Synthesis of optimal control under state constraints for crop fertigation

Mahugnon Dadjo, Alain Rapaport, Jérôme Harmand, Rosane Ushirobira, Denis Efimov

Synthesis of optimal control under state constraints for crop fertigation

M.G. Dadjo[1,3] A. Rapaport[1], J. Harmand[2], R. Ushirobira[3] and D. Efimov[3]

June 7, 2025

Abstract. We consider a generic model of fertigation for which the control variable is the irrigation flow rate. We first characterize conditions for which the dynamical system is viable by allowing maximal biomass production at harvesting time. Then, we consider the problem of minimizing the quantity of water delivered during a season under the viability constraint. We provide a complete optimal feedback synthesis as concatenations of bang and boundary arcs. We show that the optimal strategies can be radically different depending on the initial condition, leading to distinct ways to avoid water and nitrogen stresses for crops. Moreover, we characterize the possibility of having an infinity of optimal singular trajectories.

Key words. Crop modeling, water irrigation, state constraints, viability theory, optimal control, boundary arcs, singular arcs.

AMS subject classifications. 34H05, 93C15, 49J30, 92B05.

1 Introduction

In many regions of the world, and especially in the arid ones, agricultural production faces water scarcity, leading to difficulties in satisfying food population needs. Crops irrigation with treated wastewater, instead of fresh water, is a solution for preserving water resources. This practice increasingly attracts the interests of decision-makers.

The underlying idea is to preserve, at the water treatment step, the nutrients that are beneficial for crop growth, notably nitrogen, so that no additional nutrients, which may spread and contaminate soils, have to be brought. In this regard, irrigation with reused water amounts to fertigation (*i.e.*, irrigation with nutrients supplied in water). In the present work, we propose to address the problem of optimizing crop production via fertigation. Although many works about optimization with classical irrigation can be found in the literature [22, 19, 18, 13, 3], and several commercial software are available on the market [6, 17], the optimization of fertigation has been comparatively much less considered, apart some recent works [14, 12, 10]. With classical irrigation that covers the crop's water needs during the whole agricultural season, the soil's nitrogen concentration can be diluted to the point that it no longer meets the crop demands if the initial nitrogen content is insufficient. This often causes farmers to supply nitrogen at the beginning of the season with the risk of spreading unnecessary quantities of nitrogen in soils. Bringing nitrogen with water appears to be a safe solution, provided that its concentration is appropriate and does not end up with too much nitrogen in the soil at the end of the season.

Here, one also faces a dilemma between nitrogen dilution and water supply, which we will find in the optimization problem. In the former works [14, 12, 10], numerical investigations have been conducted, but a mathematical analysis of optimal control under constraints has not yet been done, as we do here.

In the present work, we consider a crop model describing hydric and nitrogen stresses inspired by existing literature [21, 15] and former works [11, 3, 10]. We assume here that the crop growth is not limited by other resources or nutrients (such as phosphorus). As reported in the literature, water and nitrogen are the common limiting factors of crop yield [20]. The originality compared to existing literature

[1]MISTEA, Univ. Montpellier, INRAE, Institut Agro, 34060 Montpellier, France.
Emails: mahugnon.dadjo@inrae.fr,alain.rapaport@inrae.fr
[2]LBE, INRAE, Univ. Montpellier, 11100 Narbonne, France.
Email: jerome.harmand@inrae.fr
[3]Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France.
Emails: rosane.ushirobira@inria.fr,denis.efimov@inria.fr

models is to consider explicitly in the model that nitrogen can be brought with irrigation, modeling the so-called fertigation.

With the help of this model and control theoretical tools, the present work aims to grasp the right balance between water and nitrogen inputs to be reached for good growth performances. The nitrogen input can be made by classical fertilization, for instance, at seed time or with fertigation. Fertigation is typically used when soil nitrogen content is insufficient to guarantee maximal growth even when the supplied water avoids the hydric stress but can be compensated by the nitrogen in the irrigation water. However, when the nitrogen concentration in supplied water is too low, an initial input has to be considered. Therefore, we are also looking for the analysis of the model to be able to characterize situations for which initial nitrogen inputs are necessary or not.

For farmers, biomass production at harvesting time is usually the main performance index, but that has to be balanced with water consumption. In this note, we consider that achieving the nominal production is a "hard constraint," and we consider the optimal control, which consists of minimizing the total amount of water (with fertigation) during a season under this constraint. To ensure this last problem is feasible with the features of the fertigation system, we consider the viability domain as the set of initial conditions, given by the moisture and nitrogen contents at seeding time, for which it is possible to obtain the maximal production at harvesting time. Our viability analysis allows us to characterize the operating parameters of the fertigation systems, which are nitrogen concentration in irrigation water and maximal irrigation rate for which the viability property is satisfied. The limits of the viability conditions that we have identified with this model provide the minimal inputs to obtain the maximal production at harvesting time, avoiding unnecessary overload of the soil with nitrogen content. In the second step, we study the optimal control problem with constraints, which consists of having the maximal production (that amounts to stay in the viability domain) while minimizing the total water consumption. Preliminary results, with numerical simulations but no proofs, are presented in the conference paper [8].

The paper is organized as follows. In the next section, we present the model and its hypotheses. In Section 3, we posit the viability problem and give conditions for the viability property to be fulfilled. In Section 4, we formulate the optimal control under constraints feasible under the former conditions. We also distinguish cases according to which constraint is saturated and in which order. In particular, we exploit a recent extension of the Bang-Bang Principle for piecewise affine dynamics [5] to define a family of control strategies. Finally, Section 5 illustrates our results and discusses several points related to non-intuitive situations and messages for practitioners. Finally, Section 6 summarizes our main results.

2 The model

In the spirit of existing models [21, 15] and former works [11, 3, 10, 7], we consider the following simplified crop model over the agronomic season represented by the time interval $[0, T]$, where S denotes the level of water in the soil (between 0 and 1) and N, B are the nitrogen and biomass per unit of surface

$$\dot{S} = k_1\Big(-\varphi(t)\, K_S(S) - (1-\varphi(t))\, K_R(S) + k_2 u\Big) \tag{1}$$

$$\dot{N} = -k_3\varphi(t)\, K_S(S) f\left(\frac{N}{S}\right) + k_4 C_N^{in} u \tag{2}$$

$$\dot{B} = \varphi(t)\, K_S(S) f\left(\frac{N}{S}\right) g(B) \tag{3}$$

where

$$K_S(S) = \begin{cases} 0, & S \in [0, S_w] \\ \dfrac{S - S_w}{S^\star - S_w}, & S \in [S_w, S^\star] \\ 1, & S \in [S^\star, 1] \end{cases} \qquad K_R(S) = \begin{cases} 0, & S \in [0, S_h] \\ \dfrac{S - S_h}{1 - S_h}, & S \in [S_h, 1] \end{cases}$$

with threshold values $0 < S_h < S_w < S^\star < 1$,

$$f(C_N) = \begin{cases} \dfrac{C_N}{\eta_C}, & C_N \in [0, \eta_C] \\ 1, & C_N > \eta_C \end{cases}$$

The crop radiation interception efficiency φ, which is a C^1 increasing function with $\varphi(0) = 0$, $\varphi(T) = 1$, is used to model the regulation between crop transpiration $\varphi(t)K_S(S)$ and soil evaporation $(1-\varphi(t))K_R(S)$, where K_S, K_R have classical expressions that can be found in the literature, while f is the plant nitrogen uptake function. The growth function g is C^1 and positive for non null biomass. The control u is the irrigation rate, that takes values in a set $U = [0, u_{\max}]$. Parameters k_i $(i = 1, \cdots, 4)$ are all positive numbers, where k_1 is a rate related to soil porosity, k_3 a stochiometric coefficient, and k_2, k_4 are related to characteristics of the irrigation system. The originality of this model is to consider irrigation water containing nitrogen at concentration denoted C_N^{in}, a practice known as fertigation. Note that adding nitrogen needed for crops in this manner requires adding water, which could require more water than needed. On the other hand, providing water needed for crops could dilute the nitrogen present in soil if the concentration C_N^{in} is not large enough. This interplay between water and nitrogen supplies requires to investigate beneficial strategies for crops that consume reasonable amounts of water, which is the main purpose of the present work.

We shall say that a control (time function) u is *admissible* if it belongs to $L^1([0, T], U)$ and the corresponding solution S remains on the domain $[0, 1]$. Clearly, any solution of (1) with a non-negative initial solution remains non-negative whatever is the control. The state constraint $S \leq 1$ can be simply satisfied by imposing the following condition on the control

$$\{S = 1\} \Rightarrow u \leq \frac{1}{k_2} \ .$$

From the agronomic viewpoint, this means that when the soil is saturated, *i.e.* $S \equiv 1$, the irrigation has to be regulated to avoid water loss. We shall denote $\mathcal{U}$ the set of admissible controls u. We shall say that the crop suffers from *hydric stress* when the value of the function K_S is not maximal (that is when $S < S^\star$), and *nitrogen stress* when the value of the function f is not maximal (that is when $N - \eta_C S < 0$). The parameters C_N^{in} and $u_{\max}$ are the operating parameters of the fertigation. This model does not consider climatic events like rain and droughts because we implicitly assume that crops grow in a greenhouse.

For the following, we shall denote by $(S^u_{t_0,S_0}, N^u_{t_0,S_0,N_0})$ the solution of (1)-(2) with $S(t_0) = S_0$, $N(t_0) = N_0$ and control $u \in \mathcal{U}$.

3 The viability analysis

From equation (3), one can see that the biomass growth is maximal when

$$K_S(S(t)) f\left(\frac{N(t)}{S(t)}\right) = 1, \quad t \in [0, T]$$

which amounts to claim that the state $(S(t), N(t))$ belongs to the set

$$E := \{(S(t), N(t)) \in [0, 1] \times \mathbb{R}_+ \mid S(t) \geq S^\star,\ N(t) \geq \eta_C S(t)\} \tag{4}$$

(see Figure 1) for any $t \in [0, T]$. Therefore, the maximal production of biomass at time T is attained exactly for solutions $(S(t), N(t))$ that remain in the set E for $t \in [0, T]$. Being in E amounts to state that both hydric and nitrogen stresses are avoided. So, we shall consider the following viability property.

Definition 1. *The domain E is* viable *if for any initial condition $(t_0, S_0, N_0) \in [0, T] \times E$, there exists an admissible control u such that the solution of (1)-(2) with $S(t_0) = S_0$, $N(t_0) = N_0$ verifies $(S(t), N(t)) \in E$ for any $t \in [t_0, T]$.*

For convenience, we define the numbers

$$C_1 = \eta_C k_1 - k_3, \quad C_2 = k_4 C_N^{in} - \eta_C k_1 k_2.$$

Proposition 1. *The domain E is viable in the sense of Definition 1 if and only if the condition*

$$C_N^{in} \geq \underline{C_N^{in}} := \frac{k_2}{k_4} \max\Big(\eta_C k_1\big(1 - K_R(S^\star)\big), k_3\Big) \tag{5}$$

is fulfilled and $u_{\max}$ is such that

$$u_{\max} \geq \underline{u_{\max}} := \begin{cases} \max\left(\dfrac{1}{k_2}, \dfrac{-C_1}{C_2}\right) & \text{if } C_1 < 0 \text{ and } C_2 > 0, \\ \dfrac{1}{k_2} & \text{otherwise} \end{cases} \tag{6}$$

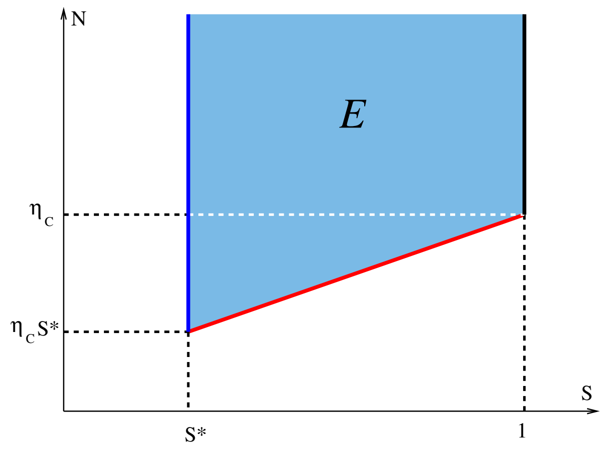

Figure 1: Illustration of the set E. The boundaries of the hydric stress is depicted in blue and of the nitrogen one in red

Proof. We have to show the existence of a control at any point on the boundary of the domain E, such that the inward pointing condition is satisfied (see [2]). Conditions (5) and (6) amount to guarantee that it is possible to have $\dot{S} \geq 0$ at $S = S^\star$, $\dot{S} \leq 0$ at $S = 1$ and $\dot{N} - \eta_C \dot{S} \geq 0$ when $N = \eta_C S$. The calculations, distinguishing between the signs of the numbers C_1 and C_2, are detailed in the technical report [9].

□

4 The optimal control problem

We assume in the rest of the paper that the operating parameters C_N^{in}, $u_{\max}$ verify the viability conditions:

$$C_N^{in} \geq \underline{C_N^{in}}, \quad u_{\max} \geq \underline{u_{\max}},$$

where $\underline{C_N^{in}}$ and $\underline{u_{\max}}$ are given in (5)-(6). According to Proposition 1, the set E defined in (4) is thus viable, and one can consider the optimization problem of minimizing the total amount of water delivered on $[0, T]$ while remaining in the set E, (*i.e.* guaranteeing the maximal biomass production at time T).

Problem $\mathcal{P}$: For $(S_0, N_0) \in E$, search

$$\inf_{u \in \mathcal{U}} \int_0^T u(\tau) d\tau \ \text{ s.t. } \ (S^u_{0,S_0}(t), N^u_{0,S_0,N_0}(t)) \in E, \, t \in [0, T].$$

Problem $\mathcal{P}$ takes the form of a standard optimal control problem with state constraints, and classical argumentation ensures the existence of optimal solutions (see, for instance, [4]). A usual approach to studying optimal solutions is to write necessary optimality conditions from the Maximum Principle with state constraints. In general, these conditions do not provide straightforwardly the optimal solution because one has to analyze the solutions of the adjoint system with the maximization condition of the Hamiltonian and terminal conditions. We failed to characterize the optimal solutions of our problem in this way due to the non-autonomous behavior of the dynamics (the Hamiltonian is no constant along optimal solutions) and the measure-multipliers of the state constraints (or jumps in the adjoint variables) to be determined, which make together the analysis quite intricate, even though our constraints are of relative degree 1 for which formulas are available [16]. We have chosen another route exploiting the particular structure of the dynamics and the constraints with a recent extension of the Bang-Bang Principle for piecewise affine systems [5].

As usual for optimal control problems linear in the control variable, we call "bang arc" a part of trajectory with constant control u that takes extreme value (0 or 1) and "singular arc" a part of trajectory for which the Hamiltonian is singular in the control u (i.e. for which the switching function is identically null). When dealing with constraints, we distinguish also "boundary arc" as a part of trajectory for which at least one constraint is saturated.

Proposition 2. *For any initial condition $(S_0, N_0) \in E$, problem $\mathcal{P}$ admits an optimal solution composed of bang arcs in the interior of E and possibly boundary arcs on the boundary of E.*

Proof. Let (S_0, N_0, B_0) be an initial condition for system (1)-(2)-(3) with $(S_0, N_0) \in E$ and $B_0 > 0$. Let $V^\star$ be the optimal value of the Problem $\mathcal{P}$, and posit

$$M^\star = \int_0^T \varphi(t)dt\,.$$

We consider the variables

$$M(t) = \int_{B_0}^{B(t)} \frac{d\xi}{g(\xi)}, \quad V(t) = \int_0^t u(t)dt$$

and the extended controlled system defined on $\mathbb{R}_+^7$ (without constraints)

$$\left\{\begin{array}{lcl} \dot{S} & = & k_1(-\varphi(t) - (1-\varphi(t))K_R(S) + k_2 u) \\ \dot{N} & = & -k_3\varphi(t) + k_4 C_N^{in} u \\ \dot{M} & = & \varphi(t) \\ \dot{V} & = & u \\ \dot{Z}_1 & = & \max(S-1, 0) \\ \dot{Z}_2 & = & \max(S - S^\star, 0) \\ \dot{Z}_3 & = & \max(\eta_C S - N, 0) \end{array}\right. \tag{7}$$

Then, one can straightforwardly check that an optimal control $u(\cdot)$ for problem $\mathcal{P}$ minimizes the time to reach the target

$$\mathcal{T} := \{(S, N, M, V, Z_1, Z_2, Z_3) \in \mathbb{R}_+^7;\ (S, N) \in E,\ M = M^\star,\ V = V^\star,\ Z_i = 0\ (i = 1, 2, 3)\}$$

from the initial condition $(S_0, N_0, 0, 0, 0, 0, 0)$ for system (7) with controls in $\mathcal{U}$, and that the minimal time is exactly T. Note that the dynamics (7) is (non-autonomous) piecewise affine with a convex control set U, and its non-differential locus is exactly on the boundary of E. On can then apply the result given in [5], which provides an extension of the well-known Bang-Bang Principle and states that a reachable target for a piecewise (non-autonomous) affine dynamics with a compact convex control set is reached in minimal time with a control that takes extreme values where the dynamics is smooth with possible singular arcs at non-differential locus. Clearly, the associated trajectory to such a control satisfy $Z_i(t) = 0$ $(i = 1, 2, 3)$ for $t \in [0, T]$ and thus satisfy the constraint $(S(t), N(t)) \in E$ for any $t \in [0, T]$. This shows the existence of an optimal control with bang arcs in the interior of E and possibly boundary arcs on the boundary of E. □

Note that this general result does not provide information about the number and the order of switches and boundary arcs. However, it guides us to look for particular structures of the optimal solutions, which we did by exploiting the cascade property of the dynamics, where the S-dynamics does not depend on the other variable N (but the optimal control has to depend on both variables because of the constraints), and using comparison arguments with the introduction of intermediate variables. Our approach is as follows: we first consider the simplest solutions with a bang and boundary controls (*i.e.* with a minimal number of commutations) that saturate only one constraint among the three constraints $S \leq 1$, $S \geq S^\star$, $N \geq \eta_c S$ and study their optimality. In a second step, we search for more complicated trajectories (*i.e.* with more commutations) that can saturate more than one constraint by combining the former strategies. Finally, we cover all possible initial conditions in E.

Let us begin with a preliminary result that will be useful in the following.

Lemma 1. *For any $(S_0, N_0) \in E$, an optimal solution of the problem $\mathcal{P}$ minimizes $N(T)$ among all admissible controls. Moreover, the optimal cost is given by the expression*

$$V^\star = \frac{N^\star(T) - N_0 + k_3 \displaystyle\int_0^T \varphi(t)dt}{k_4 C_N^{in}}$$

where $N^\star(T)$ is the minimal value of $N(T)$.

Proof. From (7), one gets

$$N(t) = N_0 - k_3 \int_0^t \varphi(\tau)d\tau + k_4 C_N^{in} \int_0^t u(\tau)d\tau, \quad t \in [0, T].$$

Therefore, minimizing $\int_0^T u(\tau)d\tau$ amounts to minimizing $N(T)$ among all admissible solutions in E. Moreover, one gets the expression of the cost as

$$\int_0^T u(\tau)d\tau = \frac{N^\star(T) - N_0 + k_3 \int_0^T \varphi(t)dt}{k_4 C_N^{in}}$$

□

Remark 1. *Note that $\eta_C S^\star$ is the smallest value of N in E. Therefore, if there exists an admissible control such that the solution remains in E and verifies $N(T) = \eta_C S^\star$, then it is necessarily optimal.*

Because of the cascade structure of the dynamics (1)-(2), its is convenient to fix an initial value $S(0) = S_0 \in [S^\star, 1]$ for the humidity variable, and distinguish several cases for the initial value $N(0) = N_0$ of the nitrogen variable, with (S_0, N_0) in E. Guided by the intuition that for high values of the initial nitrogen content N_0 one expects the nitrogen stress to be never met, we characterize first values of N_0 for which the optimal solution saturates the hydric constraint $S \geq S^\star$ only (Section 4.1). Then, we study cases for which the nitrogen constraint $N \geq \eta S$ is saturated before the hydric one (Section 4.2). For lower values of N_0, we expect cases for which the nitrogen constraint $N \geq \eta S$ only is saturated, what we study in Section 4.3. However, we found a possible gap among these sets of values of N_0, but showed that it corresponds to singular cases in the sense that a (strict) convex combination of two particular extreme solutions, which is a singular arc, is optimal (Section 4.4). Note that this does not contradict the result in [5] as this latter one does not guarantee the uniqueness of optimal trajectories.

4.1 Saturation of the hydric stress only

We consider situations where applying the control $u \equiv 0$ generates a trajectory that hits the hydric constraint first (or never saturates both constraints), defining a control strategy that ensures the hydric constraint is fulfilled at any time.

Definition 2. *The S-strategy is given by the time-varying feedback*

$$u_S(t, S) := \begin{cases} 0, & S > S^\star, \\ u_S^{bdr}(t), & S = S^\star \end{cases} \tag{8}$$

where

$$u_S^{bdr}(t) := \frac{1}{k_2}\big(\varphi(t) + (1 - \varphi(t))K_R(S^\star)\big) \tag{9}$$

Note that under (6), the control (9) does take values in $[0, u_{\max}]$ and is thus admissible. This strategy consists of no irrigation until the humidity level reaches the threshold $S^\star$ if it can, and in this latter case, maintaining the humidity S constant at the value $S^\star$. We now characterize the optimality of this strategy. Let us define the hitting time related to the hydric constraint with the null control as follows.

$$t_S := \sup\{t \in [0, T];\ S^0_{0,S_0}(t) > S^\star\}, \quad S_S^\star := S^0_{0,S_0}(t_S). \tag{10}$$

Note that when the time horizon T is large enough, one has necessarily $S_S^\star = S^\star$.

Proposition 3. *The S-strategy is optimal for Problem $\mathcal{P}$ when condition*

$$C_S := C_1\varphi(t_S) + k_1\eta_C K_R(S_S^\star)(1 - \varphi(t_S)) \geq 0 \tag{11}$$

or

$$N_0 \geq N_0^\flat := \max\left(\eta_C S_0, \eta_C S_S^\star + k_3 \int_0^{t_S} \varphi(t)dt\right) \tag{12}$$

is fulfilled. Then, the optimal value of the criterion is equal to

$$V^\star = \begin{cases} \dfrac{1}{k_2}\displaystyle\int_{t_S}^T \big(\varphi(t) + (1 - \varphi(t))K_R(S^\star)\big)dt, & t_S < T \\ 0, & t_S = T \end{cases} \tag{13}$$

Proof. From equations (1)-(2), one gets

$$\begin{array}{rcl} \dot N - \eta_C \dot S & = & k_1\eta_C\big(\varphi + (1-\varphi)K_R(S)\big) - k_3\varphi + \big(k_4 C_N^{in} - \eta_C k_1 k_2\big)u \\ & = & C_1\varphi + (1-\varphi)k_1\eta_C K_R(S) + C_2 u \end{array} \tag{14}$$

For $S > S^\star$ and $u = 0$, one obtains from (14) the following properties

1. when $C_1 \geq 0$, one has $\dot N - \eta_C \dot S \geq 0$ with the control $u = 0$, and has necessarily the property

$$N(t) - \eta_C S(t) \geq 0, \quad t \in [0, t_S] \tag{15}$$

2. when $C_1 < 0$, one has

$$\ddot N(t) - \eta_C \ddot S(t) = C_1\varphi'(t) - \varphi'(t)K_R(S(t)) + (1-\varphi(t))k_1\eta_C K_R'(S(t))\dot S(t) \leq 0. \tag{16}$$

and thus

$$\dot N(t) - \eta_C \dot S(t) \geq \dot N(t_S) - \eta_C \dot S(t_S) = C_1\varphi(t_S) + (1-\varphi(t_S))k_1\eta_C K_R(S_S^\star), \quad t \in [0, t_S]$$

Therefore, when (11) is fulfilled, the property (15) is also verified. Otherwise, as (16) shows that the map $t \mapsto N(t) - \eta_C S(t)$ is concave on $[0, t_S]$, the property (15) is verified exactly when

$$N(0) - \eta_C S(0) \geq 0 \text{ and } N(t_S) - \eta_C S(t_S) \geq 0. \tag{17}$$

From (2) with $u \equiv 0$, one gets

$$N(t_S) = N_0 - k_3 \int_0^{t_S} \varphi(t)dt$$

Then (17) is fulfilled precisely when condition (12) is verified.

At $S = S^\star$, which is attainable before T when $t_S < T$, one obtains with the control (9)

$$\dot N - \eta_C \dot S = \left(C_1 + \frac{C_2}{k_2}\right)\big(\varphi + (1-\varphi)K_R(S^\star)\big) + k_3(1-\varphi)K_R(S^\star)$$

where

$$C_1 + \frac{C_2}{k_2} = \frac{k_4}{k_2}C_N^{in} - k_3$$

which is non-negative under the condition $C_N^{in} \geq \underline{C_N^{in}}$. Therefore, one has the property

$$\dot N(t) - \eta_C \dot S(t) \geq 0, \quad t \in [t_S, T] \tag{18}$$

with the S-strategy. So, property (15) along with (18) shows that the solution $S_{0,S_0}^{u_S}$ generated by the S-strategy remains in E on $[0, T]$, when (11) or (12) is fulfilled. We will prove that it is necessarily optimal.

Let us denote by $S(\cdot)$ the solution generated by the S-strategy, and u the corresponding control function. For any other admissible solution $\tilde S$ with a control $\tilde u$, one has

$$\dot{\tilde S}(t) - \dot S(t) = -k_1(1-\varphi(t))\frac{\tilde S(t) - S(t)}{1 - S_h} + k_1k_2\tilde u(t) := F(t, \tilde S(t) - S(t)), \quad t \in [0, t_S]$$

As the function F satisfies $F(t, 0) \geq 0$ for any $t \in [0, t_S]$, we deduce that the solution $\delta = \tilde S - S$ of $\dot\delta = F(t, \delta)$, $\delta(0) = 0$ stays non-negative on this time interval. When $T > t_S$, one has on $[t_S, T]$ $S(t) = S^\star$ which is the smallest admissible value to stay in E. Therefore, any admissible solution $\tilde S$ that stays in E verifies $\tilde S(t) \geq S(t)$ for any $t \in [0, T]$. Then one can write

$$\begin{array}{rl} & S(T) - S(0) = -k_1 \displaystyle\int_0^T \varphi(t) + (1-\varphi(t))K_R(S(t))dt + k_1k_2 \int_0^T u(t)dt \\ \leq & \tilde S(T) - \tilde S(0) = -k_1 \displaystyle\int_0^T \varphi(t) + (1-\varphi(t))K_R(\tilde S(t))dt + k_1k_2 \int_0^T \tilde u(t)dt \end{array}$$

from which we obtain

$$\int_0^T u(t)dt \leq \frac{1}{k_2}\int_0^T (1-\varphi(t))(K_R(S(t)) - K_R(\tilde{S}(t)))dt + \int_0^T \tilde{u}(t)dt$$

and as the function K_R is increasing, we deduce

$$\int_0^T u(t)dt \leq \int_0^T \tilde{u}(t)dt$$

that is the optimality of the S-strategy. Finally, expression (13) follows directly from (8). □

Let us stress that conditions (11), (12) do not depend on operating parameters and are thus intrinsic to the crop-soil system. We focus now on situations for which these two conditions are both invalidated. Note that having condition (12) not fulfilled is possible (*i.e.* the existence of initial conditions $(S_0 N_0)$ in E) when $N_0^\flat > \eta_C N_0$. Thus, we shall consider now the following conditions:

$$C_S = C_1\varphi(t_S) + k_1\eta_C K_R(S_S^\star)(1-\varphi(t_S)) < 0 \tag{19}$$

and

$$N_0^\flat = \eta_C S_S^\star + k_3\int_0^{t_S}\varphi(t)dt > \eta_C S_0. \tag{20}$$

4.2 Saturation of both stresses

From the analysis in Section 4.1, we know that when applying the control $u \equiv 0$, the hydric constraint is hit first, then the nitrogen constraint remains non-saturated up to the terminal time. We consider here cases for which, when applying the control $u \equiv 0$, the nitrogen constraint is hit before the hydric one, and we define then a NS-strategy that guarantees both constraints to be satisfied, as follows. Note that we must consider cases for which $S_S^\star = S^\star$ (*i.e.* when $S = S^\star$ can be reached).

Definition 3. *The NS-strategy is given by the time-varying feedback*

$$u_{NS}(t,S,N) := \begin{cases} 0, & N > \eta_C S \text{ and } S > S^\star, \\ \max(0, u_N^{bdr}(t,S)), & N = \eta_C S \text{ and } S > S^\star, \\ u_S^{bdr}(t), & S = S^\star \end{cases} \tag{21}$$

where

$$u_N^{bdr}(t,S) := \frac{C_1\varphi(t) + k_1\eta_C K_R(S)(1-\varphi(t))}{-C_2} \tag{22}$$

and u_S^{bdr} is defined in (9).

Note that (19) implies $C_1 < 0$ and (5) gives $C_2 \geq -k_2 C_1 > 0$. Then for any $(t,S) \in [0,T]\times[0,1]$ one has $u_N^{\mathrm{bdr}}(t,S) \leq \frac{-C_1}{C_2}$, which is upper bounded by $u_{\max}$ with condition (6). The control u_{NS} is thus admissible.

From the definitions (10) of t_S and (20) of $N_0^\flat$, one has $N^0_{0,S_0,N_0}(t_S) = N^{u_{NS}}_{0,S_0,N_0}(t_S) = \eta_C S^\star$ with $N_0 = N_0^\flat$, and the following number is thus well defined.

$$N_0^\dagger := \inf\{N_0 \in [\eta_C S_0, N_0^\flat];\ \exists t \in [0,T] \text{ s.t. } N^{u_{NS}}_{0,S_0,N_0}(t) = \eta_C S^\star\} \tag{23}$$

One has the following result about the optimality of the NS-strategy.

Proposition 4. *Assume that conditions* (19)-(20) *are satisfied with $S_S^\star = S^\star$. For any $N_0 \in [N_0^\dagger, N_0^\flat]$, the NS-strategy is optimal for problem $\mathcal{P}$. The optimal cost is equal to*

$$V^\star = \frac{1}{k_4 C_N^{in}}\left(\eta_C S^\star - N_0 + k_3\int_0^{t^\star}\varphi(t)dt\right) + \frac{1}{k_2}\int_{t^\star}^T \varphi(t) + (1-\varphi(t))K_R(S^\star)dt$$

where

$$t^\star := \inf\{t \in (t_S, T];\ S^{u_{NS}}_{0,S_0,N_0}(t) = S^\star\}.$$

If $N_0^\dagger > \eta_C S_0$, then the optimal solution for $N_0 = N_0^\dagger$ verifies $N(T) = \eta_C S^\star$.

Proof. To lighten the reading, we omit here the proof (a bit long) which uses arguments similar to those of the proof of Proposition 3. However, a detailed proof is available in the technical report [9]. □

Remark 2. *When $N_0^\dagger = \eta_C$ and initial condition is $S_0 = 1$, $N_0 = \eta_C$, one has $S = N/\eta_C > S^\star$ at initial time. The control u_N^{bdr} defined in (22) is equal to $u_N^{bdr}(0,1) = -k_1\eta_C/C_2 < 0$, which is not admissible. However, the optimal control is $u = 0$ at the initial time, which immediately takes the trajectory out of the boundary $N - \eta_C S = 0$ until it joins this edge again. This is why the expression (21) of the feedback is $\max(0, u_N^{bdr}(t,S))$ and not simply $u_N^{bdr}(t,S)$ when $S = N/\eta_C > S^\star$.*

4.3 Saturation of the nitrogen stress only

The cases with $N_0 \geq N_0^\dagger$ have already been treated in Sections 4.1 and 4.2, where it has been proved that applying the control $u = 0$ until one of the constraints is saturated is optimal. For $N_0 < N_0^\dagger$, we found this strategy to be non longer optimal. Instead, we consider a radically different strategy, which consists of irrigating as much as possible since the beginning with a "bang-bang" or "bang-boundary-bang" control in such a way that the nitrogen constraint only is saturated strictly at final time T (indeed, we found that saturating the constraint $N - \eta_C S \geq 0$ on a time interval of positive length is not optimal if the trajectory does not touch $S = S_S^\star$). This new strategy is defined as follows.

Definition 4. *The N-strategy is parameterized by a commutation time $t_c \in [0,T]$, and given by the feedback*

$$u_N(t,S,t_c) := \begin{cases} u_{\max}, & t < t_c \text{ and } S < 1, \\ \bar{u}, & t < t_c \text{ and } S = 1, \\ 0 & t \geq t_c, \end{cases} \tag{24}$$

where

$$\bar{u} := \frac{1}{k_2}. \tag{25}$$

Clearly, this control is admissible with condition (6). We begin by a technical Lemma.

Lemma 2. *There exists an unique $\underline{t}_c \in [0,T]$ such that $S_{0,S_0}^{u_N(\cdot,\cdot,\underline{t}_c)}(T) = S_S^\star$ for the dynamics (7).*

Proof. Consider the map $\mu : t_c \mapsto S_{0,S_0}^{u_N(\cdot,\cdot,t_c)}(T)$, which is clearly continuous, and let us show that it is increasing. Take $t_{c,1} < t_{c,2}$ in $[0,T]$, and denote S_1, S_2 the corresponding solutions with control $u_N(\cdot,\cdot,t_{c,1})$, $u_N(\cdot,\cdot,t_{c,2})$ respectively. Clearly, the solutions S_1, S_2 coincide on $[0,t_{c,1}]$. One has $S_2(t_{c,1}) = S_1(t_{c,1})$ and $S_2(t_{c,2}) > S_1(t_{c,2})$. On the time interval $[t_{c,2},T]$, S_1, S_2 are solutions of the same dynamics with control $u = 0$. By comparison of solutions of scalar dynamics (see for instance [23]), one gets $S_2(t) > S_1(t)$ for any $t \in [t_{c,2},T]$, which implies $\mu(t_{c,2}) > \mu(t_{c,1})$. For $t_c = T$, one has $\mu(T) = S_{0,S_0}^{u_N(\cdot,\cdot,T)}(T) \geq S_0 \geq S^\star \geq S_S^\star$. For $t_c = 0$, one has $\mu(0) \leq \mu(t_S) = S_{0,S_0}^0(T) = S_S^\star$. Therefore, there exists an unique $\underline{t}_c \in [0,T]$ such that $\mu(\underline{t}_c) = S_S^\star$. □

It will be relevant to define the time

$$t_1 = \max\{t \in [0,T],\ S_{0,S_0}^{u_{\max}}(t) \leq 1\}. \tag{26}$$

Then, the value

$$N_0^\sharp := \eta_C S_S^\star + k_3 \int_0^T \varphi(t)dt - k_4 C_N^{in}\big(u_{\max}\min(\underline{t}_c, t_1) + \bar{u}\max(0, \underline{t}_c - t_1)\big) \tag{27}$$

is such that the solution for $N_0 = N_0^\sharp$ and the control u_N with commutation time $\underline{t}_c$ reaches exactly the point $(S_S^\star, \eta_C S_S^\star)$ in E at the final time T. Note that one has necessarily $N_0^\sharp \leq N_0^\flat$ because any solution with $N_0 > N_0^\flat$ cannot reach the set $\{N - \eta_C S = 0\}$ at time T (see Section 4.1). One has the following result about the optimality of the control u_N.

Proposition 5. *Assume that conditions (19)-(20) are fulfilled. If $N_0^\sharp \geq \eta_C S_0$, then for any $N_0 \in [\eta_C S_0, N_0^\sharp]$, there exists a unique $t_c^\star \in [\underline{t}_c, T]$ such that the solution with control $u_N(\cdot,\cdot,t_c^\star)$ verifies $N(T) = \eta_C S(T) \geq \eta_C S^\star$. Then, the N-strategy with $t_c = t_c^\star$ is optimal for the problem $\mathcal{P}$, and the optimal cost is $V^\star = u_{\max}\min(t_c^\star, t_1) + \bar{u}\max(0, t_c^\star - t_1)$.*

Proof. Note first that condition (19) implies $C_1 < 0$ and then condition (5) gives $C_2 \geq -k_2 C_1 > 0$.

The map $N_c(t_c) := N^{u_N(\cdot,\cdot,t_c)}_{0,S_0,N_0}(t_c)$ is clearly continuous with respect to t_c. Then, the composed function

$$\xi(t_c) := N^0_{t_c,S(t_c),N_c(t_c)}(T) - \eta_C S^0_{t_c,S(t_c)}(T)$$

where $S_c(t_c) = S^{u_N(\cdot,\cdot,t_c)}_{0,S_0}(t_c)$, is continuous with respect to t_c (from the continuity of solutions of ordinary differential equations w.r.t. the initial condition). We look now for the existence of a zero of this function.

Let $t_c \in [0, T]$ and denote S, N the solution with control $u_N(\cdot,\cdot,t_c)$. For $t \in [0, \min(t_1, t_c)]$, one has

$$\begin{array}{rcl}\dot{N}(t) - \eta_C \dot{S}(t) & = & (-k_3 + \eta_C k_1)\varphi(t) + (k_4 C_N^{in} - \eta_C k_1 k_2) u_{\max} + \eta_C k_1 (1 - \varphi(t)) K_R(S(t)) \\ & \geq & C_1 \varphi(t) + C_2 u_{\max}.\end{array}$$

From (6), one has $C_2 u_{\max} > -C_1 > 0$ and thus $\dot{N}(t) - \eta_C \dot{S}(t) > 0$ for any $t \in [0, \min(t_c, t_1)]$. If $t_c > t_1$, then for $t \in [t_1, t_c]$, one has $S(t) = 1$, and from (2) one gets

$$\dot{N}(t) = -k_3 \varphi(t) + \frac{k_4 C_N^{in}}{k_2} > -k_3 + \frac{k_4 C_N^{in}}{k_2} \geq -k_3 + \frac{k_4 \underline{C_N^{in}}}{k_2} \geq 0,$$

(except when $t = t_c = T$), the last inequality being provided by condition (5). Therefore, the function

$$\gamma(t) := N(t) - \eta_C S(t)$$

is increasing on $[0, t_c]$. We immediately deduce the inequality $\xi(T) = \gamma(T) > \gamma(0) \geq 0$.

Let us show that ξ is increasing with respect to t_c. Take $t_c' > t_c$ in $[0, T]$ and denote S', γ' the solution with control $u_N(\cdot,\cdot,t_c')$. Note that the solutions coincide on $[0, t_c]$ and thus one has $S'(t_c) = S(t_c)$, $\gamma'(t_c) = \gamma(t_c)$. Note that one has also

$$\int_{t_c}^{t_c'} u_N(t, S'(t), t_c') dt > \int_{t_c}^{t_c'} u_N(t, S(t), t_c) dt = 0$$

and one gets from equation (1) the inequality $S'(t) > S(t)$ for $t \in (t_c, t_c']$. Then, by uniqueness of the solutions of the scalar dynamics (1) with control $u = 0$, we deduce the inequality $S'(t) > S(t)$ for any $t \in (t_c', T]$. From equation (14), one can write

$$\xi(t_c') = \gamma'(T) = \gamma'(t_c) + \int_{t_c}^{T} \big(C_1 \varphi(t) + (1 - \varphi(t)) k_1 \eta_C K_R(S'(t))\big) dt + C_2 \int_{t_c}^{t_c'} u_N(t, S'(t), t_c')) dt$$

As K_R is increasing and $C_2 > 0$, we obtain

$$\xi(t_c') > \gamma(t_c) + \int_{t_c}^{T} \Big(C_1 \varphi(t) + (1 - \varphi(t)) k_1 \eta_C K_R(S(t))\Big) dt = \gamma(T) = \xi(t_c)$$

which proves that ξ is increasing. Moreover, as in the proof of Proposition 3 with $C_1 < 0$, we obtain that the function γ is concave (with control $u = 0$).

For $t_c = 0$, the control u is equal to 0 on the whole interval $[0, T]$, and one has for any $N_0 \leq N_0^\sharp$

$$\gamma(t_S) = N(t_S) - \eta_C S(t_S) = N_0 - k_3 \int_0^{t_S} \varphi(t) dt - \eta_C S_S^\star \leq N_0^\flat - k_3 \int_0^{t_S} \varphi(t) dt - \eta_C S_S^\star = 0.$$

If $t_S = T$, one has thus $\gamma(T) < 0$. If $t_S < T$ (which implies $S_S^\star = S^\star$), one has with condition (19)

$$\dot{\gamma}(t) < \dot{\gamma}(t_S) = C_1 \varphi(t_S) + (1 - \varphi(t_S) k_1 \eta_C K_R(S^\star) < 0, \quad t > t_S.$$

Therefore, one has $\gamma(T) < \gamma(t_S) \leq 0$. Whatever is t_S, one has thus $\xi(0) < 0$ when $t_c = 0$. The function ξ being continuous increasing with $\xi(T) > 0$, we deduce that the exists an unique $t_c^\star$ such that $\xi(t_c^\star) = 0$ (*i.e.* such that corresponding solution verifies $N(T) - \eta_C S(T) = 0$), which belongs to $(0, T)$. However, we must check that the corresponding solution remains in E. We have already shown that γ is increasing on $[0, t_c^\star]$, which implies $\gamma(t_c^\star) \geq 0$, and that γ is concave on $(t_c^\star, T)$ with $\gamma(T) = 0$. This implies that one has necessarily $\gamma(t) > 0$ for any $t \in [0, T)$. The constraint $N \geq \eta_C S$ is thus satisfied at any time.

The solution S is non decreasing on $[0, t_c^\star]$, and decreasing on $[t_c^\star, T]$ (as the control u is equal to 0 for $t \geq t_c^\star$). Therefore, the solution remains in E when $S(T) \geq S^\star$. If $t_S = T$, then one has necessarily

$S(T) \geq S_S^\star \geq S^\star$. If $t_S < T$, note that $T - \underline{t}_c$ is the largest time for which it is possible to reach $S^\star$ at T and thus one has $S(T) \geq S^\star$ exactly when $t_c^\star \geq \underline{t}_c$. From equation (2) one gets

$$N(T) = N_0 - k_3 \int_0^T \varphi(t)dt + k_4 C_N^{in} \int_0^{t_c} u_N(t, S(t), t_c)dt$$

and for $t_c = \underline{t}_c$, one has $S(T) = S^\star$ with the property

$$N_0 \leq N_0^\sharp \iff N(T) \leq N_0^\sharp - k_3 \int_0^T \varphi(t)dt + k_4 C_N^{in} \int_0^{\underline{t}_c} u_N(t, S(t), \underline{t}_c)dt = \eta_C S^\star \iff \xi(\underline{t}_c) \leq 0 = \xi(\underline{t}_c^\star).$$

As the function ξ is increasing, we deduce the equivalence

$$N_0 \leq N_0^\sharp \iff t_c^\star \geq \underline{t}_c.$$

We conclude that the solution with the control u_N such that $N(T) - \eta_C S(T) = 0$ satisfies also the constraint $S(t) \geq S^\star$, $t \in [0, T]$, when $N_0 \leq N_0^\sharp$ (when $t_S = T$, note that one has necessarily $\underline{t}_c = 0$).

Let us show now that the solution (S, N) with the control $u(t) = u_N(t, S(t), t_c^\star)$ is optimal. If not, there should exist another solution in E, denoted $(\tilde{S}, \tilde{N})$ with an admissible control $\tilde{u}$ such that

$$\int_0^T \tilde{u}(t)dt < \int_0^T u(t)dt.$$

Note from equation (2) that this implies the inequality

$$\tilde{N}(T) < N(T). \tag{28}$$

On the another hand, one has from equation (14)

$$N(T) - \eta_C S(T) = N_0 - \eta_C S_0 + C_1 \int_0^T \varphi(t)dt + k_1 \eta_C \int_0^T (1 - \varphi(t)) K_R(S(t))dt + C_2 \int_0^T u(t)dt = 0$$

and

$$\tilde{N}(T) - \eta_C \tilde{S}(T) = N_0 - \eta_C S_0 + C_1 \int_0^T \varphi(t)dt + k_1 \eta_C \int_0^T (1 - \varphi(t)) K_R(\tilde{S}(t))dt + C_2 \int_0^T \tilde{u}(t)dt \geq 0$$

which implies

$$k_1 \eta_C \int_0^T (1 - \varphi(t))\big(K_R(S(t)) - K_R(\tilde{S}(t))\big)dt \leq C_2 \int_0^T \tilde{u}(t) - u(t)dt < 0. \tag{29}$$

Note that one has, for a.e. $t \in [0, t_1]$

$$\begin{aligned} \dot{S}(t) - \dot{\tilde{S}}(t) &= k_1\big(- \varphi(t))(K_R(S(t)) - K_R(\tilde{S}(t))) + k_2(u_{\max} - \tilde{u}(t))\big) \\ &\geq -k_1 \varphi(t))(K_R(S(t)) - K_R(\tilde{S}(t))) \end{aligned}$$

with $S(0) = \tilde{S}(0)$, which implies that the inequality $S(t) \geq \tilde{S}(t)$ is satisfied any $t \in [0, t_1]$. For $t \in [t_1, t_c^\star]$, one has $S(t) = 1 \geq \tilde{S}(t)$. Therefore, the inequality $S(t) \geq \tilde{S}(t)$ is satisfied for any $t \in [0, t_c^\star]$, and as the function K_R is increasing, we deduce the inequality $K_R(S(t)) - K_R(\tilde{S}(t)) \geq 0$ for any $t \in [0, t_c^\star]$. This implies from (29), that there exists necessarily an non-empty interval $(t_a, t_b) \subset (t_c^\star, T)$ such that $K_R(S(t)) - K_R(\tilde{S}(t)) < 0$ for $t \in (t_a, t_b)$. As the function K_R is increasing, we deduce that one should have $\tilde{S}(t) > S(t)$ for $t \in (t_a, t_b)$. But then, as $\tilde{u}(t) \geq 0 = u(t)$ for any $t \in [t_a, T]$, the solutions S, $\tilde{S}$ cannot cross *i.e.* the inequality $\tilde{S}(t) > S(t)$ is verified for any $t > t_a$. In particular, at $t = T$, one gets

$$\tilde{N}(T) \geq \eta_C \tilde{S}(T) > \eta_C S(t) = N(T)$$

which contradicts the inequality (28).

Finally, the expression of $V^\star$ follows from the definition of the strategy u_N. □

Remark that the case for which $S_S^\star > S^\star$ (that is when $S = S^\star$ cannot be reached) is treated with Proposition 5.

4.4 The singular case

For convenience, let us posit

$$\tilde{N}_0^\sharp := \max(\eta_C S_0, N_0^\sharp) \tag{30}$$

where $N_0^\sharp$ is defined in (27). Note that when condition (11) is not fulfilled and $N_0^\dagger > \tilde{N}_0^\sharp$, the case $N_0 \in (N_0^\dagger, \tilde{N}_0^\sharp)$ is not covered by Propositions 3, 4, 5 of former sections. In accordance with Proposition 2, we characterize an optimal solution which is "bang-(boundary)-bang-boundary" as follows. It combines the feedback expressions (21) and (24) of the NS and N-strategies.

Definition 5. *The $NS^\star$-strategy is parameterized by a commutation time $t_c \in [0, T]$, and given by the feedback*

$$u_{NS^\star}(t, S, t_c) := \begin{cases} u_{\max}, & t < t_c \text{ and } S < 1, \\ \bar{u}, & t < t_c \text{ and } S = 1, \\ 0 & t \geq t_c \text{ and } N > \eta_C S \\ u_N^{bdr}(t, S) & t \geq t_c \text{ and } N = \eta_C S \end{cases} \tag{31}$$

where the controls $\bar{u}$ and u_N^{bdr} are defined in (25) *and* (22).

Proposition 6. *Assume that conditions* (19)-(20) *are satisfied with $S_S^\star = S^\star$. If $N_0^\dagger > \tilde{N}_0^\sharp$, where $N_0^\dagger$, $\tilde{N}_0^\sharp$ are defined in* (23), (30), *then for any $N_0 \in (\tilde{N}_0^\sharp, N_0^\dagger)$, there exists $\hat{t}_c \in (0, T)$ such that the solution with the control $u_{NS^\star}(\cdot, \cdot, \hat{t}_c)$ verifies $S(T) = S^\star$ and $N(T) = \eta_C S^\star$. For such a commutation time, the corresponding control is optimal and the cost is given by*

$$V^\star = \frac{1}{k_4 C_N^{in}} \left(\eta_C S^\star - N_0 + k_3 \int_0^T \varphi(t) dt \right). \tag{32}$$

Proof. We omit the proof, which is available in the technical report [9].

□

Finally, exploiting the results of Propositions 3, 4, 5 and 6, we refine the result of the former Proposition 2 as follows.

Theorem 7. *For any initial condition $(S_0, N_0) \in E$, Problem $\mathcal{P}$ admits an optimal solution which is composed of bang arcs in the interior of E and possibly boundary arcs on the boundary of E, with up to three commutations.*

Remark 3. *Under conditions of Proposition 6, one can easily check that for any $N_0 \in (\tilde{N}_0^\sharp, N_0^\dagger)$, the control*

$$u(t) = \lambda u^\sharp(t) + (1 - \lambda) u^\dagger(t), \quad \lambda = \frac{N_0^\dagger - N_0}{N_0^\dagger - \tilde{N}_0^\sharp}, \quad t \in [0, T] \tag{33}$$

is optimal, where $u^\dagger$, $u^\sharp$ denote the optimal open loop controls given by Propositions 4, 5 for the initial condition $N_0 = N_0^\dagger$, $\tilde{N}_0^\sharp$ respectively. There exists indeed an infinity of optimal solutions. Let S, N be the solution with the control u defined on (33) *for an initial condition $N_0 \in (\tilde{N}_0^\sharp, N_0^\dagger)$. Take any absolutely continuous function δ such that*

$$\int_0^T (1 - \varphi(t)) \delta(t) = 0.$$

with $\delta(t) = 0$ for $t \in [0, \tau] \cup [T - \tau, T]$ for some positive $\tau < T/2$, and for any t with $u(t) = 0$ Then, the control

$$\tilde{u}(t) = u(t) + \varepsilon \left(\frac{1 - \varphi(t)}{k_2(1 - S_h)} \delta(t) + \frac{\dot{\delta}(t)}{k_1 k_2} \right), \quad a.e.\ t \in [0, T]$$

takes values in $[0, u_{\max}]$ for a.e. $t \in [0, T]$, provided the number $\varepsilon > 0$ to be small enough. One can easily check that the control $\tilde{u}$ is also optimal.

We call this case "singular" because, as shown in Remark 3, there exist optimal controls that do not take extreme values nor saturate the constraints, which are thus necessarily singular controls, what differs from the optimal strategies in our former propositions.

5 Numerical illustrations and discussion

We have considered a class of concave functions φ, given by the following expression

$$\varphi(t) = \frac{t(1+\alpha)}{t+\alpha}, \quad \alpha > 0$$

parameterized by α, which in some sense measures how quickly the vegetation cover progresses over time, impacting the LAI (Leaf Area Index). We have chosen a plausible set of values of model parameters (see Table 1), inspired by the literature.

k_1	k_2	k_3	k_4	$S^\star$	S_w	S_h	T	α	η_C	C_N^{in}	$u_{\max}$	S_0
1	1	1.55	1	0.7	0.4	0.2	1	0.09	0.5	1.55	1.4	0.95

Table 1: Parameters values of the model

For these parameters, we have computed the lower bounds (5), (6) on the operating parameters C_N^{in}, $u_{\max}$ for the set E to be viable (see Table 2). The values in Table 1 of these parameters being above these bounds, we deduce that the set E is viable for these operating parameters.

C_N^{in}	$u_{\max}$
1.55	1

Table 2: Lower bounds on the operating parameters for the viability of the set E

Then, we have determined by solving numerically the differential equation (1) the various quantities considered in Propositions 3, 5, 4, that are reported them in Table 3.

t_S	C_S	$N_0^\flat$	$\underline{t}_c$	$N_0^\sharp$	$N_0^\dagger$	$\eta_C S_0$
0.269673	−0.801007	0.594958	0.699568	0.502550	0.551077	0.475

Table 3: Values of the various quantities considered in Propositions 3, 5 and 4

From Table 3, one has $C_S < 0$, and from Proposition 3 we get that the S-strategy is optimal for $N_0 \geq N_0^\flat$. As $t_S < T$, we have $S_S^\star = S^\star$ and from Proposition 4 we know that the NS-strategy is optimal for $N_0 \in [N_0^\dagger, N_0^\flat]$. In Table 3, we see also that one has $N_0^\sharp > \eta_C S_0$, and then the N-strategy is optimal for $N_0 \in [\eta_C S_0, N_0^\sharp]$ according to Proposition 5. Finally, as $N_0^\dagger > N_0^\sharp$, the $NS^\star$-strategy is optimal for $N_0 \in (N_0^\sharp, N_0^\dagger)$ according to Proposition 6, but also the singular control defined in 33. For this set of parameters, all four possible strategies are met depending on the initial nitrogen concentration N_0, as summarized in Table 4.

N_0	$[\eta_C S_0, N_0^\sharp]$	$(N_0^\sharp, N_0^\dagger)$	$[N_0^\dagger, N_0^\flat]$	$[N_0^\flat, \to)$
optimal control	N-strategy	$NS^\star$-strategy	NS-strategy	S-strategy
	anticipate stress		*act on stress*	

Table 4: Optimality of the various strategies depending on N_0

For various values of N_0, we have determined numerically the optimal solutions with the help of the Bocop software, which is based on a direct method [1]. We can see in Figures 2 to 7 that the solutions are in perfect accordance with the theoretical results, as predicted by Table 4. Of course, the advantage of the analytical analysis is that it provides feedback strategies instead of open loops given by the numerical method, which is easier and more robust to apply in practice.

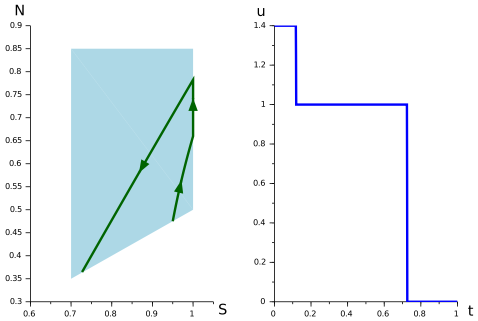

Figure 2: Optimal solution for $N_0 = \eta_C S_0$

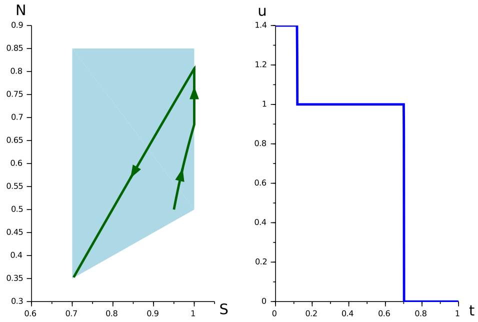

Figure 3: Optimal solution for $N_0 = N_0^\sharp$

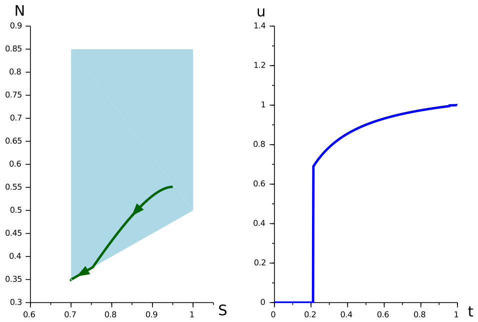

Figure 4: Optimal solution for $N_0 = N_0^\dagger$

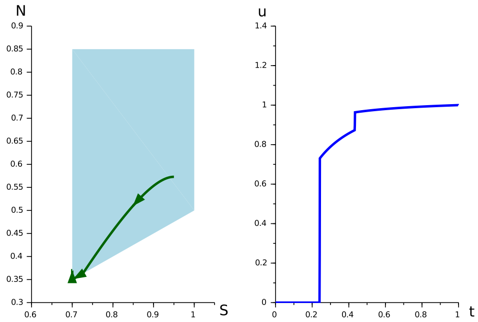

Figure 5: Optimal solution for $N_0 \in (N_0^\dagger, N_0^\flat)$

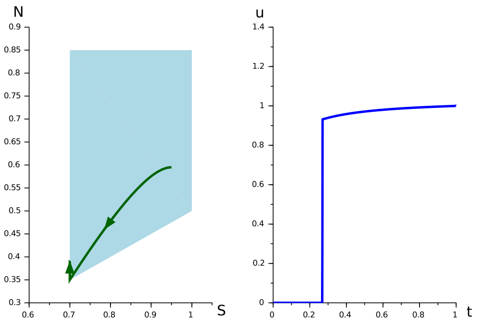

Figure 6: Optimal solution for $N_0 = N_0^\flat$

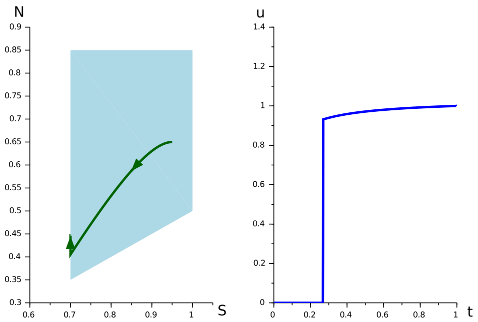

Figure 7: Optimal solution for $N_0 > N_0^\flat$

For the singular case, that is for $N_0 \in (N_0^\sharp, N_0^\dagger)$, we have plotted various solutions

1. on Figure 8 , the solution given by the $NS^\star$-strategy (cf Proposition 6) which is in accordance with the (extended) Bang-Bang Principle. Note this solution presents three commutations, while former ones have one or tow. This is the maximum number given by Theorem 7.

2. on Figure 9, the solution as the convex combination of trajectories and control of the particular cases given in Figures 3 and 4, given by expression (33). This solution is singular on the whole time interval.

3. on Figure 10, the solution provided by the Bocop software, which is also singular on the whole time interval. The numerical software has provided a kind of regularization of the solution given in (33). The control is smooth but is a pure open loop.

Note that all these solutions end at the corner point $(S^\star, \eta_C S^\star)$ at time T and are thus optimal (cf Remark 1).

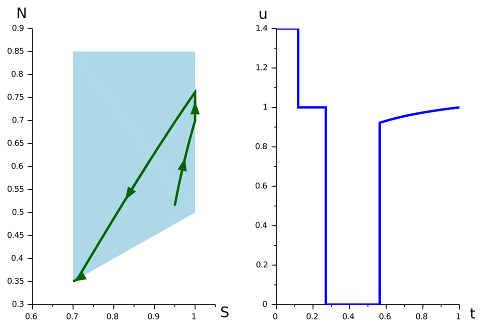

Figure 8: Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$ with the $NS^\star$-strategy

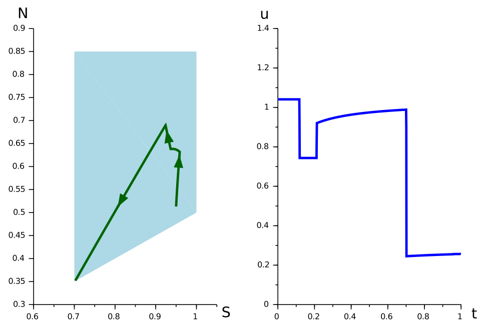

Figure 9: Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$ with the control (33)

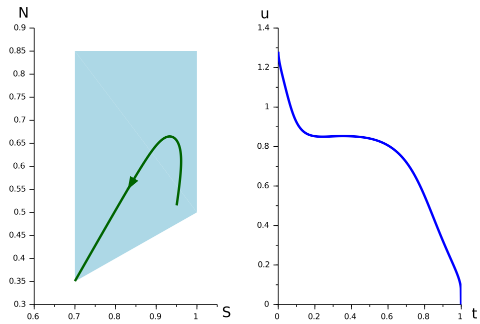

Figure 10: Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$ obtained with the numerical software Bocop

Let us comment on these optimal solutions.

- When the initial quantity of nitrogen is high ($N_0 > N_0^\flat$), it is not surprising that the nitrogen constraint is not saturated. Indeed, the S-strategy coincides with the optimal one already found in the literature for the simplified model with no nitrogen stress [3]. It consists of no irrigation until the humidity threshold $S^\star$ is met and then keeping the humidity level equal to this threshold up to the final time. A sensor measuring the humidity level S is required to apply this control strategy.

- For the lower value of initial nitrogen $N_0 \in (N_0^\dagger, N_0^\flat)$, the lack of irrigation causes the system to face the nitrogen stress before the hydric one. Then, the optimal solution consists of keeping the system at the edge of this constraint until it reaches the humidity threshold and then keeping the humidity level precisely equal to this threshold (as for the S-strategy). In this way, the NS-strategy generalizes the S-strategy in the context of nitrogen stress. Here, as online information, the measurement of the nitrogen content N (or its concentration N/S) is required, in addition to the humidity level S.

- A surprising feature occurs when keeping the system at the edge of the nitrogen stress does not allow it to reach the humidity threshold (because the time horizon has been reached before) for small initial nitrogen content $N_0 < N_0^\sharp$. Then, the NS-strategy is no longer optimal, and the best one is radically different. The N-strategy consists of irrigating since the beginning to maintain the humidity level at its maximal level up to a precise time $t_c^\star$, from which stopping the irrigation conducts the system to touch the N-stress precisely at the terminal time. Unlike the S-strategy, the N-strategy is not a particular instance of the NS-strategy. On the contrary, the N-strategy requires anticipating the future and providing water from the beginning. From a practical viewpoint, this "open-loop" strategy requires the precise determination of the optimal commutation time $t_c^\star$ from the model's data an initial N_0, without the need for future online measurement.

- Another non-intuitive feature is the possibility for a subset of initial conditions $N_0 \in (N_0^\sharp, N_0^\dagger)$ to reach the "corner" point $(S^\star, \eta_C S^\star)$, defined as the intersection of the two constraints, precisely at the final time, and different strategies are possible. The $NS^\star$-strategy does not require irrigation on a time window, but differently to the other cases, this time window is not at the beginning or the end of the season. Other possibilities are singular controls given by particular irrigation strategies, which no longer consist of keeping the system at boundaries of the set E or without irrigation. Such strategies are most demanding to implement because they require the prior determination of the whole time profile depending on the initial N_0. However, it can obtained with smooth controls but remains open-loops. For these cases, the final nitrogen content is the smallest among all possible ones.

On Figure 11, we have plotted the optimal value of the water consumption V and the corresponding final nitrogen $N(t)$, as function of the initial nitrogen content. This shows that although the optimal

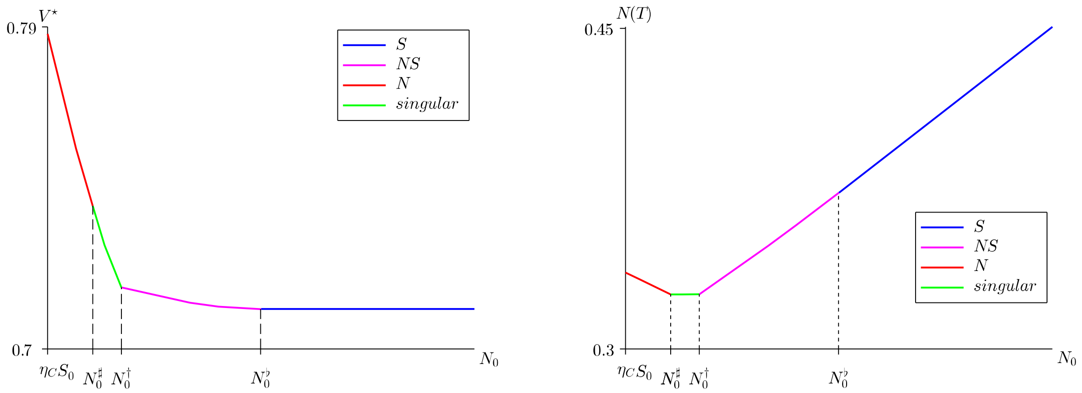

Figure 11: Minimal water consumption (left) and final nitrogen content (right) as a function of N_0 (the colors indicate which strategy is optimal)

irrigation strategies can be quite different for $N_0 < N_0^\sharp$ or $N_0 > N_0^\dagger$, the minimal water consumption $V^\star$ is constantly decreasing with initial N_0, down to the constant level for which the system receives enough nitrogen never to face nitrogen stress, that is when $N_0 \geq N_0^\flat$. Note that the part of the curve for $N_0 \in (N_0^\sharp, N_0^\dagger)$ is linear, justified by the expression (32) given in Proposition 6. Due to the particular structure of the singular cases with $N_0 \in (N_0^\sharp, N_0^\dagger)$ for which the final nitrogen content $N(T)$ is always the minimal value $\eta_C S^\star$ of N in E, the map $N_0 \mapsto N(T)$ is non-monotonic and flat for $(N_0^\sharp, N_0^\dagger)$.

If one intends to minimize both the water consumption and the residual nitrogen content (while ensuring maximal biomass production), a Pareto diagram might be helpful. In Figure 12, one can see that the Pareto front is for $N_0 \in [N_0^\dagger, N_0^\flat]$, for which a compromise between water consumption and final nitrogen content has to be chosen. Note that this locus corresponds to the optimality of the NS-strategy.

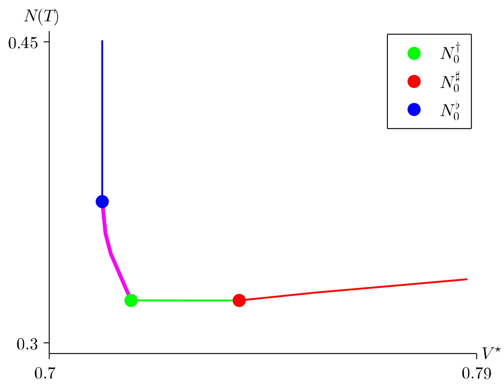

Figure 12: Set of optimums $(N(T), V^\star)$. The Pareto front is depicted in magenta

We have presented here a case study for which all the four N, S, NS, $NS^\star$ strategies strategies can be optimal with one, two or three commutations. For other sets of parameters, one can have $N_0^\dagger < N_0^\sharp$

and then there is no singular case, or $N_0^\sharp < \eta_C S_0$ and then the N-strategy is never optimal, or $N_0^\dagger = N_0^\flat$ and in this last situation the NS-strategy is never optimal. To preserve a reasonable paper length, we have not depicted here other such simpler cases.

6 Conclusion

Considering the problem of minimal irrigation under maximal biomass production for a simplified crop model, we have provided conditions for this optimal control problem with state constraints to be feasible, and given the different structures of the optimal strategy depending on the initial nitrogen content. We have characterized optimal solutions as concatenations of bang and boundary arcs, these latter ones being on the boundary of the constraints set only, and shown that they have at most three commutations. This analysis illustrates the extended Bang-Bang Principle for non-autonomous piecewise affine dynamics. We have also exhibited cases for which there exists an infinity of optimal singular trajectories, related to the non-smooth feature of the constraints set.

These strategies can be implemented as simple feedback measuring online the humidity level S if the limit of water stress is met before the harvesting time or the nitrogen concentration N/S if the limit of the nitrogen stress is met before the harvesting time. In the other cases, both quantities need to be measured. We have also exhibited a particular set of initial conditions that conducts to the smallest residual nitrogen in the soil, which could serve as a theoretical target for the practitioners.

This study revealed that when the crop radiation efficiency increases rapidly with time, and the nitrogen concentration in the irrigation water is low, then the minimal residual nitrogen content in the soil is obtained when the initial nitrogen content belongs to a particular interval of values, and not for the smallest initial nitrogen content. Therefore, having a low initial nitrogen content in the soil could lead paradoxically to larger residual contents. This can be explained by the larger water consumption needed to maintain maximal production. This non-intuitive feature is due to the interplay between nitrogen dilution and water supply in fertigation. This interval for the initial nitrogen content could serve as a theoretical target for practitioners. However, the practical application of the fertigation strategies would have to face some technical and sanitary issues (such as time-varying fluctuations, pipe clogging, and undesirable nano-particles or pathogens), that are not addressed in the model.

Future perspectives of this work could concern non-viable cases, looking then to viability kernels, that are the largest subsets of initial conditions for which the optimal control problem under constraint is feasible [2], and the consideration of non deterministic climatic disturbances (rains, droughts).

Acknowledgments

The authors thank INRAE and Inria for M. Dadjo's PhD grant. This work has been achieved within the REUSE and Viability INRAE network frameworks.

References

[1] BOCOP: an open source toolbox for optimal control, Team Commands, Inria Saclay. `http://bocop.org`, 2017.

[2] J.-P. Aubin. *Viability Theory.* Birkäuser, 2009.

[3] K. Boumaza, N. Kalboussi, A. Rapaport, S. Roux, and C. Sinfort. Optimal control of a crop irrigation model under water scarcity. *Optimal Control Applications and Methods*, 42(6):1612–1631, 2021.

[4] L. Cesari. *Optimization - Theory and applications.* Springer, 1983.

[5] R. Chenevat, C. Bruno, S. Roux, and A. Rapaport. About the Bang-Bang Principle for spatially-temporally regional affine dynamics under constraints. Tech. report `https://hal.science/hal-04606183`, HAL open science, 2024.

[6] B. Cheviron, R.W. Vervoort, R. Albasha, R. Dairon, C. Le Priol, and J.-C. Mailhol. A framework to use crop models for multi-objective constrained optimization of irrigation strategies. *Environmental Modelling & Software*, 86:145–157, 2016.

[7] M.-G Dadjo, D. Efimov, J. Harmand, A. Rapaport, and R. Ushirobira. An adaptive observer for time-varying nonlinear systems - application to a crop irrigation model. In *62nd IEEE Conference on Decision and Control (CDC23)*, Singapore, Dec. 2023.

[8] M.-G Dadjo, A. Rapaport, J. Harmand, R. Ushirobira, and D. Efimov. Minimal water consumption for a crop fertirrigation model. In *European Control Conference (ECC24)*, Stockholm, Sweden, June 2024.

[9] M.G. Dadjo, A. Rapaport, J. Harmand, R. Ushirobira, and D. Efimov. Water-minimizing strategies under viability constraint for a crop fertirrigation model. Tech. report `https://hal.science/hal-04267286`, HAL open science, April 2025.

[10] A. Haddon, A. Rapaport, S. Roux, and J. Harmand. Model based optimization of fertilization with treated wastewater reuse. *Advances in Water Resources*, 181(104561), 2023.

[11] N. Kalboussi, S. Roux, K. Boumaza, C. Sinfort, and A. Rapaport. About modeling and control strategies for scheduling crop irrigation. *IFAC-PapersOnLine*, 52(23):43–48, 2019.

[12] M. Kefi, N. Kalboussi, A. Rapaport, J. Harmand, and H. Gabtni. Model-based approach for treated wastewater reuse strategies focusing on water and its nitrogen content: A case study for olive growing farms in peri-urban areas of sousse, tunisia. *Water*, 15(4):755, 2023.

[13] S. Lopes, F. Fontes, R. Pereira, M.-R. de Pinho, and A. Goncalves. Optimal control applied to an irrigation planning problem. *Mathematical Problems in Engineering*, 5076879, 2016.

[14] O. Neto, A. Haddon, F. Aichouche, J. Harmand, M. Mulas, and F. Corona. Predictive control of activated sludge plants to supply nitrogen for optimal crop growth. *IFAC-PapersOnLine*, 54(3):200–205, 2021.

[15] N. Pelak, R. Revelli, and A. Porporato. A dynamical systems framework for crop models: Toward optimal fertilization and irrigation strategies under climatic variability. *Ecological Modelling*, 365:80–92, 2017.

[16] H. Schättler. Local Fields of Extremals for Optimal Control Problems with State Constraints of Relative Degree 1. *Journal of Dynamical and Control Systems*, 12(4):563–599, 2006.

[17] N. Schütze and G. Schmitz. Occasion: new planning tool for optimal climate change adaption strategies in irrigation. *Journal of Irrigation and Drainage Engineering*, 136(12):836–846, 2010.

[18] N. Schüze, M. de Paly, and U. Shamir. Novel simulation-based algorithms for optimal open-loop and closed-loop scheduling of deficit irrigation systems. *Journal of Hydroinformatics*, 14(1):136–151, 2011.

[19] U. Shani, Y. Tsur, and A. Zemel. Optimal dynamic irrigation schemes. *Optimal Control Applications and Methods*, 25(2):91–106, 2004.

[20] T. Sinclair and T. Rufty. Nitrogen and water resources commonly limit crop yield increases, not necessarily plant genetics. *Global Food Security*, 1(2):94–98, 2012.

[21] P. Steduto, T. Hsiao, D. Raes, and E. Fereres. AquaCrop - the FAO crop model to simulate yield response to water: I. concepts and underlying principles. *Agronomy Journal*, 101(3):426–437, 2009.

[22] B.J. Sunantara and J. Ramírez. Optimal stochastic multicrop seasonal and intraseasonal irrigation control. *Journal of Water Resources Planning and Management*, 123(1):39–48, 1997.

[23] W. Walter. *Ordinary Differential Equations*. Springer, 1998.