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About identifiability and observability for a class of dynamical systems

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Abstract

We present novel results and constructive methods on the observability, identifiability and joined observability-identifiability analysis of autonomous dynamical systems, with a special focus on nonlinear systems. While most existing works treat rational systems, the study of general nonlinearities remains less developed, up to our knowledge. Moreover, even in rational systems, a rigorous and unified theoretical argument ensuring the identifiability of parameters from input-output equations is still lacking. To address these gaps, we extend existing results in the literature. The key ingredient of our methods is a test of the linear independence of some functions of the observations and their derivatives. We furthermore describe constructive procedures to retrieve the unknowns based on the resolution of linear systems of equations. Finally, we apply our approach to several illustrative cases, including two non-rational cases that highlight its relevance. In particular, we show how our results allow the treatment of general nonlinearities and may complement existing techniques.

Key words: Deterministic systems; first-order systems; nonlinear systems; non-analytic systems; identifiability; identification algorithms; parameter identification; observability; linear independence.

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1 Introduction

We consider autonomous systems of ODEs, together with some observations, with a special focus on nonlinear systems, for which we study their identifiability and observability. We moreover analyze cases where all parameters and states are unknown, investigating also the concept of joint observability-identifiability (see [5]).

When a system is identifiable or observable, several techniques may help us estimate practically the unknowns with a certain accuracy (see, for example, [9], [13], [18], [20], [28]), but do not guarantee obtaining exact values. Let f be the function that describes the dynamics of the ODE system we are considering and $h(x, \theta)$ the function that describes the observations in terms of the solution x and the parameter vector θ. A common approach to recover theoretically the unknowns consists in studying if the function $\Gamma_r : (x, \theta) \mapsto (h(x, \theta), L_f h(x, \theta), ..., L_f^r h(x, \theta))$ is invertible, for some $r \in \mathbb{N}$ (see, for instance, [7], [12], [14], [17]). In particular, when the dynamics are nonlinear, r may be greater than $n + m - 1$, and it may be very difficult studying this mapping. This is related to the so-called *higher-order observability* (see e.g. [21]). To address this challenge, some authors focus on expressions that are linear in all parameters (see [10], [11], [12]).

Reference [1] is an exhaustive survey of methods for testing identifiability, mainly for general nonlinear systems, which have have provided a solid basis for analyzing identifiability. On the other hand, [2] offers a deep insight on available softwares in the literature. Among available tools, STRIKE-GOLDD ([27]) analyzes local identifiability of general nonlinear systems under analyticity assumptions, extending the *Observability Rank Condition* but often requiring high differentiation orders. Other software, such as SIAN ([16]) and Struc-

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turalIdentifiability.jl ([8]), addresses global identifiability of rational systems with significantly improved computational performance. In general, most examples in the literature are rational, and general nonlinearities are typically excluded or approximated by rational expressions (e.g., [17]).

For rational systems, important developments in differential algebra focus on *input-output* (IO) *equations* and *input-output identifiability*, i.e., recovering the parameters from the IO equations coefficients (see [23]). A common gap is assuming that IO identifiability implies identifiability. Several methods in the literature rely on this assumption, which, though useful, does not always hold. For instance, in [3], the DAISY algorithm reduces the amount of differentiation, computing the IO equations and recovering the parameters after proving IO identifiability. This improves computational efficiency, although it lacks a rigorous justification for global identifiability. The authors suggest in [25, Example 3] recovering the parameters by choosing suitable times, but do not justify why these times exist. Similarly, in [19], the authors recover functions of the parameters solving some linear systems which they assume, but do not prove, that have a unique solution. This gap is addressed in [23]: The authors prove that IO identifiability implies identifiability if the system has no rational first integrals. While this is a valuable theoretical condition, verifying the absence of first integrals is complex, particularly for high-dimensional nonlinear systems. In [22], the authors use a linear independence argument on linear systems. Also, in [6], linear independence arguments are used for assessing identifiability of rational systems under known initial conditions and data at the initial time.

These formal identification methods typically involve two steps: First, "eliminating" unmeasured variables to obtain a set of equations only on the observation functions, their derivatives, the parameters, and the input (if any); second, inverting these equations with respect to the parameters. While the first step is well documented in the literature, with several algorithms proposed to obtain the minimal set of equations, the second step has fewer systematic developments. Our approach aims at gathering rigurously both steps into one, offering a more systematic way to address the second step, leading to a more cohesive approach to identifiability and observability analysis. Moreover, typically, all the mentioned algorithms do not analyze singular values for parameters or initial conditions.

The results presented here (1) cover nonlinear systems beyond the rational and analytic settings, (2) avoid high-order derivatives, (3) explicitly account for possible singular values, and (4) provide general conditions for recoverability of parameters. We establish results for a class of ODE systems with linear relations between parameter functions and the observations (as in IO equations), proving joint observability-identifiability under suitable hypotheses. Constructive methods to recover the unknowns are provided, relying on the linear independence of specific functions. These results are illustrated with examples in Section 3, including one analytic non-rational system and one non-analytic system, highlighting the scope of our framework.

2 A general framework

Consider the autonomous dynamical system

$$\begin{cases} \dot{x}(t;\xi,\theta) = f(x(t;\xi,\theta),\theta), \quad x(0;\xi,\theta) = \xi, \\ y_{(\xi,\theta)}(t) = h(x(t;\xi,\theta),\theta), \end{cases} \tag{1}$$

where $f(\cdot,\cdot) : \Omega \times \Theta \to \mathbb{R}^n$ is a function of (x,θ) which is locally Lipschitz-continuous w.r.t. $x \in \Omega \subset \mathbb{R}^n$ and continuous w.r.t. $\theta \in \Theta \subset \mathbb{R}^b$; θ are constant parameters of the system; Ω is a positively invariant set w.r.t. the ODE system of System (1); $x(\cdot;\xi,\theta) : \mathcal{I} \to \Omega$ denotes the unique, global solution of the ODE system of System (1) with initial condition $\xi \in \Omega$, in particular, $\mathcal{I} = [0,+\infty)$; and the output $y_{(\xi,\theta)}(t)$, $t \in \mathcal{S} \subset \mathcal{I}$, is described by some known function $h(\cdot,\cdot) : \Omega \times \Theta \to \mathbb{R}^m$.

We aim to know if, given the output $y_{(x_0,\theta_0)}$, for some (x_0,θ_0), we can determine (x_0,θ_0) univocally.

Definition 1 (Identifiability/observability)

(a) *System* (1) *is identifiable on* Θ *in* $\mathcal{S} \subset \mathcal{I}$ *with initial conditions in* Ω *if, for any* $\xi \in \Omega$*, two different* $\theta_1, \theta_2 \in \Theta$ *produce a different output at some time* $t \in \mathcal{S}$*, i.e.,* $h(x(t;\xi,\theta_1),\theta_1) \neq h(x(t;\xi,\theta_2),\theta_2)$.

(b) *System* (1) *is observable on* Ω *in* $\mathcal{S} \subset \mathcal{I}$ *with parameters in* Θ *if, for any* $\theta \in \Theta$*, two different* $x_0, x_0' \in \Omega$ *produce a different output at some time* $t \in \mathcal{S}$*, i.e.,* $h(x(t;x_0,\theta),\theta) \neq h(x(t;x_0',\theta),\theta)$.

(c) *System* (1) *is said to be identifiable (resp. observable) if* (a) *(resp.* (b)*) is satisfied for* $\mathcal{S} = \mathcal{I}$.

A natural extension of this framework is to treat the parameters as states of an augmented system. If we consider $\dot{\theta} = 0$, we can study both the identifiability and observability as a case of observability in higher dimension: If the extended system is observable, the original system is both observable and identifiable. However, the reciprocal is generally not true. This relates to *joined observability-identifiability* (see [5], [26]).

Definition 2 (Joined observability-identifiability)

(a) *System* (1) *is jointly observable-identifiable on* $\Omega \times \Theta$ *in* $\mathcal{S} \subset \mathcal{I}$ *if two different* $(x_0,\theta_1), (x_0',\theta_2) \in \Omega \times \Theta$ *produce a different output at some time* $t \in \mathcal{S}$*, i.e.,* $h(x(t;x_0,\theta_1),\theta_1) \neq h(x(t;x_0',\theta_2),\theta_2)$.

(b) *System* (1) *is said to be jointly observable-identifiable if* (a) *is satisfied for* $\mathcal{S} = \mathcal{I}$.

For convenience, we denote $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$ and $\alpha_1 : \alpha_N \coloneqq \alpha_1, \dots, \alpha_N$, for any $\alpha_i \in \mathbb{N}_0$. Now, given $\theta \in \Theta$, denote $h_\theta(\cdot) = h(\cdot, \theta)$ and $f_\theta(\cdot) = f(\cdot, \theta)$, and assume $h_\theta \in \mathcal{C}^d(\Omega; \mathbb{R}^m)$, $f_\theta \in \mathcal{C}^{\max\{0,d-1\}}(\Omega; \mathbb{R}^n)$, for some $d \in \mathbb{N}_0$. Then, $y_{(\xi,\theta)} \in \mathcal{C}^d(\mathcal{I}; \mathbb{R}^m)$. In particular,

$$\dot{y}_{(\xi,\theta)}(t) = L^1_{f_\theta} h_\theta(x(t; \xi, \theta)),$$

where $L^1_{f_\theta} h_\theta$ is the Lie derivative of h_θ with respect to the vector field f_θ. If we continue differentiating $y_{(\xi,\theta)}$,

$$y^{(k)}_{(\xi,\theta)}(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} h(x(t; \xi, \theta)) = L^k_{f_\theta} h_\theta(x(t; \xi, \theta)), \ \forall\, t \geq 0,$$

$k \in \{1 : d\}$. We denote $y^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}t^k}$, $k \in \mathbb{N}$, when $y \in \mathcal{C}^k(\mathcal{I})$, and $y^{(0)} = y$. Define $\mathcal{L}_{f_\theta, h_\theta, d} : \Omega \to \mathbb{R}^{m \times d}$ as

$$\mathcal{L}_{f_\theta, h_\theta, d}(\xi) = \left(y^{(0)}_{(\xi,\theta)}(0), \dots, y^{(d)}_{(\xi,\theta)}(0)\right).$$

In the following, we denote $y_{(\xi,\theta)}$, $\dot{y}_{(\xi,\theta)}$ and $y^{(k)}_{(\xi,\theta)}$ as y, $\dot{y}$ and $y^{(k)}$, respectively, when the context is clear.

Before presenting our main results, let us give some results about linear independence of families of functions, a concept on which our approach is based.

2.1 About linear independence of functions

Definition 3 *Given $\mathcal{I} \subset \mathbb{R}$, functions $\phi_i : \mathcal{I} \to \mathbb{R}$, $i \in \{1 : q\}$, $q \in \mathbb{N}$ are said to be linearly independent if the only constants $\{a_i\}_{i=1}^q \subset \mathbb{R}$ such that $a_1\phi_1(t) + \cdots + a_q\phi_q(t) = 0$, for all $t \in \mathcal{I}$, are $a_1 = \cdots = a_q = 0$.*

Lemma 4 *Let $\phi_i : \mathcal{S} \subset \mathbb{R} \to \mathbb{R}$, $i \in \{1 : q\}$, $q \in \mathbb{N}$, $\mathcal{S} \subset \mathcal{I}$. These functions are linearly independent if, and only if, there exist q different times $\{t_i\}_{i=1}^q \subset \mathcal{S}$ such that the matrix $(\phi_i(t_j))_{i,j=1:q}$ has full rank.*

The proof is given in Appendix A.

Remark 5 *Lemma 4 gives a necessary and sufficient condition for linear independence that does not require analyticity nor differentiability, as opposed to the classical condition of having non-null Wrońskian (see [4]).*

2.2 Main results

We recall the classical result on observability based on Lie derivatives (see [5], [9], [15]), adapted to our framework, and hence will not provide a proof for Theorem 6.

Theorem 6 *Let $h_{\theta,i} \in \mathcal{C}^{d_i}(\Omega; \mathbb{R})$, for some $d_i \in \mathbb{N}_0$, $i \in \{1 : m\}$, and $f_\theta \in \mathcal{C}^{d-1}(\Omega; \mathbb{R}^n)$, $d = \max\{1, d_1 : d_m\}$, for any $\theta \in \Theta$. If*

$$\mathcal{L}_{f_\theta, h_\theta, \{d_1:d_m\}} \ : \ \xi \mapsto \left(\mathcal{L}_{f_\theta, h_{\theta,1}, d_1}(\xi), \dots, \mathcal{L}_{f_\theta, h_{\theta,m}, d_m}(\xi)\right)$$

is injective in Ω, then System (1) is observable on Ω in any semi-open interval $[a, b) \subset \mathcal{I}$ with parameters in Θ.

For $(\xi, \theta) \in \Omega \times \Theta$, $\{d_i\}_{i=1}^m \subset \mathbb{N}_0$, we define the notation

$$\begin{aligned}\mathbf{y}^{(d_1:d_m)}_{(\xi,\theta)}(t) \coloneqq \Big(& y^{(0)}_{(\xi,\theta),1}(t), \ \dots, y^{(d_1)}_{(\xi,\theta),1}(t), \\ & \dots, y^{(0)}_{(\xi,\theta),m}(t), \dots, y^{(d_m)}_{(\xi,\theta),m}(t)\Big) .\end{aligned}$$

We present now our main result, which provides a sufficient condition to ensure recoverability of parameters.

Theorem 7 *Let $h_{\theta,i} \in \mathcal{C}^{d'_i}(\Omega; \mathbb{R})$, for some $d'_i \in \mathbb{N}_0$, $i \in \{1 : m\}$, and $f_\theta \in \mathcal{C}^{d'-1}(\Omega; \mathbb{R}^n)$, $d' = \max\{1, d'_1 : d'_m\}$, for any $\theta \in \Theta$. Consider $\mathcal{D} \subset \mathbb{R}^{d'_1 + \cdots + d'_m + m}$ such that, for all $(\xi, \theta) \in \Omega \times \Theta$, $\mathbf{y}^{(d'_1:d'_m)}(t) \in \mathcal{D}$, for all $t \in \mathcal{I}$. Let $\mathcal{S} \subset \mathcal{I}$ be such that every connected component of $\mathcal{S}$ contains an open interval. Assume there exist $g : \mathcal{D} \to \mathbb{R}^{q+p}$ and $r : \Theta \to \mathbb{R}^q$, for some $q, p \in \mathbb{N}$, satisfying:*

- (C1) *$g = (g_{1,0}, \dots, g_{1,q_1}, \dots, g_{p,0}, \dots, g_{p,q_p})$ and $r = (r_{1,1}, \dots, r_{1,q_1}, \dots, r_{p,1}, \dots, r_{p,q_p})$, with $q_1 + \cdots + q_p = q$, satisfy that*

$$g_{j,0}(\mathbf{y}^{(d'_1:d'_m)}) = \textstyle\sum_{l=1}^{q_j} r_{j,l}(\theta) g_{j,l}(\mathbf{y}^{(d'_1:d'_m)}), \qquad (2)$$

  in $\mathcal{S}$, for all $j \in \{1 : p\}$, for any $(\xi, \theta) \in \Omega \times \Theta$,
- (C2) *r is injective, and*
- (C3) *for any $j \in \{1 : p\}$, $(\xi, \theta) \in \Omega \times \Theta$, we have that $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)}(t))$, $l \in \{1 : q_j\}$, are linearly independent functions with respect to $t \in \mathcal{S}$.*

Then, System (1) is identifiable on Θ in $\mathcal{S}$ with initial conditions in Ω.

PROOF. Given $\xi \in \Omega$, let $\theta_1, \theta_2 \in \Theta$ such that $h_{\theta_1}(x(t; \xi, \theta_1)) = h_{\theta_2}(x(t; \xi, \theta_2))$, i.e., $y_{(\xi,\theta_1)}(t) = y_{(\xi,\theta_2)}(t)$, for all $t \in \mathcal{S}$. Then, since every connected part of $\mathcal{S}$ contains some open interval, this implies that $y^{(k)}_{(\xi,\theta_1),i}(t) = y^{(k)}_{(\xi,\theta_2),i}(t)$, for all $t \in \mathcal{S}$, $k \in \{0 : d'_i\}$, $i \in \{1 : m\}$. Since $g_{j,0}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_1)}) - g_{j,0}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_2)}) \equiv 0$ in $\mathcal{S}$, we obtain, from (2), $\sum_{l=1}^{q_j} \left(r_{j,l}(\theta_1) - r_{j,l}(\theta_2)\right) g_{j,l}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_1)}) \equiv 0$, in $\mathcal{S}$, for every $j \in \{1 : p\}$. Given the linear independence of $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, $\mathcal{S} \subset \mathcal{I}$, for every $j \in \{1 : p\}$, $(\xi, \theta) \in \Omega$, then $r(\theta_1) = r(\theta_2)$. Since r is injective, we have $\theta_1 = \theta_2$. Hence, System (1) is identifiable on Θ in $\mathcal{S} \subset \mathcal{I}$ with initial conditions in Ω. $\square$

System (2) in (C1) is analogous to IO equations in controlled rational systems, while (C2) gives IO identifiability. A condition similar to (C3) is considered in [22] for linear systems with one output, and [23] requiring that no rational first integrals exist, which can be difficult to verify. We aim to generalize and simplify this condition.

Next we show that, under certain conditions, given $y_{(x_0,\theta_0)}$ in some time set, we can recover $(x_0,\theta_0)\in\Omega\times\Theta$.

Theorem 8 *Assume we know $y_{(x_0,\theta_0)}$ in $\mathcal{S}\subset\mathcal{I}$, $\mathcal{S}$ such that every connected component contains an open interval. If the hypotheses of Theorems 6 and 7 are satisfied, then System (1) is jointly observable-identifiable in $\mathcal{S}$ and we can reconstruct the pair (x_0,θ_0) univocally using the values of $y_{(x_0,\theta_0)}$ and its derivatives at, at most, $q+1=q_1+\dots+q_p+1$ suitable values of $t\in\mathcal{S}$.*

PROOF. Let $\phi_{j,l}(t)=g_{j,l}(\mathbf{y}^{(d'_1:d'_m)}_{(x_0,\theta_0)}(t))$, $t\in\mathcal{S}$, $l\in\{0:q_j\}$, $j\in\{1:p\}$. By hypothesis, $\{\phi_{j,l}\}_{l=1}^{q_j}$ are linearly independent in $\mathcal{S}$, for each $j\in\{1:p\}$. Hence, as seen in Lemma 4, for every $j\in\{1:p\}$, there exist q_j different times $\{t_{j,\ell}\}_{\ell=1}^{q_j}\subset\mathcal{S}$ such that $\det((\phi_{j,l}(t_{j,\ell}))_{l,\ell=1:q_j})\neq 0$. Then, there exists a unique solution σ to

$$\big(\phi_{j,1}(t_{j,\ell})\ \dots\ \phi_{j,q_j}(t_{j,\ell})\big)\,\sigma=\phi_{j,0}(t_{j,\ell}),\ \forall\,\ell\in\{1:q_j\}. \tag{3}$$

Attending to (2), it satisfies $\sigma_{j,l}=r_{j,l}(\theta_0)$, $l\in\{1:q_j\}$, $j\in\{1:p\}$. Since r is injective, such that $r^{-1}:r(\Theta)\to\Theta$, we can recover our original parameter vector $\theta_0=r^{-1}(\sigma_{1,1},\dots,\sigma_{p,q_p})$.

Finally, to recover x_0, take some time $\tilde{t}\in\mathcal{S}$, which can be some $\tilde{t}\in\{t_{1,1},\dots,t_{p,q_p}\}$. Since $\mathcal{L}_{f_{\theta_0},h_{\theta_0},\{d_1:d_m\}}$ is injective in Ω, there exists a unique $\tilde{\xi}\in\Omega$ such that $\tilde{\xi}=\mathcal{L}^{-1}_{f_{\theta_0},h_{\theta_0},\{d_1:d_m\}}(\mathbf{y}^{(d_1:d_m)}_{(x_0,\theta_0)}(\tilde{t}))$, noticing that $\mathbf{y}^{(d_1:d_m)}_{(x_0,\theta_0)}(\tilde{t})=\mathbf{y}^{(d_1:d_m)}_{(\tilde{\xi},\theta_0)}(0)$. We can recover $x_0\in\Omega$ integrating backwards the ODE system in System (1) knowing $\tilde{\xi}$, $\tilde{t}$ and θ_0 (since f_{θ_0} is Lipschitz in Ω positively invariant w.r.t. the ODE system of (1)). If $0\in\mathcal{S}$, we directly choose $\tilde{t}=0$.

This is, we have recovered θ_0 and x_0 from the data univocally knowing $y_{(x_0,\theta_0)}$ in $\mathcal{S}$, using its value and the value of its derivatives at $q+1$ (at most) different times. □

Given a system of autonomous ODEs and some observations, we can check the hypotheses in Theorems 6 and 7 to confirm its observability and/or identifiability. If the hypotheses of Theorem 8 hold, the system is jointly observable-identifiable, and we can recover x_0 and θ_0. Moreover, the proof of Theorem 8 yields a constructive method that will be applied to some cases in Section 3.

Remark 9 *To recover (x_0,θ_0) using the procedure in the proof of Theorem 8, for each set $\{\phi_{j,l}\}_{l=1}^{q_j}$, $j\in\{1:p\}$, we need q_j different times. Since some of these may coincide among the different sets of functions, the total number of required times lies between $\tilde{q}=\max\{q_1:q_p\}$ and $q=q_1+\dots+q_p$, along with maybe $\tilde{t}=0$, if $0\in\mathcal{S}$. Moreover, recall that we do not need $y_{(x_0,\theta_0)}$ in $\mathcal{S}$, but it is sufficient having the values of $y_{(x_0,\theta_0)}$ and its derivatives at the aforementioned different times, where the needed order of the derivatives are the same as for Theorem 8.*

The time points required in the proof of Theorem 8 can be anywhere in $\mathcal{S}\subset\mathcal{I}$, and can be challenging finding them. In Lemma 10, we give sufficient conditions to identify System (1) in any $[a,b)\subset\mathcal{S}$, similarly to Theorem 6; then, these times can be chosen in any of these intervals.

Lemma 10 *Assume that the hypotheses of Theorem 7 are satisfied for some $\mathcal{S}\subset\mathcal{I}$ and, for any $j\in\{1:p\}$, $l\in\{1:q_j\}$ and $(\xi,\theta)\in\Omega\times\Theta$, the functions $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$ are analytic on $\mathcal{I}$. Then, System (1) is identifiable on Θ in any semi-open interval $[a,b)\subset\mathcal{S}$ with initial conditions in Ω. If $\mathcal{S}$ is connected, it is sufficient for $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $j\in\{1:p\}$, $l\in\{1:q_j\}$, to be analytic on $\mathcal{S}$.*

PROOF. Given $\xi\in\Omega$, let $\theta_1,\theta_2\in\Theta$ such that $h_{\theta_1}(x(t;\xi,\theta_1))=h_{\theta_2}(x(t;\xi,\theta_2))$, i.e., $y_{(\xi,\theta_1)}(t)=y_{(\xi,\theta_2)}(t)$, for all $t\in[a,b)$. Then, $y^{(k)}_{(\xi,\theta_1),i}(t)=y^{(k)}_{(\xi,\theta_2),i}(t)$, for all $t\in[a,b)$, $k\in\{0:d'_i\}$, $i\in\{1:m\}$. Since, for any $j\in\{1:p\}$, $l\in\{1:q_j\}$ and $(\xi,\theta)\in\Omega\times\Theta$, the functions $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$ are analytic in $\mathcal{I}$, then the function $G_{j,\theta}(\mathbf{y}^{(d'_1:d'_m)}):=\sum_{l=1}^{q_j}r_{j,l}(\theta)g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$ is also analytic in $\mathcal{I}$. Given that $[a,b)\subset\mathcal{S}$, $G_{j,\theta}(\mathbf{y}^{(d'_1:d'_m)})=g_{j,0}(\mathbf{y}^{(d'_1:d'_m)})$, in $[a,b)$, for any $\theta\in\Theta$. Since $g_{j,0}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_1)})=g_{j,0}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_2)})$, in $[a,b)$, we have $G_{j,\theta_1}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_1)})=G_{j,\theta_2}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_2)})$, in $[a,b)$. This implies that, for every $j\in\{1:p\}$,

$$R_j:=\textstyle\sum_{l=1}^{q_j}\big(r_{j,l}(\theta_1)-r_{j,l}(\theta_2)\big)g_{j,l}(\mathbf{y}^{(d'_1:d'_m)}_{(\xi,\theta_1)})\equiv 0$$

in $[a,b)$. Due to the analyticity of $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l\in\{1:q_j\}$, in $\mathcal{I}$, R_j is also analytic in $\mathcal{I}$. If $R_j\equiv 0$ in $[a,b)$, then $R_j\equiv 0$ in $\mathcal{I}$ ([24, Theorem 8.5]). Therefore, since $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l\in\{1:q_j\}$, are linearly independent in $\mathcal{S}\subset\mathcal{I}$, they are also linearly independent in $\mathcal{I}$, and hence, for R_j to be 0 in $\mathcal{I}$, for all $j\in\{1:p\}$, we need $r(\theta_1)=r(\theta_2)$. Since r is injective, this implies $\theta_1=\theta_2$.

If $\mathcal{S}$ is connected, $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l\in\{1:q_j\}$, analytic in $\mathcal{S}$ implies R_j is analytic in $\mathcal{S}$, and $R_j\equiv 0$ in $[a,b)\subset\mathcal{S}$ implies $R_j\equiv 0$ in $\mathcal{S}$ (if $\mathcal{S}$ is not connected, we can only assure $R_j\equiv 0$ in the connected component containing $[a,b)$). Thus, we conclude analogously using the linear independence of $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l\in\{1:q_j\}$, in $\mathcal{S}$. Hence, System (1) is identifiable on Θ in any $[a,b)\subset\mathcal{S}$ with initial conditions in Ω. □

Using Theorem 6 and Lemma 10, we may recover x_0 and θ_0 knowing $y_{(x_0,\theta_0)}$ only in some $[a,b) \subset \mathcal{I}$. This will be shown in Lemma 11. As in Theorem 8, its proof provides a constructive method to recover the unknowns, which will be applied to the examples in Section 3.

Lemma 11 *Assume we know* $y_{(x_0,\theta_0)}$ *in some* $[a,b) \subset \mathcal{I}$*. If the hypotheses of Theorem 6 and Lemma 10 are satisfied for some* $\mathcal{S} \subset \mathcal{I}$ *such that* $[a,b) \subset \mathcal{S}$*, then System* (1) *is jointly observable-identifiable in* $[a,b)$*, and we can reconstruct* (x_0,θ_0) *univocally using the values of* $y_{(x_0,\theta_0)}$ *and its derivatives at a finite amount of suitable values in* $[a,b)$*. In particular, the required number of time points is between* $\tilde{q} = \max\{q_1 : q_p\}$ *and* $q = q_1 + \cdots + q_p$*.*

PROOF. We recover θ_0 analogously to the proof of Theorem 8. We only need to see that, given $j \in \{1 : p\}$, since the functions $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, are linearly independent in some $\mathcal{S} \subset \mathcal{I}$ and analytic in $\mathcal{I}$, they are linearly independent in $[a,b) \subset \mathcal{S}$. Given $j \in \{1 : p\}$, assume that $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, are linearly dependent in $[a,b)$, i.e., there exist $\{a_{j,l}\}_{l=1}^{q_j} \subset \mathbb{R}$ not all of them null such that,

$$G_j(t) = \textstyle\sum_{l=1}^{q_j} a_{j,l} g_{j,l}(\mathbf{y}^{(d'_1:d'_m)}(t)) = 0, \ \forall\, t \in [a,b).$$

Since $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, are analytic in $\mathcal{I}$, then so it is G_j. Due to [24, Theorem 8.5], this implies that $G_j \equiv 0$ in $\mathcal{I}$ and, thus, $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, are linearly dependent in $\mathcal{I}$, and hence in $\mathcal{S}$, which is a contradiction. Moreover, if $\mathcal{S}$ is connected, it is enough asking for $g_{j,l}(\mathbf{y}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, analytic in $\mathcal{S}$, since, then, so it is G_j. Again due to [24, Theorem 8.5], $G_j \equiv 0$ in $[a,b) \subset \mathcal{S}$ connected implies $G_j \equiv 0$ in $\mathcal{S}$, which leads to the same contradiction. Therefore, for each $j \in \{1 : p\}$, the functions $\{\phi_{j,l}\}_{l=1}^{q_j}$, with $\phi_{j,l} = g_{j,l}(\mathbf{y}_{(x_0,\theta_0)}^{(d'_1:d'_m)})$, $l \in \{1 : q_j\}$, are linearly independent in $[a,b)$ and we can conclude analogously to Theorem 8, along with Remark 9, choosing $t_{1,1}, \ldots, t_{p,q_p} \in [a,b)$. On the other hand, to recover the initial condition, we proceed as in the proof of Theorem 8. Hence, we are able to recover (x_0,θ_0) univocally when knowing $y_{(x_0,\theta_0)}$ in $[a,b)$, using its values and the values of its derivatives at some finite set of times in $[a,b)$; concretely, between $\tilde{q}$ and q ddistinct suitable time points. □

Remark 12 *As in Remark 9, we do not need* $y_{(x_0,\theta_0)}$ *in* $[a,b)$*, but only the values of this function and its derivatives at the times indicated in the proof of Lemma 11, which also determines the amount of differentiation.*

Remark 13 *The required number of time points in Lemma 11 could be reduced to one using higher-order derivatives of* y*, similarly to some classic approaches (e.g., [14]). However, relying on higher-order derivatives tends to be less robust to noisy data, even when the theoretical output is analytic.*

We have given conditions for a system to be observable, identifiable, or jointly observable-identifiable. Besides, under some analyticity properties, we can have data only in some $[a,b) \subset \mathcal{I}$. The following cases illustrate this.

3 Illustrative cases

In the following cases, we shall implicitly consider solutions with non-constant observation variables (some steady states or particular initial states could generate constant outputs, which we will avoid, since, in those cases, the system will be typically non-observable or non-identifiable).

3.1 Linearly parameterized rational systems

We revisit the class of rational systems considered in [10]:

$$\dot{x} = n(x)^{-1}\left(\theta^{\mathrm{T}}\varphi(x) + \textstyle\sum_{i=0}^{m} \rho_i(x)u^i\right), \quad y = x, \qquad (4)$$

where $x(0) = x_0 \in \Omega \subset \mathbb{R}$, Ω positively invariant w.r.t. the ODE of System (4); $\theta \in \Theta = \mathbb{R}^b$ is the vector of unknown parameters; every φ_j in $\varphi = (\varphi_1, \ldots, \varphi_b)^{\mathrm{T}}$ is a polynomial function; n, ρ_i, $i \in \{0 : m\}$, are also polynomial functions such that $n(x) > 0$, $x \in \Omega$; and u is a scalar time function (which acts as a control in [10]). If $t \mapsto u(t)$ is analytic (as required in [10]), for any $(\xi,\theta) \in \Omega \times \Theta$, then, the solution $x(\cdot)$ is unique and analytic. Although this system is non-autonomous, one can recover the general autonomous framework considering that t is another (known) state variable such that $\dot{t} = 1$.

Let $\varphi = (\varphi_1, \ldots, \varphi_b)^{\mathrm{T}}$ and s be the highest degree among the φ_j, $j \in \{1 : b\}$. One has $\varphi_j(x) = \sum_{i=0}^{s} a_{i,j}x^i$, $j \in \{1 : b\}$, for $a_{i,j} \in \mathbb{R}$, $i \in \{0 : s\}$, $j \in \{1 : b\}$. Let $A = (a_{i,j})_{i=0:s,\ j=1:b}$. We need the next assumptions, analogously to [10]: (A1) The highest degree s of polynomials φ_j, $j \in \{1 : b\}$, satisfies $s \geq b-1$, and (A2) $u(\cdot)$ is an analytic function such that the solution of the ODE in System (4) is non-constant and defined for all $t \geq 0$.

Proposition 14 *Under assumptions* (A1) *and* (A2)*, System* (4) *is identifiable on* Θ *with initial conditions in* Ω *if, and only if,* A *has full rank. Moreover, we can determine* θ *univocally in terms of* y*,* u *and* $\dot{y}$*.*

PROOF. Let us rewrite the ODE in System (4):

$$n(x)\dot{x} - \textstyle\sum_{i=0}^{s} \rho_i(x)u^i = (1, x, \ldots, x^s)A\theta, \qquad (5)$$

i.e., opposed to the linearly parameterized expression in System (4), we sort the expression in terms of monomials of x, which is more suitable for our methodology. Notice that, if A does not have full rank, for any $\theta \in \Theta$, there exists some $\tilde{\theta} \in \Theta$ such that $A\theta = A\tilde{\theta}$. Hence, for System (4) to be identifiable, we need that A has full rank. Differentiating y once and substituting x and $\dot{x}$ in (5),

$$n(y)\dot{y} - \textstyle\sum_{i=0}^{s} \rho_i(y)u^i = \sum_{i=0}^{s} \left(\sum_{j=1}^{b} \theta_j a_{i,j}\right) y^i. \quad (6)$$

Then, we have an equation in the form of (2), i.e., (C1) of Theorem 7 is satisfied. Let $r(\theta) = (\sum_{j=1}^{b} \theta_j a_{0,j}, \ldots, \sum_{j=1}^{b} \theta_j a_{s,j}) = A\theta$. This implies that (C2) of Theorem 7, i.e., the injectivity of r, is also satisfied if $A \in \mathbb{R}^{(s+1)\times b}$ has full rank. Finally, proving (C3) of Theorem 7 is straightforward, since equation (6) is a polynomial on y, and y is analytic non-constant due to (A2), so $1, y, \ldots, y^s$ are linearly independent. Hence, we can recover univocally θ differentiating the observations just once, instead of differentiating s times as performed in [10]. Then, we need at most $s+1$ different times $t_1, \ldots, t_{s+1} \geq 0$, which can be chosen in any $[a, b) \subset \mathcal{I}$, to solve univocally the following linear system:

$$\big(1\ y(t_\ell)\ \ldots\ y^s(t_\ell)\big)\sigma = \psi(t_\ell), \quad \forall\, \ell \in \{1 : s+1\},$$

where $\psi(t_l) = n(y(t_l))\dot{y}(t_l) - \sum_{i=0}^{s} \rho_i(y(t_l))u(t_l)^i$, $l \in \{1 : s+1\}$. Thus, if A has full rank, there exists a submatrix $\tilde{A} \in \mathbb{R}^{b\times b}$ such that, given $r(\theta) = \sigma$, $\theta = \tilde{A}^{-1}\sigma$. □

3.2 A non-rational system

We revisit Example 4 of [17], which is the motivating example the authors use to discuss the approach proposed in [25] based on differential algebra. Let the following system for a non-isothermal reactor:

$$\begin{aligned}&\dot{c}_A = -k_{10}\mathrm{e}^{-\frac{E}{T}}c_A,\ \dot{c}_B = k_{10}\mathrm{e}^{-\frac{E}{T}}c_A,\ \dot{T} = -h_1k_{10}\mathrm{e}^{-\frac{E}{T}}c_A,\\ &y_1 = c_A,\ y_2 = T,\end{aligned} \quad (7)$$

for $(c_A(0), c_B(0), T(0))^{\mathrm{T}} \in \Omega = (0,\infty)\times[0,\infty)\times(0,\infty)$ and $(k_{10}, h_1, E)^{\mathrm{T}} \in \Theta = (0,\infty)^3$. The set Ω is clearly positively invariant for System (7). Our goal is to recover the parameters $\theta = (k_{10}, h_1, E)^{\mathrm{T}}$. The authors of [17] show that this system can be transformed into a rational system with respect to the variable states, and claim that the parameters can be estimated through linear regression. They also propose an alternative approach involving several computations using Padé approximations, followed by an application of the methodology in [25]. In both cases, no proof is provided for the exact determination of the parameters. For our approach, we differentiate y_1 and y_2 once, getting $\dot{y}_1 = -k_{10}\mathrm{e}^{-E/y_2}y_1$, $\dot{y}_2 = -h_1k_{10}\mathrm{e}^{-E/y_2}y_1$. After some computations, we derive the following rational relations among the variables, similarly to [17], in the form of (2) in (C1) of Theorem 7:

$$\log(-\dot{y}_1/y_1) = \log(k_{10}) - E/y_2, \quad \dot{y}_1/\dot{y}_2 = h_1. \quad (8)$$

Let now $r(\theta) = (\log(k_{10}), -E, h_1)$, which is injective from Θ to $r(\Theta)$, satisfying (C2) of Theorem 7. Finally, checking (C3) of Theorem 7 consists of proving the linear independence when we consider $(\xi, \theta) \in \Omega\times\Theta$ only of 1 and $1/y_2$, since it is trivial for the second equation in (8). The linear independence of 1 and $1/y_2$ is also direct since y_2 is a non-constant function. Then, the hypotheses of Theorem 7 hold differentiating only once, and we can therefore identify k_{10}, h_1 and E. For h_1, it is clear that, for any $t \geq 0$, $h_1 = \dot{y}_1(t)/\dot{y}_2(t)$. On the other hand, we need to find $t_1, t_2 \geq 0$ such that we can solve univocally:

$$(1\ 1/y_2(t_l))\sigma = \log(-\dot{y}_1(t_l)) - \log(y_1(t_l)),\ l \in \{1, 2\},$$

and then $k_{10} = \mathrm{e}^{\sigma_1}$, $E = -\sigma_2$. For this particular case, it can be easily seen that y_2 is strictly monotonic in Ω, and hence we can choose t_1 and t_2 in $[0, \varepsilon)$, for any $\varepsilon > 0$.

3.3 A one-dimensional non-rational example

Consider the following non-rational example:

$$\dot{x} = \alpha x, \quad y = x + \varepsilon\sin(x), \quad (9)$$

with $x(0) = x_0 \in \Omega = \mathbb{R}\setminus\{0\}$ and $\theta = (\alpha, \varepsilon)^{\mathrm{T}} \in \Theta = (\mathbb{R}\setminus\{0\})\times[-1, 1]$. The set Ω is clearly positively invariant. The observation y consists of x plus small oscillations, representing perturbations in the data modeled as a periodic signal with unknown (small) amplitude.

For any $\theta \in \Theta$, $\xi \mapsto y_{(\xi,\theta)}(0) = \xi + \varepsilon\sin(\xi)$ is strictly increasing from $\mathbb{R}$ to $\mathbb{R}$, and hence it is injective. Moreover, it vanishes only at $\xi = 0$. Due to Theorem 6, System (9) is observable on Ω in any $[a, b) \subset \mathcal{I}$ with parameters in Θ. Now, differentiating y twice, we get $y = x + \varepsilon\sin(x)$, $\dot{y} = (1 + \varepsilon\cos(x))\alpha x$, and $\ddot{y} = (1 + \varepsilon\cos(x))\alpha^2 x - \varepsilon\sin(x)\alpha^2x^2$. This implies $\ddot{y} = \alpha\dot{y} - \alpha^2x^2y + \alpha^2x^3$, which no longer depends on ε. Since $\alpha \neq 0$ and $y(t) \neq 0$, for all $t \geq 0$, and differentiating y once more, we have that $x^2 = (\alpha\dot{y} - \ddot{y} + \alpha^2x^3)/(\alpha^2y)$ and $y^{(3)} = 3\alpha^3x^3 + \alpha\ddot{y} - \alpha^2x^2\dot{y} - 2\alpha^3x^2y$. This implies

$$yy^{(3)} - \dot{y}\ddot{y} = x^3(\alpha^3y - \alpha^2\dot{y}) + 3\alpha y\ddot{y} - \alpha\dot{y}^2 - 2\alpha^2y\dot{y}. \quad (10)$$

Notice that $\alpha^3y(t) - \alpha^2\dot{y}(t) \neq 0$ for almost any $t \geq 0$. In particular, one can see that $\alpha y = \dot{y}$ if, and only if, $x = \tan(x)$. This equality holds for an infinite countable amount of values of x, but $x(t)$, $t \geq 0$, is analytic, non-constant, and hence $\alpha y \neq \dot{y}$ almost everywhere. Hence, from equation (10) and performing a last derivative,

$$\begin{cases} x^3 = \dfrac{yy^{(3)} - \dot{y}\ddot{y} - 3\alpha y\ddot{y} + \alpha\dot{y}^2 + 2\alpha^2y\dot{y}}{\alpha^3y - \alpha^2\dot{y}}, \\ yy^{(4)} - \ddot{y}^2 = x^3(3\alpha^4y - 2\alpha^3\dot{y} - \alpha^2\ddot{y}) + \alpha\dot{y}\ddot{y} \\ \qquad +3\alpha yy^{(3)} - 2\alpha^2\dot{y}^2 - 2\alpha^2y\ddot{y}, \end{cases} \quad (11)$$

where the equation for x^3 is satisfied almost everywhere. By continuity, System (11) further simplifies to

$$\begin{aligned}&-\dot{y}y^{(4)} + \ddot{y}y^{(3)} = 6y\dot{y}\alpha^4 - (3\dot{y}^2 + 11y\ddot{y})\alpha^3\\&+2(3yy^{(3)} + 2\dot{y}\ddot{y})\alpha^2 + (4\ddot{y}^2 - yy^{(4)} - 5\dot{y}y^{(3)})\alpha,\end{aligned} \tag{12}$$

for all $t \geq 0$. Let $\phi_0 = -\dot{y}y^{(4)} + \ddot{y}y^{(3)}$, $\phi_1 = y\dot{y}$, $\phi_2 = 3\dot{y}^2 + 11y\ddot{y}$, $\phi_3 = 3yy^{(3)} + 2\dot{y}\ddot{y}$, and $\phi_4 = 4\ddot{y}^2 - yy^{(4)} - 5\dot{y}y^{(3)}$. If $\{\phi_i\}_{i=1}^4$ are linearly independent, there exist $t_1, t_2, t_3, t_4 \geq 0$ such that there is a unique solution $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ to a system analogous to (3). Then, $\alpha = \sigma_4 = \mathrm{sg}(\sigma_4)\sqrt{\sigma_3/2} = \sqrt[3]{-\sigma_2} = \mathrm{sg}(\sigma_4)\sqrt[4]{\sigma_1/6}$. If we know e.g. σ_4, using Systems (9) and (11), we can recover $\varepsilon = (y - \psi_{\sigma_4})/\sin(\psi_{\sigma_4})$, for almost every $t \geq 0$, where

$$\psi_{\sigma_4} = \sqrt[3]{\frac{yy^{(3)} - \dot{y}\ddot{y} - 3\sigma_4 y\ddot{y} + \sigma_4\dot{y}^2 + 2\sigma_4^2 y\dot{y}}{\sigma_4^3 y - \sigma_4^2\dot{y}}}.$$

Notice that $\sin(\psi_{\sigma_4}) = \sin(x) = 0$ if, and only if, $t \geq 0$ is such that $x(t) = \pi n$, for $n \in \mathbb{Z}$. Hence, if we know σ_4, we can recover ε using almost any $t \geq 0$, such that $\alpha y(t) \neq \dot{y}(t)$ and $\psi_{\sigma_4}(t) \neq \pi n$, for any $n \in \mathbb{Z}$. Notably, (C1) and (C2) of Theorem 7 hold, with (12) and $\varepsilon = (y - \psi_{\sigma_4})/\sin(\psi_{\sigma_4})$ conforming the linear system (2), and $r(\theta) = (6\alpha^4, -\alpha^3, 2\alpha^2, \alpha, \varepsilon)$. Finally, we must prove that $\{\phi_i\}_{i=1}^4$ are linearly independent to recover α (and then ε and x_0). Let a, b, c, d such that $a\phi_1 + b\phi_2 + c\phi_3 + d\phi_4 = 0$. Expressing $\{\phi_i\}_{i=1}^4$ in terms of x, α and ε (omitting intermediate computations for brevity), we obtain

$$\begin{aligned}0 = {}&-d\alpha^3\varepsilon x^4\sin(x) - 2d\alpha^3\varepsilon^2 x^3\sin(x)^2\\&+\alpha^2\varepsilon(-3c + 11d\alpha)x^3\cos(x) + 5d\alpha^3\varepsilon^2 x^3\\&+\alpha^2\varepsilon^2(-5c + 13d\alpha)x^2\sin(x)\cos(x)\\&+\alpha\varepsilon(-11b - 11c\alpha + 14d\alpha^2)x^2\sin(x)\\&+\alpha\varepsilon^2(-14b - 11c\alpha + 8d\alpha^2)x\sin(x)^2\\&+\varepsilon(a + 17b\alpha + 7c\alpha^2 - 3d\alpha^3)x\cos(x)\\&+(a + b\alpha(3\varepsilon^2 + 14) + c\alpha^2(2\varepsilon^2 + 5) - d\alpha^3(\varepsilon^2 + 2))x\\&+\varepsilon^2(a + 11b\alpha + 3c\alpha^2 - d\alpha^3)\sin(x)\cos(x)\\&+\varepsilon(a + 11b\alpha + 3c\alpha^2 - d\alpha^3)\sin(x).\end{aligned} \tag{13}$$

It is easy to verify $a = b = c = d = 0$ if, and only if, each coefficient of the monomials in x, $\sin(x)$, and $\cos(x)$ equals zero. Hence, we analyze if these monomials are linearly independent. Since x, $\sin(x)$, and $\cos(x)$ are analytic, it is enough proving their linear independence in any finite set of time points in $\mathbb{R}$ to conclude linear independence in any $[a, b) \subset [0, \infty)$. Let $\{\tau_{1,i}\}_{i=1}^5$ such that $x(\tau_{1,m}) = 2m\pi$, and $\{\tau_{2,i}\}_{i=1}^5$ such that $x(\tau_{2,m}) = 2m\pi + \pi/2$, $m \in \{1 : 5\}$. By construction, we obtain $\sin(x(\tau_{1,m})) = \cos(x(\tau_{2,m})) = 0$ and $\sin(x(\tau_{2,m})) = \cos(x(\tau_{1,m})) = 1$. Then, for each $m \in \{1 : 5\}$, we have

$$\begin{cases}\begin{aligned}0 = {}&\alpha^2\varepsilon(-3c + d\alpha(5\varepsilon + 11))x(\tau_{1,m})^3 + (a(\varepsilon + 1)\\&+b\alpha(3\varepsilon^2 + 17\varepsilon + 14) + c\alpha^2(2\varepsilon^2 + 7\varepsilon + 5)\\&-d\alpha^3(\varepsilon^2 + 3\varepsilon + 2))x(\tau_{1,m}),\\0 = {}&-d\alpha^3\varepsilon x(\tau_{2,m})^4 + 3d\alpha^3\varepsilon^2 x(\tau_{2,m})^3 + \alpha\varepsilon(-11b\\&-11c\alpha + 14d\alpha^2)x(\tau_{2,m})^2 + (a + b\alpha(-11\varepsilon^2 + 14)\\&+c\alpha^2(-9\varepsilon^2 + 5) + d\alpha^3(7\varepsilon^2 - 2))x(\tau_{2,m})\\&+\varepsilon(a + 11b\alpha + 3c\alpha^2 - d\alpha^3).\end{aligned}\end{cases}$$

These are polynomial expressions in x of degree at most 4. Vanishing at five values implies all their coefficients must be zero, and it follows that $a = b = c = d = 0$. Thus, $\{\phi_i\}_{i=1}^4$ are linearly independent, satisfying (C3) of Theorem 7, and we can recover $\alpha = \sigma_4$. Then, we recover $\varepsilon = (y - \psi_{\sigma_4})/\sin(\psi_{\sigma_4})$, and x_0 from (11) or $y_{\xi,\theta}(0) = \xi + \varepsilon\sin(\xi)$, proving that System (9) is jointly observable-identifiable on $\Omega \times \Theta$ in any $[a, b) \subset [0, \infty)$.

Remark 15 *System* (9) *can also be analyzed using existing software. For instance, STRIKE-GOLDD can confirm (local) identifiability under analyticity assumptions, although it does not provide explicit structural relations such as* (13) *that may guide the recovery of parameters. Similarly, algorithms for rational systems, such as StructuralIdentifiability.jl or DAISY, can be applied after performing a suitable change of variables, e.g.* $z_1 = \varepsilon\sin(x)$, $z_2 = \varepsilon\cos(x)$*. In this way, one obtains relations analogous to* (13)*, although proving the linear independence of monomials in three variables can become more demanding without explicit expressions for* z_1 *and* z_2*. It is worth noting that finding an appropriate change of variables is not always straightforward. For example, the transformation* $z_1 = \sin(x)$, $z_2 = \cos(x)$ *also rationalizes the system, but StructuralIdentifiability.jl concludes non-identifiability in this case. In addition, in [23] the authors prove that IO-identifiability implies identifiability if the system has no rational first integrals. Interestingly, the rationalized version of* (9) *possesses the rational first integral* $z_1^2 + z_2^2 = z_1(0)^2 + z_2(0)^2$*, and yet the system remains identifiable. This indicates that the reciprocal of their result does not hold in general. Determining first integrals can be non-trivial, hence we illustrate the value of analyzing the system directly in its original non-rational form.*

3.4 A system with non-analytic observations

We propose the following system with non-analytic observations:

$$\begin{aligned}&\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1,\\&y_1 = x_1|x_1|,\ y_2 = x_1^2,\ y_3 = \alpha x_1|x_1| + \beta x_1^2,\end{aligned} \tag{14}$$

with $(x_1(0), x_2(0))^{\mathrm{T}} \in \Omega = \mathbb{R}^2 \setminus \{(0,0)^{\mathrm{T}}\}$, $\theta = (\alpha, \beta)^{\mathrm{T}} \in \Theta = [\alpha^-, \alpha^+] \times \mathbb{R}$, where α^-, α^+ are known and $0 < \alpha^- < \alpha^+$. The set Ω is positively invariant.

Differentiating y_2, we obtain that, for any $\theta \in \Theta$, $\xi \mapsto (y_{(\xi,\theta),1}(0), y_{(\xi,\theta),2}(0), \dot{y}_{(\xi,\theta),2}(0)) = (\xi_1|\xi_1|, \xi_1^2, 2\xi_1\xi_2)$ is injective in $\Omega \setminus \{(0,\xi_2) : \xi_2 \in \mathbb{R} \setminus \{0\}\}$, with $\xi_1 = \mathrm{sg}(y_1(0))\sqrt{y_2(0)}$ and $\xi_2 = y_2(0)/2\xi_1$. Nevertheless, notice that, if $\xi_1 = x_1(0) = 0$, $\xi_2 = x_2(0) \neq 0$, then $\dot{x}_1(0) > 0$ or $\dot{x}_1(0) < 0$, and there exists $\varepsilon > 0$ such that $\tilde{\xi_1} \coloneqq x_1(\varepsilon) \neq 0$. Then, $\tilde{\xi_2} \coloneqq x_2(\varepsilon) = y_2(\varepsilon)/2\tilde{\xi_1}$, and we can recover ξ_2 integrating the system backwards. Hence, System (14) is observable on Ω in any $[a,b) \subset \mathcal{I}$ with parameters in Θ.

Notice now that $y_3 = \alpha y_1 + \beta y_2$, so (C1) of Theorem 7 is satisfied. Moreover, $r(\theta) = (\alpha, \beta)$ is clearly injective, i.e., (C2) is also satisfied. Interestingly, when checking the identifiability of the system, differentiation-based algorithms (e.g., STRIKE-GOLDD) may classify the system as non-identifiable [1] . This is due to the fact that the Wrońskian of y_1 and y_2 is zero for all time. However, since y_1 is non-analytic, this does not imply linear dependence. It is easy to check that $x_1(t) = x_1(0)\cos(t) + x_2(0)\sin(t)$, for all $t \geq 0$. Hence, $x_1(t)$ changes sign when $t \in [0,T)$, for $T > \pi$. If we have measurements in $[0,T)$, then y_1 and y_2 are linearly independent, and (C3) is satisfied. Thus, System (14) is jointly observable-identifiable on $\Omega \times \Theta$ in $[0,T)$ such that $T > \pi$.

4 Conclusions

We have presented novel results and constructive methods to study the joined observability-identifiability of a class of autonomous dynamical systems, focusing on nonlinear (both rational and non-rational) systems. These results complement existing methods and in some cases may require fewer differentiations than traditional strategies. They also provide general conditions to ensure the recoverability of the parameters from a specific system of equations, analogous to the IO equations in controlled rational systems. The key ingredient that we put ahead for the identifiability analysis is the linear independence of a set of (non-linear) functions of the observations and its derivatives. Moreover, we provide different constructive procedures to recover the unknowns, depending on the analyticity of the system and the observations. We illustrate the applicability of our framework through several examples, highlighting in particular its ability to handle non-rational systems, while also offering insights that can be useful in the rational case. Future work may include (i) the extension to more complex non-rational models, (ii) the assessment of robustness under noisy measurements, and (iii) the development of systematic criteria for non-identifiability, in analogy with the role played by IO equations in rational systems.

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A Proof of Lemma 4

We prove "⇒" by induction on the number of linearly independent functions, following Definition 3.

Case $q = 2$: If ϕ_1 and ϕ_2 are linearly independent in $\mathcal{S}$, given $t_1 \in \mathcal{S}$ such that $\phi_i(t_1) \neq 0$, for some $i \in \{1,2\}$,

$$\det \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1(t_1) & \phi_2(t_1) \end{pmatrix}$$

is not identically zero in $\mathcal{S}$; otherwise, a non-trivial linear relation between ϕ_1 and ϕ_2 would hold, contradicting the independence. Hence, there exists $t_2 \in \mathcal{S}$ such that $\phi_2(t_1)\phi_1(t_2) - \phi_1(t_1)\phi_2(t_2) \neq 0$.

Induction step: Suppose the result holds for $\{\phi_i\}_{i=1}^{q-1}$, $q \geq 3$. Then, there exist $\{t_i\}_{i=1}^{q-1} \subset \mathcal{S}$ such that $D_{q-1} = \det((\phi_i(t_j))_{i,j=1:q-1}) \neq 0$. If $\{\phi_i\}_{i=1}^{q}$ are linearly independent,

$$\det\left((A_{i,j})_{i,j=1:q}\right) = (-1)^{q-1} D_{q-1}\phi_q(t) + \textstyle\sum_{i=1}^{q-1} d_i \phi_i(t),$$

with $A_{1,j} = \phi_j(t)$ and $A_{i,j} = \phi_j(t_{i-1})$, for $i \in \{1 : q-1\}$, $j \in \{1 : q\}$, is not identically null in $\mathcal{S}$, where

$$d_k = (-1)^{k-1} \det\left((\phi_i(t_j))_{i=1:k-1,k+1:q,\ j=1:q-1}\right),$$

$k \in \{1 : q-1\}$. Else, since $D_{q-1} \neq 0$, there would exist q coefficients $a_i = d_i$, $i \in \{1 : q-1\}$, $a_q = (-1)^{q-1} D_{q-1}$, not all of them null, such that $a_1\phi_1 + \ldots a_q\phi_q \equiv 0$ in $\mathcal{S}$,

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[1] MATLAB's symbolic variables are assumed to be complex, and hence, for correct interpretation of this example, y_1 should be rewritten as $y_1 = \sqrt{x_1^2}$.

leading to a contradiction. Then, there exists some time $t_q \in \mathcal{S}$ such that

$$\det\left((\phi_i(t_j))_{i,j=1:q}\right) \neq 0. \tag{A.1}$$

Conversely for "$\Leftarrow$": If such $\{t_i\}_{i=1}^q \subset \mathcal{S}$ exist, then $\{(\phi_i(t_1), \dots, \phi_i(t_q))^{\mathrm{T}}\}_{i=1}^q$ are linearly independent. Therefore, no non-trivial combination $\sum_{i=1}^q a_i \phi_i(t)$ can vanish identically in $\mathcal{S}$, and hence $\{\phi_i\}_{i=1}^q$ are linearly independent. □

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