit is $\delta_N = 5$, and the cluster positions are measured after $T = 500$ time units (on average 2500 agents). In each case, the figure shows the average of the most frequent configuration (defined by the number of primary clusters) over 100 replicas for each value of $\epsilon$. The definition of the clusters with their effective weight is the same as in Fig. 4. Comparing these results with the

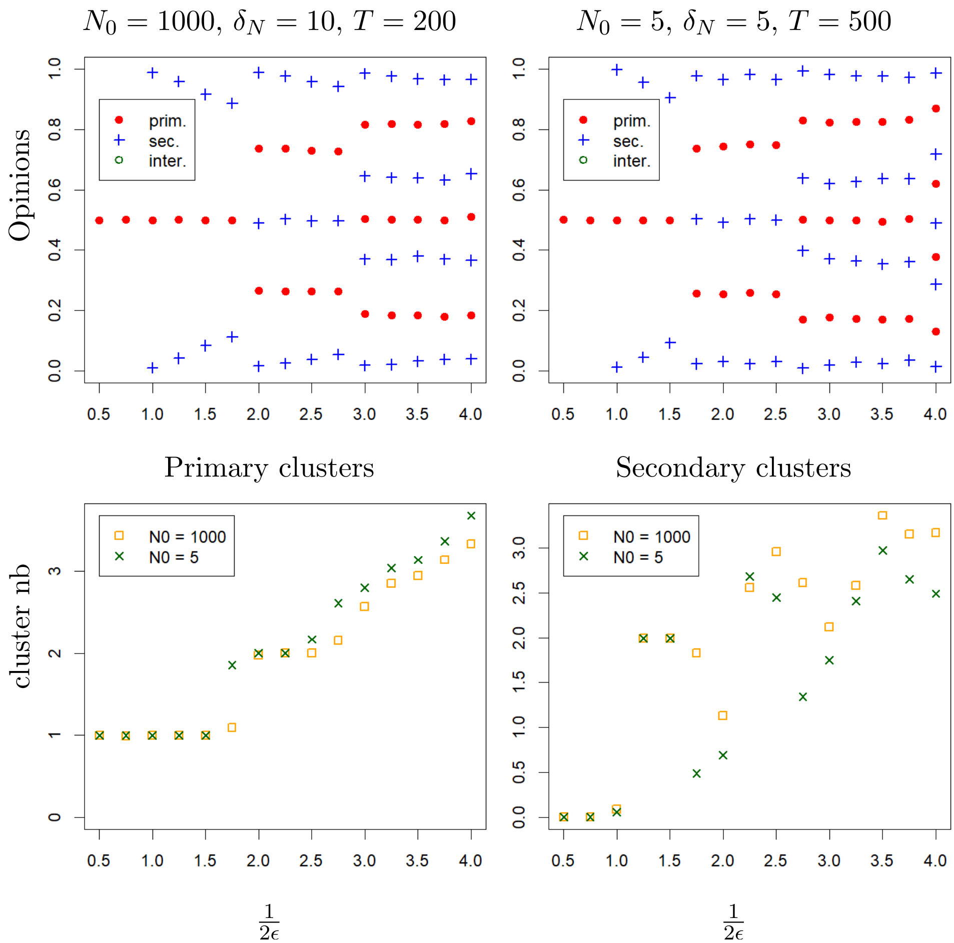


Fig. 5. Positions of clusters (top panels) and average number of clusters (bottom panels) for the agent model having an initial number of agents $N_0 = 1000$, $\delta_N = 10$ and after $T = 200$ time units in the left panel; while $N_0 = 5$, $\delta_N = 5$ and after $T = 500$ time units in the right panel. The figure shows the average of the most frequent configuration (defined by the number of primary clusters) over 100 replicas for each value of $\epsilon$.

results from the density model yields the following main observations:

- The average positions of clusters for $N_0 = 1000$ and $\delta_N = 10$ (left panel of Fig. 5) are very close to the result obtained with the density growing model, even though $\delta_N/N_0 = \omega = 1$ for that density model. The only noticeable difference is that the agent model shows additional secondary clusters at the transitions between 2 and 3, and 3 and 4 primary clusters (for $\frac{1}{2\epsilon} \in \{2, 3\}$),
- The transitions take place at lower values of $\frac{1}{2\epsilon}$ when the initial number of agents is $N_0 = 5$ (right panel) than when $N_0 = 1000$. Indeed, for $N_0 = 5$, three transitions

are observable: between 1 and 2 primary clusters for $\frac{1}{2\epsilon} \in [1.5, 1.75]$, between 2 and 3 primary clusters for $\frac{1}{2\epsilon} \in [2.5, 2.75]$, and between 3 and 4 primary clusters for $\frac{1}{2\epsilon} \in [3.75, 4]$. In contrast, for $N_0 = 1000$, only two transitions are observable: between 1 and 2 primary clusters for $\frac{1}{2\epsilon} \in [1.75, 2]$, and between 2 and 3 primary clusters for $\frac{1}{2\epsilon} \in [2.75, 3]$.



These observations suggest that, in general, a low initial number of agents $N_0$ leads to a higher number of primary clusters in the most frequent configurations (for the same average number of agents $\delta_N$ added at each time unit).

This observation is completed by the bottom panels of Fig. 5, showing the average number of clusters for all the configurations. Indeed, the left panel shows that, for $1/2\epsilon \geq 2.5$, the average number of primary clusters is higher for $N_0 = 5$ than for $N_0 = 1000$ of 0.1 to 0.4, which can be interpreted as there is one more cluster in 10–40% of the simulations. However, on the contrary, the right panel shows that the number of secondary clusters is generally higher for $N_0 = 1000$. This suggests that, for $N_0 = 5$, when the primary clusters are more numerous, they tend to be more often too close to each other to allow the emergence of secondary clusters.

### 3.2. *Simulations with the smooth influence function*

The top panels of Fig. 6 show the positions of the clusters as a function of $\epsilon$ for the density smooth model, for the case of growing (left) and non-growing (right) population. The bottom panels show the simulations of the density model for $\epsilon = 0.2$. It is striking that the secondary clusters are almost absent in the non-growing case. There are some additional secondary clusters in the growing case, which are located at the extremes of the opinion interval. Moreover, the transitions between configurations of numbers of primary clusters take place at lower values of $\frac{1}{2\epsilon}$ than in the standard influence.

These results suggest that the secondary clusters found in the growing case with smooth influence are generated by a different process than the one generating the secondary clusters in the non-growing case, which seems mainly due to the discontinuity of the standard influence function. We summarize our results in Table 1.

Figure 7 is the equivalent of Fig. 5, replacing the standard influence with the smooth one. It shows the average positions of the clusters for the most frequent configuration over 100 replicas of simulations of the agent model in the top panels. The average number of primary and secondary clusters is shown in the bottom panels. The left top panel shows the results of the simulations for an initial number of agents $N_0 = 1000$ and adding on average $\omega N_0 = 10$ agents at each iteration. The right top panel shows the results for $N_0 = 5$ and adding on average $\omega N_0 = 5$ agents at each iteration. The positions of the clusters (top panels) are the average for the most frequent configuration defined by the number of primary clusters over 100 replicas (for each value of $\epsilon$). The average number of clusters (bottom line) is performed on all configurations. The left panel shows the average number of primary

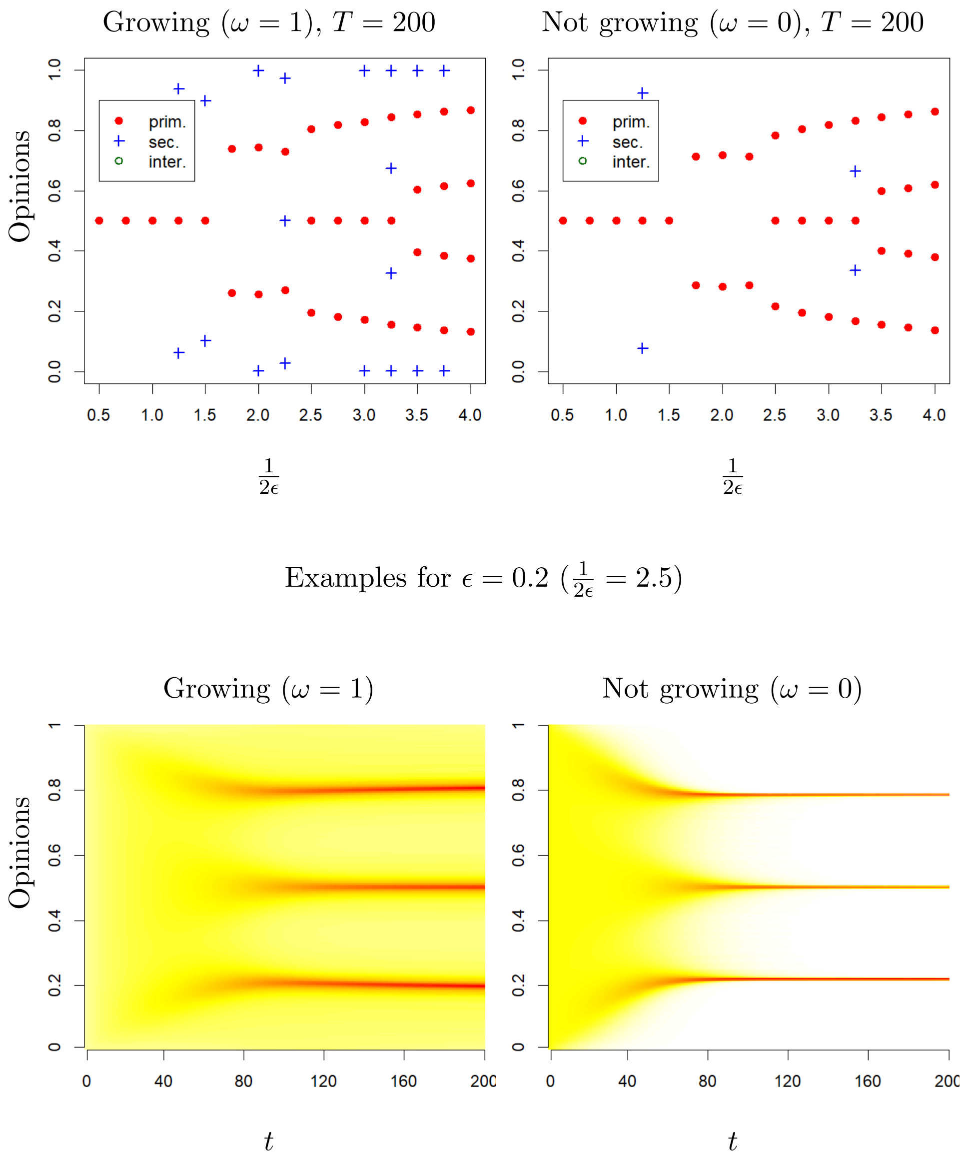


Fig. 6. Positions of clusters for the smooth influence from the density model for both growing and non-growing cases for different values $\frac{1}{2\epsilon}$. The clusters are defined like previously and computed at the simulation's last iteration ($T = 200$).

Table 1. Processes generating secondary clusters.

| | Fixed population | Growing population |
|---|---|---|
| Standard influence | Discontinuity | Discontinuity + Arrival of new agents to influence-free areas |
| Smooth influence | | Arrival of new agents to influence-free areas |

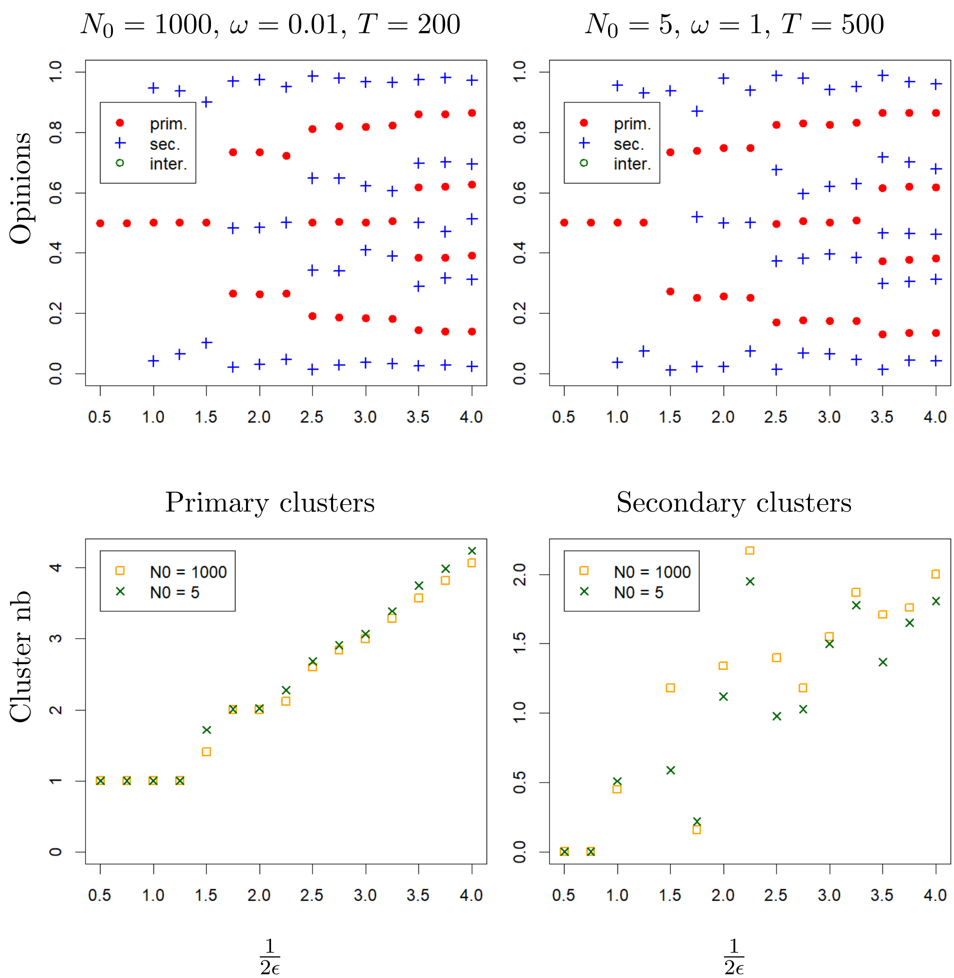


Fig. 7. Positions of clusters (top panels) and average number of clusters (bottom panels) from the agent model for an initial number of agents $N_0 = 1000$, $\omega = 0.01$ and after $T = 200$ iterations (left panel) and $N_0 = 5$, $\omega = 1$ and after $T = 500$ iterations (right panel).

clusters, and the right panel shows the average number of secondary clusters. The definition of the clusters with their effective weight is the same as in Fig. 4.

The average positions of the clusters of the most frequent configurations are similar in these top panels. The main difference is that the transition between 1 and 2 primary clusters takes place for $\frac{1}{2\epsilon} \in [1.5, 1.75]$ for $N_0 = 1000$, like the density model, and for $\frac{1}{2\epsilon} \in [1.25, 1.5]$ for $N_0 = 5$. The transitions from 2 to 3, and from 3 to 4 primary clusters take place in the same intervals $\frac{1}{2\epsilon} \in [2.25, 2.5]$ and $\frac{1}{2\epsilon} \in [3.25, 3.5]$, respectively, like in the density model.

The positions of the secondary clusters are not regular, probably because, in general, only a small part of the secondary clusters that are visible on the figures are present in each configuration participating in the average. For instance, for $N_0 = 1000$ and $\frac{1}{2\epsilon} = 3.75$, the average number of secondary clusters is 0.65. Therefore, in most simulations, there is at most only one secondary cluster. As a consequence, each position of a secondary cluster is averaged on a small sample, which explains the irregularities.

The average numbers of primary clusters in all the configurations (left bottom panel) are very similar for $N_0 = 1000$ and $N_0 = 5$, the number being very slightly higher for $N_0 = 5$. Note that the number grows almost linearly for $\frac{1}{2\epsilon} > 2.5$.

There are more differences in the average numbers of secondary clusters (right bottom panel). Most of the time, the average number of secondary clusters is smaller for $N_0 = 5$, which seems difficult to explain only by the small difference of the number of primary clusters. Hence this point may require further investigations. Overall, the number of secondary clusters is significantly lower with the smooth influence than with the standard one, confirming our claim that the secondary clusters, in the smooth case, are emerging only because of the regular arrival of new agents in locations of the opinion interval that are not under the influence of a primary cluster.

## 4. Discussion

The behavior of the bounded confidence model on a growing population is significantly different from the noisy version [18, 19] of the model because the distribution of agents tends to a density with several fixed peaks (primary and secondary), while in the noisy bounded confidence, this is generally not the case. From our simulations, the model can reach different steady states in density, like the model on a fixed population, depending on random events taking place in the initial phase of the simulation. However, checking the model on very long simulations is difficult as the number of agents keeps growing.

The most striking novelty of the model on a growing population is the emergence of secondary clusters, which are larger than when the population is fixed, and more importantly, they appear systematically, which is not the case in the fixed population. Nevertheless, when considering the density models, the maps of the primary and secondary clusters when the confidence bound varies are similar in the growing and non-growing population cases. This may suggest that the secondary clusters appear with a similar process whether the population is growing or not, the growing population simply reinforcing an existing process.

The results obtained with smooth interactions suggest otherwise. Indeed, in this case, the model on a fixed population shows almost no secondary cluster, while the secondary clusters are still present in the simulations of the model on a growing population, albeit less numerous.

This suggests that when the population is fixed, the secondary clusters are generated by the discontinuity of the standard influence and when the population is growing, they come from another process (summarized in Table 1). Our work suggests the following explanations:

- The standard influence function shows its maximum effect at the border of the attraction basin of a developing cluster. As a result of this strong attraction, the density of opinion is likely to be depleted inside the basin in the vicinity of its border. Therefore, the opinions located just outside the attraction basin can become too far from the opinions within the basin, and are thus not attracted. This situation does not occur with the smooth influence function, because the opinions located at the border of the attraction basin move slowly and have more chances to remain close enough to attract opinions beyond the border while the opinions are progressively gathering into a peak.
- With both standard or smooth influence, the primary clusters are often further than $2\epsilon$ from each other or further than $\epsilon$ from the limit of the opinion interval. This leaves some regions of the opinion interval which are not under the influence of a primary cluster. When the population is growing, the opinions that are regularly recruited in these areas, progressively feed secondary clusters.
- These secondary clusters can maintain themselves only if they are significantly smaller than their neighboring primary clusters. Indeed, the regular arrival of new opinions in a shared zone of influence is likely to generate a lot of interactions back and forth between the clusters that bring them closer and closer until they ultimately merge. However, suppose that one of the clusters is much smaller than the other. In that case, an opinion arriving in their common zone of influence is very likely to be attracted only by the bigger cluster. Therefore, the back-and-forth interactions are very unlikely, and both clusters can maintain themselves. Moreover, their difference in size keeps increasing because the bigger cluster is more likely to attract the new agents (this process resembles the "preferential attachment" in social networks).
- Finally, we showed that varying the initial number of agents in the population leads to significantly different results. In particular, when this number is small (e.g. 5–10), the model shows a first phase during which the position of the primary clusters can change significantly, possibly because of interactions with secondary clusters. These phenomena would be interesting subjects for future investigations.

## 5. Conclusions

In this work, we have analyzed the bounded confidence opinion dynamics when new agents are progressively added to the population. The main new feature appearing when the population is growing is the emergence of significant and stable secondary clusters, located approximately at one confidence bound distance of the main clusters. Those secondary clusters appear erratically and include much fewer agents when the population is fixed. The secondary clusters in the growing version can maintain themselves only if they are significantly smaller than their neighboring primary clusters. Otherwise, the regular arrival of new opinions in a shared zone of influence is likely to generate a lot of interactions back and forth between the clusters that bring them closer and closer until they ultimately merge. When the number of agents is much smaller in the secondary cluster, the new coming agents arriving in the zone of shared influence with a primary cluster are likely to interact only with agents of the primary cluster, thus leaving the secondary cluster unchanged.

Through simulations, we derive the bifurcation diagram of the growing population model and compare it to the diagram obtained with an evolving probability density instead of agents, and with their equivalent with a fixed population. Then, our analysis includes the study of an agent-based model, a density version and we also test the models with a smooth influence function.

The analysis using the smooth influence function helps us uncover that the main mechanism responsible for the systematic presence of secondary clusters in the growing version is the arrival of new agents in regions that are already free of attraction by the primary clusters while the secondary clusters appearing in the model with a fixed population are due to the discontinuity in the interaction function. A relevant follow-up to this work is to study the effect of various types of network topology when the network is growing.

## Appendix A. Effective Weight of Clusters

The method for computing the effective weight of the clusters in a distribution of opinions involves three main steps:

- Computing a smooth distribution with Gaussian kernel operator of variance about $\alpha\epsilon$ (we generally choose $\alpha = 0.1$),
- Selecting the local maxima of the smooth distribution that dominates the distribution in a vicinity of $\beta\epsilon$ (we generally choose $\beta = 0.5$),
- Computing the effective weight of each local maximum as its weight multiplied by the effective number of clusters.

We now describe each step in more details.

### A.1. *Computing a smooth distribution with a Gaussian kernel*

Let $A = (a_1, \ldots, a_N)$ be the distribution of opinions. Let $(x_1, \ldots, x_p)$ be such that for $i \in (1, \ldots, p)$, $x_i$ is the middle of interval $[\frac{i-1}{p}, \frac{i}{p}]$:

$$x_i = \frac{i - 0.5}{p}. \tag{A.1}$$

For any number $x$, let the Gaussian function $G(x_i, x)$ be (with $\alpha = 0.1$ a parameter):

$$G(x_i, x) = \exp\Big(-\Big(\frac{x - x_i}{\alpha\epsilon}\Big)^2\Big). \tag{A.2}$$

For each $x_i$, the smoothed distribution value is defined as

$$S(x_i) = \sum_{j=1}^{N} G(x_i, a_j). \tag{A.3}$$

This smoothing erases strong irregularities of the histogram of opinions and provides a more accurate view of the respective weights of the clusters than when associating the clusters to the maxima of the opinion histogram.

### A.2. *Selecting local maxima of the smooth distribution and computing the effective weight of each of them*

The couple $(x_i, S(x_i))$ defines a local maxima of the smoothed distribution if, for all $j \neq i$ such that $|x_i - x_i| < \beta\epsilon$, $S(x_j) < S(x_i)$.

### A.3. *Computing the effective weight associated with a local maximum*

Let $(x_{i_1}, \dots, x_{i_m})$ be the set of values defining the local maximums. Then $w_{i_j}$, the weight of the maximum is

$$w_{i_j} = \frac{S(x_{i_j})}{\sum_{k=1}^{m} S(x_{i_k})}. \tag{A.4}$$

From weights, we compute $n_g$ the effective number of clusters [13] as follows:

$$n_e = \frac{1}{\sum_{k=1}^{m} w_{i_k}^2}. \tag{A.5}$$

If there are $n$ clusters of the same weight $\frac{1}{n}$, then $n_e = n$, and if there are major clusters and minor clusters, $n_e$ tends to be close to the number of major clusters.

The effective $W_{i_j}$ of maximum $i_j$ is finally defined as

$$W_{i_j} = n_e w_{i_j}. \tag{A.6}$$

In general, the primary clusters define local maxima with an effective weight of around 1, while for the secondary clusters the corresponding effective weight is lower than 0.2. In the example shown on the right panel of Fig. A.1, there are no clusters of effective weight comprised between 0.2 and 0.7. Therefore, the approach discriminates well between secondary (low effective weight) and primary (high effective weight) clusters.

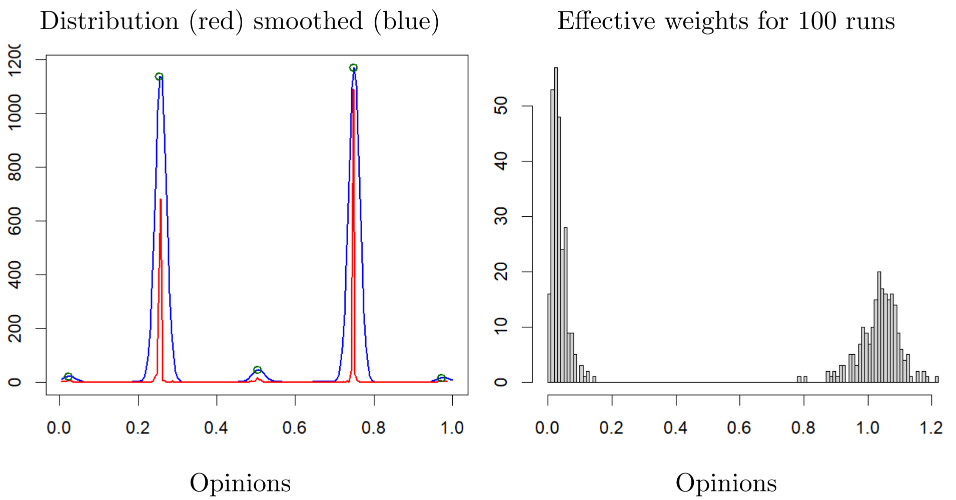


Fig. A.1. (Color online) On the left panel, an illustration of the method for identifying the clusters. The red curve is the distribution of the opinions, the blue curve is the smoothed distribution, the small circles show the locations of the detected local maxima. The effective weights of these clusters are in the order from left to right: 0.02, 1.02, 0.04, 1.05, 0.01. The right panel shows the distribution of the values of the cluster effective weights for 100 simulations for $\epsilon = 0.2$, $N_0 = 5$, $\omega = 1$ at $T = 100$ rounds.

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