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Practical observability and observers for nonlinear systems subject to bounded disturbances

M. G. Dadjo, A. Rapaport, R. Ushirobira, D. Efimov, J. Harmand

***Abstract*—This paper investigates the problem of observability and state reconstruction for nonlinear systems subject to unknown disturbances. We propose definitions of practical observability in this context and two kinds of practical observers. Under certain assumptions, we use the Lyapunov stability analysis method to show how to build an observer so that the estimation error converges in a "practical" sense.**

I. INTRODUCTION

The problem of observer design for nonlinear systems is far from having a general solution [2]. As opposed to linear systems, where, for instance, checking the rank of the Kalman observability matrix is necessary and sufficient [9], there is no universal definition of observability for nonlinear systems. This property also depends on the inputs, an essential aspect in nonlinear settings. Most solutions are proposed for the case of uniform input observability (this greatly simplifies the designer's job since input selection can be avoided). Nevertheless, many existing results are devoted to the simultaneous choice of the input and state estimation, with posterior regulation [17].

In all these cases, it is assumed that if some conditions are satisfied, then a unique state vector can be recovered by a suitable method. Therefore, numerous observers have been synthesized in the literature, providing state estimators with various performances and for different models [9], [17]. However, in many cases, these hypotheses of state observability cannot be verified, especially if the system model contains a bounded uncertainty, non-invertible nonlinearities, or variable singular gains.

In this work, we consider classes of controlled nonlinear systems under unknown disturbances, which are not observable in common sense and without the hypothesis on the uniformity of observability in the control inputs. We aim to characterize cases where estimating the state vector remains possible with a bounded error in a "practical" sense. In such a setting, the goal is to construct a "practical observer," which converges towards a small neighborhood of the actual state variable. Several design variants of these observers have been already proposed in the literature [7].

This paper is structured as follows. We begin with background on classical observability for nonlinear systems in Section II emphasizing advancements on practical observers. In Section III, we propose the definition of practical observability and three illustration examples. Next, we detail our proposed framework for practical observers. Simulation results are then presented to demonstrate the effectiveness of the developed methods across various scenarios in Section V. Finally, we conclude with a discussion on potential future directions and open problems in this dynamic field.

Notation

- $\mathbb{R}_+$, $\mathbb{R}_+^*$ denote the sets of non-negative real numbers and positive real numbers, respectively.
- $\mathbb{R}^p$ and $\mathbb{R}^{n\times m}$ denote the set of vectors of dimension p, and the set of matrices with real coefficients of dimension $n \times m$, respectively.
- A matrix M is symmetric if $M = M^\top$ and we use the $\star$ notation to avoid rewriting symmetric terms, that is $\begin{bmatrix} M_a & M_b \\ \star & M_c \end{bmatrix} = \begin{bmatrix} M_a & M_b \\ M_b^\top & M_c \end{bmatrix}$
- $\mathcal{S}_n$ denotes the set of symmetric positive definite matrices of dimension $n \times n$.
- I_n denotes the identity matrix of dimension $n \times n$.
- $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced norm for matrices.
- For a Lebesgue measurable function $v\colon \mathbb{R}_+ \to \mathbb{R}^m$, define the norm $\|v\|_{[t_1,t_2)} := \operatorname{ess\,sup}_{t\in[t_1,t_2)}\|v(t)\|$ for $[t_1,\, t_2) \subseteq \mathbb{R}_+$. We denote by $\mathcal{L}_\infty^m$ the set of functions v with $\|v\|_\infty := \|v\|_{[0,\infty)} < +\infty$.
- For any $\xi \in \mathbb{R}^n$, $B_\varepsilon(\xi) = \left\{ z \in \mathbb{R}^n, \|\xi - z\| \le \varepsilon \right\}$.
- $\mathrm{int}(E)$ denotes the interior of the topological space E.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and α is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ decreases to zero as $t \to +\infty$ for each fixed $s \in \mathbb{R}_+^*$.

II. PRELIMINARIES

This section recalls notions of observability for a system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), v(t)),\ t \in \mathbb{R}_+, \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathsf{X} \subset \mathbb{R}^n$ is the state vector, $u(t) \in \mathsf{U} \subset \mathbb{R}^m$ is the control (that can be assigned by a designer), $v(t) \in$

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$\mathsf{V} \subset \mathbb{R}^q$ is an external perturbation and $y(t) \in \mathsf{Y} \subset \mathbb{R}^p$ is the output vector available for measurements. The map $f \colon \mathsf{X} \times \mathsf{U} \times \mathsf{V} \to \mathbb{R}^n$ is Lipschitz continuous with respect to x and continuous with respect to other arguments. The map $h \colon \mathsf{X} \to \mathsf{Y}$ is continuously differentiable. We assume that the set X is *forwardly invariant* by (1) whatever are Lebesgue measurable functions u, v taking values in U, V, respectively. Therefore, for given $x_0 \in \mathsf{X}$, and functions u and v, the corresponding solution $x(t, u, v)$ to (1) is uniquely defined for all $t \in \mathbb{R}_+$. We shall denote by $y_{x_0,u,v}$ the output function of (1) for the initial condition $x(0) = x_0$, the control u and the disturbance v.

A. Observability

In absence of disturbance (say v is identically null), the system (1) is classically said to be *observable for a given input function* u if two different initial conditions $x_0^a \neq x_0^b$ are distinguishable in finite time, meaning

$$\text{there exists } \tau > 0 \text{ such that } y_{x_0^a,u,0}(\tau) \neq y_{x_0^b,u,0}(\tau),$$

or equivalently, if $y_{x_0^a,u,0}(t) = y_{x_0^b,u,0}(t)$ for any $t \in \mathbb{R}_+$ implies $x_0^a = x_0^b$ (see, *e.g.*, [1]). The system is *observable in time* α *(for a given* u*)* if the initial condition x_0 can be uniquely reconstructed from the measurements of the output y over a finite time interval $[0, \alpha]$ (see [1]), which implies that there exists a map Ω such that the state $x(t)$ is recovered as

$$x(t) = \Omega\left(y_{x_0,u,0}\left([t-\alpha, t]\right), u([t-\alpha, t])\right), \; t \geq \alpha. \tag{2}$$

If the above property is satisfied for any admissible input u, then the system is called *uniformly observable (in time* α*)*.

In the presence of unknown (bounded) disturbances $v \not\equiv 0$, the problem becomes much more challenging, apart from particular cases of exact decoupling of the "unknown inputs" [7], [10], [11] (property of *observability for unknown inputs*). In such cases, the influence of disturbances can be canceled in new coordinates, and an exact state estimation remains possible.

B. Practical Observers in the literature

When the system is not observable for unknown input, and the uncertainty v is bounded, concepts of practical observers (see, e.g., [7]) have been introduced to estimate the state up to some bounded error, with systems of the general form

$$\dot{z}(t) = g(z(t), y(t), u(t)), \quad \widehat{x}(t) = \ell(z(t), y(t)),$$

where $z(t) \in \mathbb{R}^r$ is the internal state of the observer and $\widehat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. If for any $\varepsilon > 0$ there exists a system of this form such that on has

$$\forall x(0),\, z(0),\, \exists T > 0 \text{ such that } \|\widehat{x}(t) - x(t)\| \leq \varepsilon,\; \forall t \geq T$$

then, this family of estimators parameterized by ε is a *weak practical observer* in the sense proposed in [7] (the authors considered the case without the control input u).

A weak practical observer is a *strong practical observer* if the family of systems verifies

$$\begin{aligned} &\forall \varepsilon > 0,\, \exists \lambda_\varepsilon > 0,\, \exists k_\varepsilon > 0,\, \forall x(0), z(0),\, \forall t \geq 0, \\ &\qquad \|\widehat{x}(t) - x(t)\| \leq \varepsilon + e^{-\lambda_\varepsilon t}\left(k_\varepsilon \|\widehat{x}(0) - x(0)\| - \varepsilon\right) \end{aligned}$$

with $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0$. This concept of a practical observer has been developed and found useful in the context of "interval observers", when the uncertainty v belongs to a known product of bounded intervals, [6], [15], and otherwise later exploited in [8] (without control inputs).

To conclude this section, note that there is no universal definition for "practical observers." Instead, this definition may exist for certain classes of dynamical systems.

III. PRACTICAL OBSERVABILITY

To the best of our knowledge, there is no result in the literature dealing with practical observability of systems in the general form of (1), that is, to formally quantify the possibility of state estimation with a bounded error only. To this end, in the present work, we propose to relax property (2) as follows:

Definition 1:

1) Let $\varepsilon > 0$. The system (1) is ε*- practically observable* (ε-PO) for a given control u if there exist $\alpha > 0$ and a set-valued map $\Omega_\varepsilon : \mathcal{Y}_{[t-\alpha,t]} \times \mathcal{U}_{[t-\alpha,t]} \rightrightarrows \mathsf{X}$ [1] such that, for any $x_0 \in \mathsf{X}$ and disturbance v, one has
$$\Omega_\varepsilon\left(y_{x_0,u,v}([t-\alpha, t]), u([t-\alpha, t])\right) \subset B_\varepsilon(x(t)),\; t \geq \alpha. \tag{3}$$
2) The system (1) is *strongly practically observable* (SPO) if for any $\varepsilon > 0$, there exists a control u_ε such that (1) is ε-PO.

It is well known that nonlinear systems can have "bad" inputs for which they are not observable [9]. These inputs should be avoided (see, for instance, recent works [12], [13]). This is the reason why our definitions of practical observability depend on the control u.

To illustrate cases where this notion of practical observability is fulfilled, let us consider a nonlinear system (1) written as

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t), u(t)), v(t)) \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t), u(t), v(t)) \\ y(t) &= x_1(t),\; t \geq 0, \end{aligned} \tag{4}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, with $n = n_1 + n_2$, and V is a compact set. For the system (4), since x_1 is measured directly, it is necessary to recover the variable x_2 only. To this end, we may resolve the first equation with respect to x_2, considering x_1 and $\dot{x}_1$ as known variables (the output and its derivative). Suppose such an inversion admits a unique solution, and it depends on known variables only (*i.e.*, $\dot{y}$, y and u). In that case, the system is observable in the infinitesimal sense (i.e.

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[1] Here $\mathcal{Y}_{[t-\alpha,t]}$, resp. $\mathcal{U}_{[t-\alpha,t]}$ is the set of continuous, resp. measurable functions from $[t-\alpha, t]$ to Y, resp. U.

the algebraic observability criterion [3] is fulfilled). If this inverse map depends on v, then we can consider the practical observability in the sense of (3), provided that the structure of V allows to hide the influence of v in ε. However, everything depends on the properties of the dynamics of x_2 since we may recover this state component with an accuracy depending on the characteristics of the right-hand side of (4) and the presence of uncertainty. In some cases, the inversion can be ill-defined. For instance, the variable x_2 may appear in f_1 within a function $\varphi : \mathbb{R}^{n_2} \to \mathbb{R}^w$ for some integer $w \geq 1$, which has no unique inverse. Popular examples including saturation or dead-zone functions have a set-valued inverse in the constancy domain. In other cases, the variable x_2 can be multiplied by a singular matrix gain of other known variables, whose inversion is possible only on a subspace, and the remaining variables stay implicitly defined, hence implying practical observability of the state of (4).

Let us consider four examples that illustrate different situations depicted in the discussion above.

Example 1: (case of interval uncertainty) Let $\delta \in \mathbb{R}_+^*$. Consider

$$\dot{x}_1 = x_2 + \frac{x_1}{1+x_1} + v, \tag{5}$$

$$\dot{x}_2 = u + v, \tag{6}$$

$$y = x_1, \tag{7}$$

with $\mathsf{X} = (\mathbb{R} \setminus \{-1\}) \times \mathbb{R}$, $\mathsf{U} = \mathbb{R}$, and $\mathsf{V} = [-\delta, \delta]$. One has

$$x_2(t) = x_2(t-\alpha)e^{-\alpha} + \int_{t-\alpha}^{t} e^{\tau - t}\Big(\dot{y}(\tau) - \frac{y(\tau)}{1+y(\tau)} + u(\tau)\Big)d\tau, \ t > \alpha$$

Since also $x_2(t-\alpha) \in \dot{y}(t-\alpha) - \frac{y(t-\alpha)}{1+y(t-\alpha)} + [-\delta, \delta]$, we obtain: $x_2(t) \in \omega\left(y_{[t-\alpha,t]}, u_{[t-\alpha,t]}\right) + [-\delta, \delta]\, e^{-\alpha}$ where

$$\omega(y_{[t-\alpha,t]}, u_{[t-\alpha,t]}) = \Big(\dot{y}(t-\alpha) - \frac{y(t-\alpha)}{1+y(t-\alpha)}\Big)e^{-\alpha} + \int_{t-\alpha}^{t} e^{\tau - t}\Big(\dot{y}(\tau) - \frac{y(\tau)}{1+y(\tau)} + u(\tau)\Big)d\tau \tag{8}$$

Therefore, for any $\varepsilon \in (0, 2\delta)$ and for any u, the system is ε-PO with $\alpha > 0$ such that $2\delta e^{-\alpha} = \varepsilon$, and

$$\Omega_\varepsilon(y_{[t-\alpha,t]}, u_{[t-\alpha,t]}) = \{y(t)\} \times B_\varepsilon\left(\omega(y_{[t-\alpha,t]}, u_{[t-\alpha,t]})\right).$$

The system is then SPO.

Example 2: (singular gain)

$$\dot{x}_1 = -x_1\left(x_2 - x_3\right), \tag{9}$$

$$\dot{x}_2 = x_3 - x_2, \tag{10}$$

$$\dot{x}_3 = 0, \tag{11}$$

$$y = x_1, \tag{12}$$

with $\mathsf{X} = (0, +\infty) \times (0, 1) \times (0, \rho)$, where $\rho \in (0, 1]$. Note that in this example there is no control or uncertainty, which amounts to consider $\mathsf{U} = \mathsf{V} = \{0\}$. It is straightforward to show that X is positively invariant. From the derivation of the output, one gets $x_2 - x_3 = -\frac{\dot{y}}{y}$ and then $\frac{d}{dt}(x_2 - x_3) = -(x_2 - x_3)$. The system is not observable because it is not possible to distinguish between x_2 and x_3 having the same difference, but $x_2 - x_3$ converges exponentially to 0. Therefore, for any $\varepsilon > \rho$, there exists α such that for any initial condition in X one has $x_i(t) \in (0, \varepsilon)$ for $i = 2, 3$ and $t \geq \alpha$. Hence, it is ε-PO (but not SPO).

Example 3: (non-invertible output dynamics on a sub-domain)

$$\dot{x}_1 = \min\{1, x_2\},$$

$$\dot{x}_2 = x_2(2 - x_2 + u + v),$$

$$y = x_1,$$

with $\mathsf{X} = \mathbb{R}_+ \times [0, 3+\delta]$, $\mathsf{U} = [-1, 1]$, and $\mathsf{V} = [-\delta, \delta]$.

One can easily check that X is forwardly invariant. Note that $x_2(t) < 1$ implies $\dot{y}(t) < 1$ and then one has $x_2(t) = \dot{y}(t)$. Otherwise, whatever is the control u, there exists a perturbation v such that the solution verifies $x_2(t) \geq 1$ for all $t \in \mathbb{R}_+^*$: take, for instance, $v \equiv \delta$ and one has $\dot{x}_2 \geq x_2(1 - x_2 + \delta)$ and $x_2 = 1$ is thus repulsive. It is not possible to distinguish $x_2(t)$ in the interval $[1, 1+\delta]$. The system is thus not SPO.

With the constant control $u = -1$, the solution x_2 converges asymptotically to the interval $[1-\delta, 1+\delta]$. The system is then ε-PO for any $\varepsilon > \delta$ with this control: either $x_2(t)$ takes a value below 1 at a time $t \in \mathbb{R}_+^*$, and x_2 is exactly reconstructed, either x_2 stays equal or above 1 and reaches the interval $[1, 1+\varepsilon]$ in finite time, for any v. One has

$$\Omega_\varepsilon(y(t), \dot{y}(t)) = \begin{cases} \{y(t)\} \times [1, 1+\varepsilon], & \dot{y}(t) = 1 \\ \{(y(t), \dot{y}(t))\}, & \dot{y}(t) < 1 \end{cases}$$

Note that when ε is arbitrary closed to δ, α tends to $+\infty$: for an initial condition $x_2(0) \in (1+\delta, 3+\delta)$ and $v \equiv \delta$, the solution $x_2(t)$ is outside the interval $[1, 1+\delta]$ for any t.

Example 4: (non-uniformity in the control)

$$\dot{x}_1 = -x_1\left(x_2 + \frac{v}{1+u}\right),$$

$$\dot{x}_2 = x_2(1+u) + v,$$

$$y = x_1,$$

with $\mathsf{X} = (0, +\infty) \times (\delta, +\infty)$, $\mathsf{U} = \mathbb{R}_+$, and $\mathsf{V} = [-\delta, \delta]$. It is straightforward to check that X is positively invariant. Let x be a solution in X for a control u and a disturbance $v(t) > -\delta$ for all $t \geq 0$, and $\widetilde{x}$ be the solution for the initial condition $\widetilde{x}(0) = (x_1(0), x_2(0) + \gamma)$, where $\gamma > 0$ is chosen in a way that the disturbance $\widetilde{v}(t) = v(t) - \gamma(1 + u(t)) \in V$, $t \in \mathbb{R}_+$, for the same control u. Clearly, one has $\widetilde{x}(t) = (x_1(t), x_2(t) + \gamma)$, $t \in \mathbb{R}_+$. Therefore, the system is not ε-PO for this control for any $\varepsilon < \gamma$. However, direct computations show that

$$x_2(t) \in -\frac{\dot{y}(t)}{y(t)} + \left[-\frac{\delta}{1+u(t)}, \frac{\delta}{1+u(t)}\right], \ t > 0.$$

Therefore, for any $\varepsilon \in (0, \delta)$ one can take the (constant) control $u_\varepsilon = \frac{\delta}{\varepsilon} - 1 > 0$ so that $x_2(t) \in B_\varepsilon\left(-\dot{y}(t)/y(t)\right)$

for any $t \in \mathbb{R}_+^*$. The system is thus SPO, but differently to Example 1, the control u_ε depends on the parameter ε.

IV. PRACTICAL OBSERVERS

The definition of practical observability implies a new formulation of the corresponding practical observer design problem.

A. Definition

Let us consider a family of dynamical systems parameterized by $\varepsilon > 0$:

$$\begin{array}{rcl}\dot{z}(t) & = & g_\varepsilon(z(t), u(t), y(t)) \\ \widehat{x}(t) & = & \ell_\varepsilon(z(t), y(t)),\ t \geq 0,\end{array} \tag{13}$$

where $z(t) \in \mathsf{Z} \subset \mathbb{R}^r$ (for some integer r) is the internal state of the observer, and $\widehat{x}(t) \in \mathbb{R}^n$ is the state estimate.

1) Let $\varepsilon > 0$. The system (13) is an *ε-practical observer* (ε-PObs) for the system (1) with a control $u_\varepsilon(\cdot)$, if there exist functions $\beta_\varepsilon \in \mathcal{KL}$, $\gamma_\varepsilon \in \mathcal{K}$ and numbers λ_ε, $\mu_\varepsilon > 0$, such that for any $(x_0, z_0) \in \mathsf{X} \times \mathsf{Z}$ and disturbance $v(\cdot)$, one has
$$\begin{aligned}\|\widehat{x}(t) - x(t)\| \leq \varepsilon + \beta_\varepsilon\left(\|\widehat{x}(0) - x_0\|, \lambda_\varepsilon t\right) \\ + \gamma_\varepsilon\left(\mu_\varepsilon \|v\|_{[0,t)}\right),\ t > 0.\end{aligned}$$
2) The system (13) is a *strong practical observer* (SPObs) if it is an ε-practical observer for any $\varepsilon > 0$ with $\lambda_\varepsilon \to \infty$, $\mu_\varepsilon \to 0$ when $\varepsilon \to 0$.

Let us underline that in the present definition of practical observers, the control u may depend on ε (denoted by u_ε). At the same time, only asymptotic convergence is required, unlike other definitions of practical observers that require an exponential convergence property, uniformly in u [7], [8]. Moreover, the influence of the uncertainty v is quantified separately. In this sense, this definition is weaker than the one in the literature.

In the framework of the theory of input-to-output stable systems, the above requirement may be viewed as a practical state-independent input-to-output stability [18], where additional attention is given to the steady-state estimation error through the ε-dependent gains λ_ε and μ_ε.

B. Observer design

Consider a nonlinear system of the form:

$$\begin{aligned}&\dot{x}(t) = Ax(t) + \varphi(y(t), u(t)) + D(y(t), u(t))\phi(Hx(t)) + v(t), \\ &y(t) = Cx(t),\ \forall t \in \mathbb{R}_+,\end{aligned} \tag{14}$$

where $x(t) \in \mathsf{X} \subset \mathbb{R}^n$, $u(t) \in \mathsf{U} \subset \mathbb{R}^m$, $y(t) \in \mathsf{Y} \subset \mathbb{R}^p$ and $v(t) \in \mathsf{V} \subset \mathbb{R}^l$. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ and $H \in \mathbb{R}^{w\times n}$ are given, $\varphi : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$, $\phi : \mathbb{R}^w \to \mathbb{R}^q$, $D : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^{n\times q}$ are locally Lipschitz continuous functions (with vector and matrix values).

Below, we will assume the detectability of the pair (A, C), which does not imply the observability of (14). Moreover, we suppose that the system is not observable but that there exists another map $\Phi : \mathbb{R}^w \to \mathbb{R}^q$, so that the practical observability is satisfied when ϕ is replaced by Φ, under the following assumptions:

Assumption 1: Given an input function $u : \mathbb{R}_+ \to \mathsf{U}$, there exists $\Upsilon \in \mathcal{S}_q$ such that

$$D^\top(y, u(t))D(y, u(t)) \leq \Upsilon,\ \forall y \in Y,\ \forall t \in \mathbb{R}_+.$$

Assumption 2: There exists $\varepsilon_1 > 0$ such that

$$\|\phi(Hx) - \Phi(Hx)\| \leq \varepsilon_1,\ \forall x \in \mathsf{X},$$

where the function Φ verifies the quadratic constraint for any $s_1, s_2 \in \mathbb{R}^w$:

$$\begin{bmatrix} s_1 - s_2 \\ \Phi(s_1) - \Phi(s_2)\end{bmatrix}^\top \begin{bmatrix} E & R \\ R^\top & -2\Upsilon\end{bmatrix}\begin{bmatrix} s_1 - s_2 \\ \Phi(s_1) - \Phi(s_2)\end{bmatrix} \geq -\varepsilon_2$$

for some matrices $E \in \mathcal{S}_w$, $R \in \mathbb{R}^{w\times q}$ and $\varepsilon_2 \geq 0$.

Here, Assumption 1 introduces a particular input under which the estimation will be possible, while Assumption 2 clarifies the origins of practical observability: the map Φ that is introduced instead of ϕ is at a certain distance ε_1 from ϕ, and there is an ε_2 relaxation in the quadratic constraint that will be used to design an observer with Φ instead of ϕ.

The practical observer is introduced in the form for $t \geq 0$:

$$\begin{aligned}\dot{\widehat{x}}(t) \;=\; & A\widehat{x}(t) + \varphi(y(t), u(t)) + L(y(t) - C\widehat{x}(t)) \\ & + D(y(t), u(t))\Phi\left(H\widehat{x}(t) + M(y(t) - C\widehat{x}(t))\right),\end{aligned} \tag{15}$$

where $\widehat{x}(t) \in \mathbb{R}^n$ is the state estimate, the matrices $M \in \mathbb{R}^{w\times p}$ and $L \in \mathbb{R}^{n\times p}$ being the observer gains to be designed.

Remark 1: Note that ϕ may also be uncertain. We need to know only that the map Φ that should be close to ϕ (in a similar way as in [8]).

The main result of this section is as follows. We omit its proof for brevity.

Theorem 1: Under Assumptions 1 and 2, if there exist matrices $P \in \mathcal{S}_n$, $M \in \mathbb{R}^{w\times p}$ and $U \in \mathbb{R}^{n\times p}$ such that the following linear matrix inequalities are satisfied:

$$\begin{bmatrix} \Psi & \star & \star & \star & \star \\ R^\top(MC - H) & \Upsilon & 0 & 0 & 0 \\ -P & 0 & \gamma I_n & 0 & 0 \\ P & 0 & 0 & I_n & 0 \\ H - MC & 0 & 0 & 0 & E^{-1}\end{bmatrix} \geq 0,$$
$$\Psi = C^\top U^\top + UC - A^\top P - PA - \alpha P$$

for some $\alpha > 0$ and $\gamma > 0$, then (15) is an ε-PObs for (14) with $L = P^{-1}U$ and $\varepsilon = \sqrt{\frac{2\gamma\lambda_{\max}(\Upsilon)\varepsilon_1^2 + \varepsilon_2}{\alpha\lambda_{\min}(P)}}$ provided that the input u satisfies Assumption 1.

This theorem illustrates a situation where the proposed notions of PO and PObs can be relevant for a system that is practically observable under the substitution of a part in its dynamics by another component with a bounded error with respect to the original term.

Remark 2: Among M, P, U and α, γ such that the LMI of Theorem 1 is fulfilled, one can look for the optimization problem that consists in minimizing

$$\varepsilon = \sqrt{(2\gamma\lambda_{\max}(\Upsilon)\varepsilon_1^2 + \varepsilon_2)/(\alpha\lambda_{\min}(P))}.$$

This presents some similitude with H^∞ robust control for which one searches for the best attenuation factor γ. However, here, one has to optimize several parameters. Development of this optimization problem might be a purpose of future work.

V. SIMULATION EXAMPLE

First, let us show the effectiveness of the observer (15) in Example 1. We write the dynamics of the system (5), (6), (7) as follows (in similar form as (14)):

$$\dot{x} = Ax + \varphi(y, u) + \nu$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \varphi(y,u) = \begin{bmatrix} \frac{y}{1+y} \\ u \end{bmatrix},\ \nu = \begin{bmatrix} v & v \end{bmatrix}^\top,\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Note that for any $\alpha > 0$, $\left(A + \frac{\alpha}{2} I_2, C\right)$ is observable. Then for $\alpha > 0$, there exist $L \in \mathbb{R}^2$ and $P = P^\top > 0$ such that

$$\begin{aligned} \widetilde{Q} \ &:= \ \left(A - LC\right)^\top P + P\left(A - LC\right) + \alpha P \\ &= \ \left(A + \tfrac{\alpha}{2} I_2 - LC\right)^\top P + P\left(A + \tfrac{\alpha}{2} I_2 - LC\right) < 0 \end{aligned}$$

The condition of Theorem 1 is then fulfilled when

$$\begin{bmatrix} \widetilde{Q} & P \\ P & -\gamma I_2 \end{bmatrix} < 0$$

which is satisfied for $\gamma\widetilde{Q} + P^2 \leq 0$. This ensures the estimation error of the observer

$$\dot{\widehat{x}} = A\widehat{x} + \varphi(y,u) + L\left(y - C\widehat{x}\right)$$

to converge to a ball of radius $\sqrt{\frac{2\gamma\delta}{\alpha\lambda_{min}(P)}}$.

In this example, the equations of the observer are

$$\begin{aligned} \dot{\widehat{x}}_1 &= \widehat{x}_2 + \frac{y}{1+y} + L_1\left(y - \widehat{x}_1\right), \\ \dot{\widehat{x}}_2 &= u(t) + L_2\left(y - \widehat{x}_1\right). \end{aligned}$$

Simulations depicted on Fig. 1 have been run for

$$u(t) = 2 + \cos(3t),\ v(t) = 20\cos(3t)$$

and for $x_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\widehat{x}_0 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$. Here we take $\delta = 20$, $\alpha = 2$, $L = \begin{bmatrix} 4.25 \\ 4.5 \end{bmatrix}$ and compute $P = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$ with $\lambda_{\min(P)} = \frac{6-2\sqrt{5}}{2} \simeq 0.76$. With $\gamma = 1.5$, one obtains $\varepsilon = \sqrt{\frac{60}{6-2\sqrt{5}}} \simeq 6.28$.

Example 5:

Consider the system (14) with $n = 3$ and

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ \varphi(y,u) = \begin{bmatrix} u \\ 0 \\ u \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^\top,\ H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},\ \phi(Hx) = \tanh(x_3),$$

and

$$v(t) = \begin{bmatrix} v_1(t) & v_2(t) & v_3(t) \end{bmatrix}^\top.$$

Only the first state variable is directly measurable, while the second is reconstructable in a linear way, but the third one is not, since its computaiton from the second equation depends on the uncertainty v. Nevertheless, one can practically recover the third state in the sense of 3. Since $|\tanh(z_1) - \tanh(z_2)| \leq |z_1 - z_2|$ we have

$$\begin{bmatrix} z_1 - z_2 \\ \phi(z_1) - \phi(z_2) \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 - z_2 \\ \phi(z_1) - \phi(z_2) \end{bmatrix} \geq -\varepsilon_2$$

for all $\varepsilon_2 > 0$. One can take $E = 1$, $R = 0$, and $\Upsilon = 1$. Note that (A, C) is detectable but not observable. With A, C, D and H as above, the LMI in Theorem 1 is feasible for $\alpha = 2$, $\gamma = 5$ (see [20]) and a solution is

$$L = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ P = \begin{bmatrix} 1.0998454 & 0.7511368 & 0.2529068 \\ 0.7511368 & 1.3978366 & 0.4615796 \\ 0.2529068 & 0.4615796 & 0.70502 \end{bmatrix},$$

and $M = -0.6568818$. An ε-practical-observer is synthesized as

$$\begin{aligned} \dot{\widehat{x}}_1 &= -\widehat{x}_1 + \widehat{x}_2 + u - (y - \widehat{x}_1) \qquad (16) \\ \dot{\widehat{x}}_2 &= -2\widehat{x}_1 - 3\widehat{x}_2 - \tanh\left(\widehat{x}_3 + M(y - \widehat{x}_1)\right) + \\ &\qquad\qquad\qquad + 2\left(y - \widehat{x}_1\right) \\ \dot{\widehat{x}}_3 &= -2\widehat{x}_3 + u + (y - \widehat{x}_1) \end{aligned}$$

with $\varepsilon = \sqrt{\frac{2\times5\times1\times\varepsilon_1^2 + \varepsilon_2}{2\times\lambda_{\min}(P)}}$ where $\lambda_{\min}(P) = 0.4094098$. For instance, for $\varepsilon_1 = 0.25$ and $\varepsilon_2 = 0.25$ we have $\varepsilon = 1.0337366$. To simulate our proposed pratical observer, the uncertain disturbance $v(t) = \begin{bmatrix} 0 & 0.2\cos(0.2t) & 0.5\sin(t) \end{bmatrix}^\top$ and the input $u(t) = 2(1 + \cos(0.3t))$ have been chosen with the initial condition $x_0 = \begin{bmatrix} 2 & 2 & 4 \end{bmatrix}^\top$, $\widehat{x}_0 = \begin{bmatrix} 1 & 1 & 3 \end{bmatrix}^\top$. The simulation results are shown in Fig. 2. The practical observer has been run to reconstruct the unmeasured state variables x_2 and x_3, which remain within a ball of radius ε. The states estimated by the practical observer do not converge exactly to the actual states but stay approximately within their neighborhood.

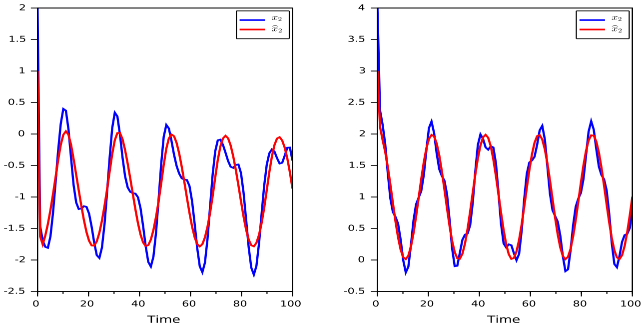

Fig. 2. Practical observer for example 5

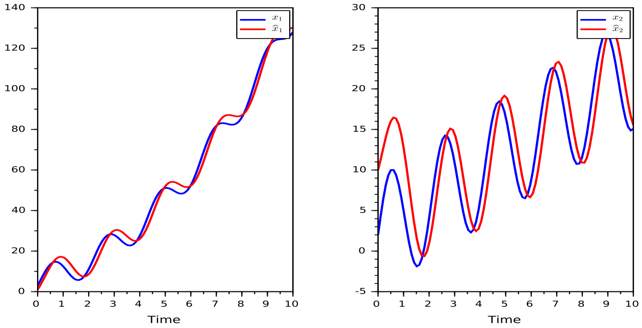

Fig. 1. Practical observer for example 1

VI. CONCLUSION

In this work, we have introduced the notions of ε-practical observability and strong practical observability for nonlinear systems subject to bounded disturbances. The goal was to relax the uniformly observable property and to recover the state vector up to a ball of radius ε. These notions have been illustrated in four different examples. The construction of an ε-practical observer has also been proposed for a class of nonlinear systems. The analysis of the estimation error was conducted for this class of systems using the Lyapunov function method. The design of a ε-practical observer has been illustrated in the examples.

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