Resilience and vulnerability in the framework of viability theory and stochastic controlled dynamical systems

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Resilience and vulnerability in the framework of viability theory and stochastic controlled dynamical systems

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Abstract: This thesis proposes mathematical definitions of the resilience and vulnerability concepts, in the framework of stochastic controlled dynamical system, and particularly that of discrete time stochastic viability theory. It relies on previous works defining resilience in the framework of deterministic viability theory.

The proposed definitions stem from the hypothesis that it is possible to distinguish usual uncertainty, included in the dynamics, from extreme or surprising events. Stochastic viability and reliability only deal with the first kind of uncertainty, and both evaluate the probability of exiting a subset of the state space in which the system's properties are verified. Stochastic viability thus appears to be a branch of reliability theory. One of its central objects is the stochastic viability kernel, which contains all the states that are controllable so their probability of keeping the properties over a given time horizon is greater than a threshold value. We propose to define resilience as the probability of getting back to the stochastic viability kernel after an extreme or surprising event. We use stochastic dynamic programming to maximize both the probability of being viable and the probability of resilience at a given time horizon.

We propose to then define vulnerability from a harm function defined on every possible trajectory of the system. The trajectories' probability distribution implies that of the harm values and we define vulnerability as a statistic over this latter distribution. This definition is applicable with both the aforementioned uncertainty sources. On one hand, considering usual uncertainty, we define sets such that vulnerability is below a threshold, which generalizes the notion of stochastic viability kernel. On the other hand, after an extreme or surprising event, vulnerability proposes indicators to describe recovery trajectories (assuming that only usual uncertainty comes into play then). Vulnerability indicators related to a cost or to the crossing of a threshold can be minimized thanks to stochastic dynamic programming.

We illustrate the concepts and tools developed in the thesis through an application to preexisting indicators of reliability and vulnerability that are used to evaluate the performance of a water supply system. We focus on proposing a stochastic dynamic programming algorithm to minimize a criterion that combines criteria of cost and of exit from the constraint set. The concepts are then articulated to describe the performance of a reservoir.

Keywords : resilience, vulnerability, stochastic viability theory, reliability, stochastic dynamic programming.

Résilience et vulnérabilité dans le cadre de la théorie de la viabilité et des systèmes dynamiques stochastiques contrôlés

Thèse soutenue par Charles ROUGÉ le 17 décembre 2013 à l'Université Blaise Pascal. Travaux encadrés par Guillaume DEFFUANT et Jean-Denis MATHIAS.

Résumé : Cette thèse propose des définitions mathématiques des concepts de résilience et de vulnérabilité dans le cadre des systèmes dynamiques stochastiques contrôlés, et en particulier celui de la viabilité stochastique en temps discret. Elle s'appuie sur les travaux antérieurs définissant la résilience dans le cadre de la viabilité pour des dynamiques déterministes.

Les définitions proposées font l'hypothèse qu'il est possible de distinguer des aléas usuels, inclus dans la dynamique, et des événements extrêmes ou surprenants dont on étudie spécifiquement l'impact. La viabilité stochastique et la fiabilité ne mettent en jeu que le premier type d'aléa, et s'intéressent à l'évaluation de la probabilité de sortir d'un sous-ensemble de l'espace d'état dans lequel les propriétés d'intérêt du système sont satisfaites. La viabilité stochastique apparaît ainsi comme une branche de la fiabilité. Un objet central en est le noyau de viabilité stochastique, qui regroupe les états contrôlables pour que leur probabilité de garder les propriétés sur un horizon temporel défini soit supérieure à un seuil donné. Nous proposons de définir la résilience comme la probabilité de revenir dans le noyau de viabilité stochastique après un événement extrême ou surprenant. Nous utilisons la programmation dynamique stochastique pour maximiser la probabilité d'être viable ainsi que pour optimiser la probabilité de résilience à un horizon temporel donné.

Nous proposons de définir ensuite la vulnérabilité à partir d'une fonction de dommage définie sur toutes les trajectoires possibles du système. La distribution des trajectoires définit donc une distribution de probabilité des dommages et nous définissons la vulnérabilité comme une statistique sur cette distribution. Cette définition s'applique aux deux types d'aléas définis précédemment. D'une part, en considérant les aléas du premier type, nous définissons des ensembles tels que la vulnérabilité soit inférieure à un seuil, ce qui généralise la notion de noyau de viabilité stochastique. D'autre part, après un aléa du deuxième type, la vulnérabilité fournit des indicateurs qui aident à décrire les trajectoires de retour (en considérant que seul l'aléa de premier type intervient). Des indicateurs de vulnérabilité lié à un coût ou au franchissement d'un seuil peuvent être minimisés par la programmation dynamique stochastique.

Nous illustrons les concepts et outils développés dans la thèse en les appliquant aux indicateurs pré-existants de fiabilité et de vulnérabilité, utilisés pour évaluer la performance d'un système d'approvisionnement en eau. En particulier, nous proposons un algorithme de programmation dynamique stochastique pour minimiser un critère qui combine des critères de coût et de sortie de l'ensemble de contraintes. Les concepts sont ensuite articulés pour décrire la performance d'un réservoir.

Mots-clefs : résilience, vulnérabilité, viabilité stochastique, fiabilité, programmation dynamique stochastique.

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Table des matières

1 Introduction **1**

1.1 Démarche et objectifs 1

1.1.1 Le cadre viabiliste pour la résilience 1

1.1.2 Démarche de la thèse 2

1.1.3 Démarche de l'introduction 4

1.2 Thème (A) : Description du système 5

1.2.1 Formulation de système dynamique stochastique contrôlé 5

1.2.2 Une typologie des aléas 8

1.2.3 Clarification des aspects descriptif et normatif 9

1.3 Thème (B) : Des concepts aux indicateurs et à leur calcul 11

1.3.1 Une typologie des indicateurs 11

1.3.2 Utilisation de la programmation dynamique stochastique 12

1.4 Thème (C) : Complémentarité de résilience et vulnérabilité 14

1.4.1 Liens entre concepts et aspects normatifs du système 14

1.4.2 Rapport à la persistance de propriétés 15

1.4.3 Description des trajectoires de restauration 16

2 Resilience : a stochastic viability framework **17**

2.1 Introduction 18

2.2 Background : resilience in the deterministic case 21

2.2.1 Problem statement 21

2.2.2 The viability kernel 23

2.2.3 Viability-based definition of resilience 24

2.3 Resilience computations in uncertain discrete-time systems 25

2.3.1 Incorporating uncertainty 27

2.3.2 Robustness to uncertainty 28

2.3.3 Probability of resilience 29

2.3.4 Resilience-related indicators 32

2.4 Application to an uncertain lake model 33

2.4.1 The problem of lake eutrophication 34

2.4.2 Resilience for a given strategy 37

2.4.3 Resilience with an optimal management strategy 43

2.5 Discussion 51

2.6 Conclusions 52

2.7 Appendix : Relationship between V_d, V_s and resilience 53

2.8 Appendix : Dimensional analysis of the lake problem 55

2.9 Appendix : Derivation of the discrete time equation (2.27) 56
2.9.1 Uncertainty 56
2.9.2 Controls 56
2.9.3 Derivation of the discrete-time evolution equation 56

3 Stochastic viability and reliability 59
3.1 Introduction 60
3.2 The controlled time-variant reliability problem 62
3.2.1 Time-invariant reliability 62
3.2.2 Time-variant reliability 63
3.2.3 The controlled time-variant reliability problem 65
3.3 Stochastic viability as controlled time-variant reliability 66
3.3.1 Deterministic viability 66
3.3.2 The stochastic viability problem 68
3.3.3 A dynamic programming solution 69
3.3.4 Approximating the date of first out-crossing 70
3.4 Application 71
3.4.1 A simple population model 71
3.4.2 Constant carrying capacity 72
3.4.3 Decreasing carrying capacity 76
3.5 Discussion 77
3.6 Conclusion 79
3.7 Appendix : Equation (3.22) as a time-invariant problem 81
3.8 Appendix : Computation of $p_f(t, x_0, u(.))$ for $t < T$ 82
3.8.1 Forward 82
3.8.2 Backward 82

4 Vulnerability 85
4.1 Introduction 86
4.2 The dynamical system framework for vulnerability 88
4.2.1 System dynamics 89
4.2.2 Harm 90
4.2.3 Vulnerability to computable uncertainty 91
4.2.4 Specific vulnerability 91
4.2.5 Low-vulnerability kernels (LVKs) 92
4.3 Examples of vulnerability indicators 93
4.3.1 Example of harm functions 93
4.3.2 Examples of vulnerability indicators 94
4.3.3 Searching for the optimal action strategy 95
4.3.4 Consequences for vulnerability computations 97
4.4 Application 97
4.4.1 A simple lake eutrophication problem 97
4.4.2 Vulnerability to computable uncertainty 99
4.4.3 Specific vulnerability to a change in b 103
4.5 Discussion 109
4.5.1 Adaptation and adaptive capacity indicators 109

4.5.2 Vulnerability and risk . . . 112
4.5.3 Components of vulnerability . . . 112
4.6 Conclusions and perspectives . . . 113
4.7 Appendix : Stochastic viability kernels as LVKs . . . 116
4.7.1 Proof of equation (4.19) . . . 116
4.7.2 $LVK(V_M^\beta; \nu, \tau)$ as a stochastic viability kernel . . . 116
4.8 Appendix : Stochastic dynamic programming algorithms . . . 117
4.8.1 Viability maximization . . . 117
4.8.2 Cost minimization . . . 117

5 Water resources application **119**
5.1 Introduction . . . 120
5.2 Computing the composite vulnerability index . . . 121
5.2.1 System dynamics . . . 121
5.2.2 Harm criteria . . . 122
5.2.3 Change in state variables . . . 124
5.2.4 Composite vulnerability index . . . 125
5.3 Application to reliability and vulnerability indicators . . . 126
5.3.1 Description of the static criteria of satisfaction . . . 126
5.3.2 Trajectory-based reliability and vulnerability . . . 126
5.3.3 The composite R-V indicator . . . 128
5.3.4 The performance kernel . . . 128
5.4 Application . . . 129
5.4.1 Model formulation . . . 129
5.4.2 Performance criteria . . . 130
5.4.3 Operating policies . . . 131
5.4.4 Performance indicators . . . 134
5.5 Discussion . . . 136
5.6 Appendix : Proof of equation (5.12) . . . 139

Bibliographie **141**

Table des figures

1.1 Présentation de la thèse 6
1.2 Typologie des indicateurs 12

2.1 Notions of strategy and time horizon 23
2.2 The original viability framework of resilience 26
2.3 The stochastic viability framework for resilience 31
2.4 Attractors for the lake eutrophication problem 37
2.5 Two trajectories in the (L, P) plane 39
2.6 Possible positions of the attractors and system state 40
2.7 Stochastic viability kernels in the lake case 41
2.8 Probability of resilience at $T = 30$ and entry time t_α for strategy $f^{(1)}$ . . 42
2.9 Probability of resilience at $T = 30$ and entry time t_α for strategy $f^{(2)}$ . . 43
2.10 Six trajectories of the managed lake system 44
2.11 Time evolution of the probability of resilience 45
2.12 Expected return time under the optimal strategy at $T = 30$ 46
2.13 99^{th} percentile of the return time under the optimal strategy at $T = 30$. 47
2.14 Perturbation amplitude and resilience of a hysteretic lake 48
2.15 Probability of resilience ($T = 30$) for an irreversible lake 49
2.16 Perturbation amplitude and resilience of an irreversible lake 50

3.1 Viability and reliability in the deterministic controlled case 67
3.2 Viability and reliability in the stochastic controlled case 69
3.3 Reliability at $T = 100$ with the performance function g_1 73
3.4 Reliability kernel depending on the horizon under g_1 74
3.5 Optimal controls under g_1 75
3.6 Reliability kernel depending on the horizon, under g_2 76
3.7 Evolution of the out-crossing rate under the performance function g_2 . . 77
3.8 Evolution of the probability of failure under g_2. 78

4.1 Static harm in the lake eutrophication problem 100
4.2 Vulnerability $V^1(x_0, \tau)$: expected value of total harm 101
4.3 Examples of trajectories using the strategy σ_1^* 102
4.4 Vulnerability as the probability of leaving $K(0.25)$ 103
4.5 Probability of leaving $K(0.25)$ using the strategy σ_1^* 104
4.6 Influence of U on $V^1(x_0, \tau)$ 105
4.7 Vulnerability $V^1(x_0, \tau; Z)$ to a 15% decrease in b 107
4.8 Vulnerability $V^2(x_0, \sigma_1^*, \sigma_1^+, \tau; Z)$ to a 15% decrease in b 108
4.9 Vulnerability to a 15% decrease in b depending on U 109

4.10 Summary of the dynamical system framework for vulnerability 114

5.1 Optimal policies for $\kappa_1^*(\lambda, x_0)$ and different values of λ. 132
5.2 Optimal policies for $\kappa_2^*(\lambda, x_0)$ and different values of λ. 133
5.3 Optimal policies for $\kappa_1^*(\lambda, x_0)$ and $T = 50$. 135
5.4 Reliability-vulnerability trade-off for $D_0 = 0.5$, $I_0 = 0.5$ and $\eta = 0.3$ . . . 135
5.5 Reliability-vulnerability trade-off for $D_0 = 0.5$, $I_0 = 0.5$ and $\eta = 0.4$ . . . 136
5.6 Overall system performance for $RV(x_0, \lambda) = \kappa_1^*(x_0, \lambda)$, and $D_0 = 0.5$. . . 137

Liste des tableaux

4.1 Summary of static notions and their dynamic conceptual equivalents . . 93
4.2 Some possible vulnerability indicators. 95

– 1 –

Introduction

1.1 Démarche et objectifs

Cette thèse sur articles s'appuie sur le cadre viabiliste pour la résilience, proposé par Martin (2004) puis adopté par Deffuant et Gilbert (2011). Il dérive de la notion de temps de crise avancée par Doyen et Saint-Pierre (1997). Il est rappelé dans la Section 1.1.1. Ensuite la démarche de la thèse, exposée dans la Section 1.1.2, repose sur la constatation de certaines limites de ce cadre, et sur l'exploitation de résultats liant la programmation dynamique et la viabilité stochastique (Doyen et De Lara, 2010). A partir de là, la Section 1.1.3 énonce les thèmes transverses aux articles que la thèse contient, et qui seront discutés dans la suite de ce chapitre introductif.

1.1.1 Le cadre viabiliste pour la résilience

Ce cadre repose sur la théorie de la viabilité (Aubin, 1991; Aubin *et al.*, 2011), qui a pour but de contrôler des systèmes dynamiques pour qu'ils respectent indéfiniment des contraintes. Elle repose sur une formulation de système dynamique déterministe contrôlé, qui s'écrit en temps discret comme suit :

$$\forall t > 0, x(t+1) = f(t, x(t), u(t)) \tag{1.1}$$

où 0 représente la date initiale. x est l'état du système. Il regroupe toutes les variables que la dynamique f influence. Une telle définition s'applique à n'importe quel système évoluant dans le temps. S'il y a p variables d'état, l'espace d'état $\mathbb{X}$ est en général un sous-ensemble de $\mathbb{R}^p$ ($p \in \mathbb{N}$). Des décisions $u(t)$ sont prises pour influencer l'état $x(t+1)$, l'espace des contrôles possibles étant défini pour chaque x et t : $U(t, x)$, un sous-ensemble de $\mathbb{R}^q$ ($q \in \mathbb{N}$).

Cette formulation a été la première à donner à la résilience un cadre mathématique dans lequel les politiques d'action qui l'influencent sont prises en compte. Un autre apport majeur du cadre est l'idée que parler de la résilience revient à parler de la résilience de propriétés du système, ces dernières étant définies comme un sous-ensemble de l'espace d'état. La définition de propriétés est à rapprocher de celle du noyau de viabilité, un objet mathématique central dans la théorie de la viabilité comme dans la définition de la résilience par Martin (2004). Il se définit comme l'ensemble des états initiaux pour lesquels il existe une série de contrôles qui permettent de respecter indéfiniment les contraintes :

$$Viab = \{x_0 \in \mathbb{X} | \exists (u(0), u(1), \dots), \forall t > 0, x(t) \in K(t)\} \tag{1.2}$$

où $K(t)$ définit les contraintes à chaque instant. Si l'état du système ne se situe pas dans le noyau de viabilité, il est certain qu'il est en-dehors de l'espace des contraintes, ou qu'il va s'y trouver. La question est alors de savoir s'il peut revenir dans le noyau. L'ensemble des états tels qu'il existe une politique d'action ramenant l'état du système dans le noyau de viabilité est appelé le bassin de capture du noyau de viabilité. Dans le cadre viabilsite pour la résilience, il définit l'ensemble des états résilients du système.

Le cadre reflète la définition classique de la résilience en écologie. Elle y est définie comme la capacité d'un système à garder ses propriétés et fonctions à la suite d'un aléa, ou de les récupérer s'il les a perdues (Holling, 1973, 1996). Les propriétés sont définies, le noyau de viabilité est l'ensemble où les propriétés sont conservées, et son bassin de capture est celui où elles sont récupérées après un aléa.

1.1.2 Démarche de la thèse

Le cadre viabiliste pour la résilience fait le choix de travailler sur l'état du système après un aléa. On suppose que l'aléa est une transformation de l'espace d'état, et l'on s'intéresse à la trajectoire post-aléa. Les études de cas variées que l'on trouve dans Deffuant et Gilbert (2011) découplent effectivement la modélisation du système étudié et celle des aléas potentiels. Le cadre permet alors d'éluder cette dernière, sous l'argument que, quel que soit l'aléa, il est possible d'évaluer la résilience des propriétés en se basant sur l'état post-aléa.

Or dans cette démarche, il faut que tous les aléas que l'on puisse imaginer n'affectent que l'état du système. Il faut donc définir les aléas possibles avant l'espace d'état lui-même. Le point de départ de cette thèse a consisté en une clarification du rôle et de la modélisation de l'aléa dans le cadre viabiliste pour la résilience.

C'est pourquoi le Chapitre 2 est l'extension du cadre dans le cas stochastique en se servant de travaux effectués en temps discret (De Lara et Doyen, 2008; Doyen et De Lara, 2010). L'équation (1.1) est alors remplacée par une représentation de système dynamique stochastique contrôlé. D'autre part, la prise en compte les incertitudes remplace l'idée d'une trajectoire unique par la réalité de multiples trajectoires possibles, ce qui justifie le choix effectué de fixer un horizon fini $T > 0$ à la représentation. Elle est donnée en temps discret par :

$$\forall t \in [0, T], x(t+1) = f(t, x(t), u(t), w(t)) \tag{1.3}$$

où 0 représente toujours la date initiale et $w(t)$ est le vecteur des incertitudes, appartenant à $\mathbb{W}$, un sous-ensemble de $\mathbb{R}^s$ ($s \in \mathbb{N}$). La résilience se définit alors comme la probabilité de retour vers le noyau de viabilité stochastique à un horizon donné. Elle peut être calculée par programmation dynamique stochastique (PDS) en utilisant les travaux de Doyen et De Lara (2010) en viabilité stochastique. Des indicateurs de résilience sont définis à partir de la distribution de probabilité des temps de retour.

Le noyau de viabilité stochastique, quant à lui, est l'ensemble des états tels que la probabilité de garder sans interruption les propriétés pendant une période donnée est supérieure à un seuil. Contrairement à son pendant déterministe, le noyau de viabilité stochastique est donc un objet qui dépend de deux paramètres. Cette constatation conduit à préciser le lien intuitif entre noyau de viabilité stochastique et persistance des propriétés, cette dernière étant communément associée à la résilience (Walker *et al.*, 2004). Ce lien apparaît dans le cas déterministe (Doyen et Saint-Pierre, 1997; Martin, 2004), et constitue un élément central dans le cadre viabiliste pour la résilience, en particulier parce que ce cadre entend refléter le sens conceptuel du terme.

Explorer le lien entre noyau de viabilité stochastique et persistance des propriétés peut passer par la confrontation de la viabilité stochastique avec d'autres cadres mathématiques. C'est pourquoi le Chapitre 3 compare la viabilité stochastique avec la théorie de la fiabilité, qui s'intéresse à la probabilité de défaillance de propriétés, et utilise pour cela un vocabulaire et des méthodes qui lui sont propres. Il met en évidence que la viabilité stochastique correspond à une branche de la fiabilité. Ce travail permet à la fois d'introduire la PDS en fiabilité, et d'ouvrir la porte à des applications d'outils fiabilistes dans des problèmes dans lesquels la viabilité est utilisée.

Explorer le lien entre noyau de viabilité stochastique et persistance des propriétés peut aussi passer par l'exploration d'un concept voisin de celui de résilience, et dont l'usage s'est développé en parallèle (Miller *et al.*, 2010) : la vulnérabilité. Il se définit de la manière très générale comme une mesure des dommages futurs (Hinkel, 2011). Le Chaptire 4 opérationnalise ce concept dans le cadre des systèmes dynamiques stochastiques contrôlés, à partir de l'hypothèse que l'on peut estimer les dommages subis par le système sur chacune de ses trajectoires possibles. La vulnérabilité se définit alors comme une statistique sur la distribution des dommages. Cela conduit à proposer un certains nombre d'indicateurs de vulnérabilité, et comme corollaire, définir des ensembles de faible vulnérabilité. Les noyaux de viabilité stochastiques sont un cas particulier de tels ensembles.

Enfin, le Chaptire 5 propose une application dans le domaine de la gestion des ressources en eau, où des indicateurs de fiabilité et de vulnérabilité sont définis dans la littérature. Un indicateur composite agrégeant des critères de coût et de viabilité est proposé, tel que son optimisation par PDS est réalisable. Cet indicateur composite est un indicateur de vulnérabilité tel que défini dans le Chapitre 4. Il représente également n'importe quelle combinaison linéaire d'indicateurs de fiabilité et de vulnérabilité tels que définis dans le domaine de la gestion des ressources en eau. La zone dans laquelle la valeur de l'indicateur composite est en-dessous de seuil est un noyau de faible vulnérabilité comme défini dans le Chapire 4. Une application simple à un réservoir qui sert de source d'approvisionnement en eau illustre le cadre.

1.1.3 Démarche de l'introduction

J'ai choisi de décliner le reste de ce chapitre d'introduction en fonction de thèmes transverses aux différents chapitre. En effet, à partir d'un point de départ clairement identifié dans la Section 1.1.1, l'ensemble du travail constitue une progression vers un cadre commun pour la résilience et la vulnérabilité, deux concepts communément utilisés pour décrire l'impact d'aléas sur un système socio-écologique (SSE). Ces thèmes transverses correspondent à des objectifs à poursuivre pour édifier ce cadre commun, qui est la suite logique de cette thèse. Aucun des chapitres ne prétend donc apporter de réponse ferme et définitive à ces objectifs. Ils constituent plutôt un prolongement de la réflexion. De plus, il m'a semblé que dégager des thèmes récurrents aide à prendre du recul, et ainsi à faciliter la continuation de ce travail, que ce soit par mes encadrants, moi-même, ou toute autre personne.

Les thèmes transverses se déclinent comme suit :

(A) Baser les définitions proposées sur une représentation générale du système étudié.

(B) Définir et calculer des indicateurs opérationnels qui restent fidèles aux concepts originels dont ils découlent ;

(C) Mettre en évidence et exploiter la complémentarité des concepts, en particulier résilience et vulnérabilité.

Le cadre vers lequel la thèse entend progresser est résumé par la Figure 1.1, qui annonce également la suite de cette introduction. La description d'un SSE correspond au thème (A), et sera discutée dans la Section 1.2. La généralité et la clarté conceptuelle de la représentation entraînent celles des indicateurs opérationnels. Sa typologie sera donc effectuée dans la Section 1.3, en même temps que les méthodes de calcul et notamment la programmation dynamique stochastique (PDS) qui est la méthode employée tout au long de la thèse. Ainsi sera introduit le thème (B). La Section 1.4 propose quelques perspectives ouvertes par le travail de thèse, en citant des points de convergence et de complémentarité des concepts : c'est le thème (C).

1.2 Thème (A) : Description du système

Dans les trois sous-sections qui détaillent ce thème, il s'agira d'énoncer et justifier la formulation de système dynamique contrôlé (Section 1.2.1), une typologie des des aléas (Section 1.2.2), et la nécessité de clarifier les aspects normatifs de la description (Section 1.2.3).

1.2.1 Formulation de système dynamique stochastique contrôlé

Cette section a pour but de justifier la formulation de système dynamique stochastique contrôlé de l'équation (1.3), en particulier vis-à-vis de la formulation déterministe (1.1). La différence tient à l'intégration d'incertitudes dans la dynamique. On peut classifier les sources d'incertitudes qui affectent un SSE en quatre catégories qui sont (Williams, 2011) :

(i) Les aléas extérieurs au système. C'est souvent la source d'incertitude la plus large, et elle n'est pas toujours bien anticipée ;

(ii) L'observabilité partielle de l'état du système. Les erreurs de mesure notamment, rentrent dans cette catégorie ;

(iii) La contrôlabilité partielle, qui traduit l'écart entre les conséquences prévues de décisions et leur impact véritable ;

(iv) L'incertitude structurelle liée au fait que le modèle n'est pas une représentation exacte du système.

Pour rendre compte de ces incertitudes dans le cadre de la théorie de la viabilité, cette thèse choisit d'utiliser la formulation de l'équation (1.3), qui est plus générale

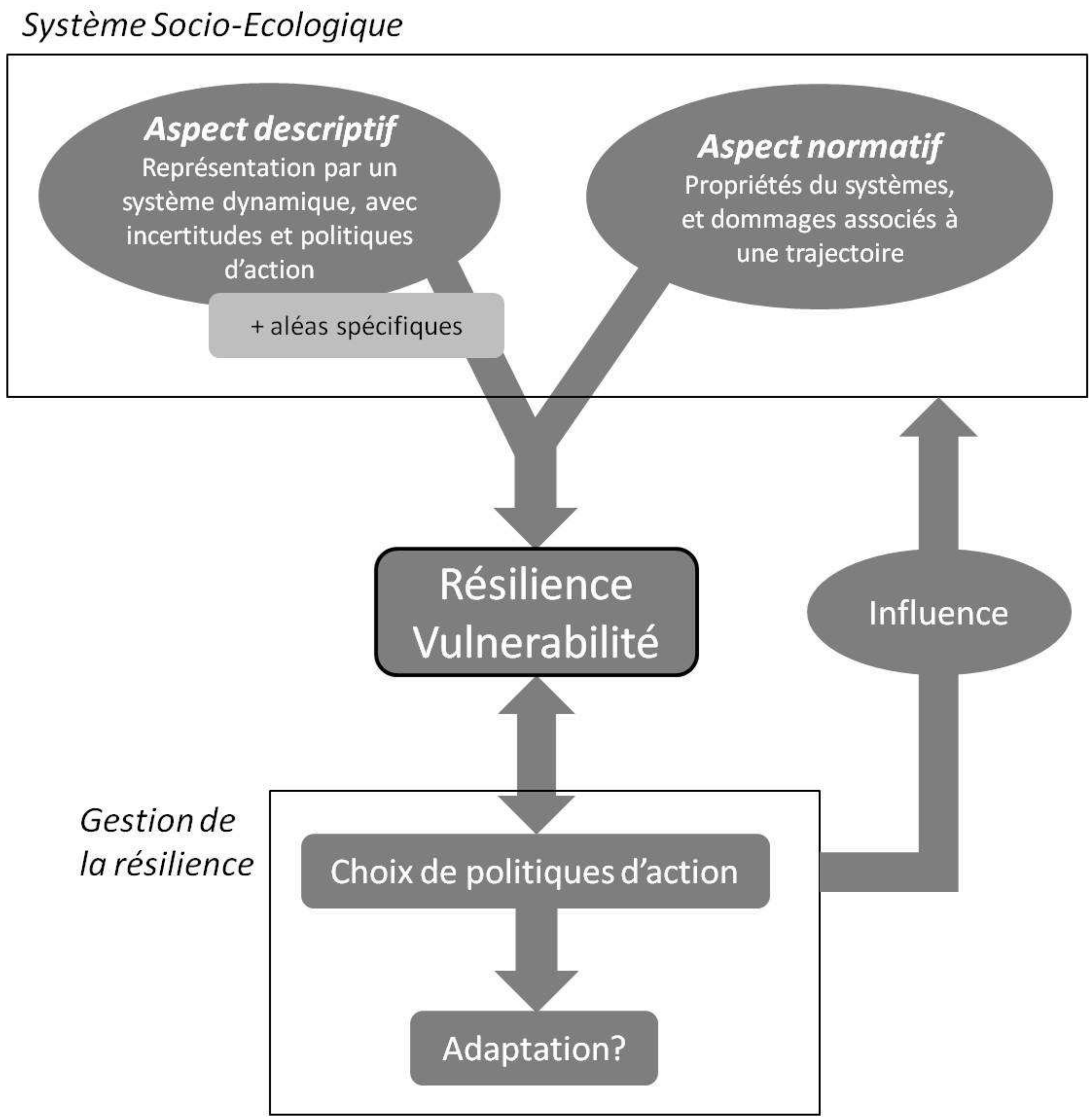

FIGURE 1.1 – La thèse propose des définitions mathématiques des concepts de résilience et vulnérabilité qui sont à l'interface entre un système et sa gestion.

que son pendant déterministe de l'équation (1.2). Comme on le verra dans le Chapitre 2, elle est également plus complexe à cause de la multiplicité des trajectoires à considérer. Elle donne donc lieu à des changements dans la définition mathématique de la résilience et à ce titre-là, la représentation mathématique du système influe sur la manière dont un concept est rendu opérationnel.

Cela est patent dans le cas de la résilience, où le cadre de Martin (2004) représente une amélioration conceptuelle par rapport à une représentation basée sur des systèmes dynamiques non contrôlés (par exemple Anderies *et al.*, 2002). La métaphore du système comme une boule roulant dans des puits de potentiels au fond duquel sont des points d'équilibres de la dynamique est adossée à cette vision non contrôlée, et a été reprise pour définir la résilience (Walker *et al.*, 2004). Si elle ne permet pas de trouver les politiques d'action pour lesquelles les propriétés d'intérêt du système sont résilientes, une telle métaphore a tout de même prouvé son utilité. Elle a par exemple eu l'avantage de populariser l'idée que les SSE pouvaient être non-linéaires, et que des effets de seuils pouvaient favoriser des transitions rapides, catastrophiques et difficilement réversibles d'un mode de fonctionnement vers un autre (Scheffer *et al.*, 2001; Scheffer et Carpenter, 2003). Ceci est par ailleurs une preuve que la représentation mathématique d'un système a un impact sur la façon de le voir.

A cet égard, il est possible de s'interroger sur la pertinence du terme de contrôle. Certes, il est mathématiquement juste puisque l'équation (1.1) vient de la théorie de la viabilité. Mais il est également difficilement dissociable de l'idée de contrôle optimal pour qui n'est pas familier avec ce cadre mathématique. Or cette idée, ou plutôt son usage généralisé, a été critiquée à juste titre dans la communauté qui s'intéresse aux SSE sous l'angle de la résilience (Folke *et al.*, 2002a; Walker *et al.*, 2002). Cela contribue à expliquer que la métaphore de la boule et du puits soit toujours dominante, alors même que le cadre initial de Martin a presque dix ans. Car paradoxalement, le concept de résilience est étroitement lié à l'idée que des décisions sont prises à différentes échelles d'espace et de temps pour que le système s'adapte ou même se transforme (Walker *et al.*, 2004; Folke, 2006; Folke *et al.*, 2010). De plus, le cadre de la viabilité lui-même a été créé pour ne contrôler le système que pour éviter que celui-ci franchisse des seuils dommageables, sinon irréversibles. Il s'adosse donc à une philosophie beaucoup moins éloignée de l'approche "résilience" que de l'optimisation d'un objectif.

Malgré l'existence de représentations concurrentes, au moins la formalisation de la résilience est-elle plus avancée que celle de la vulnérabilité. Venant de plusieurs branches des sciences sociales, alors que le concept de résilience tel qu'utilisé dans cette thèse est influencé par son utilisation en écologie (Miller *et al.*, 2010), le concept de vulnérabilité est très peu formalisé. De fait, les cadres mathématiques préexistant à cette thèse sont rares, récents et correspondent à des formulations de système dynamique de portée générale relativement faible (Ionescu *et al.*, 2009; Wolf *et al.*, 2013). Cela est à mettre en relation avec l'idée que les indicateurs de vulnérabilité n'atteignent pas toujours leurs buts avoués (Hinkel, 2011), ou avec la constatation que la dimension temporelle de la vulnérabilité est souvent négligée (Liu *et al.*, 2008).

En résumé de cette sous-section, l'existence d'une formalisation mathématique a un impact sur la manière dont ce concept est représenté, et cela même dans des cas où des équations explicites ne peuvent être écrites. Ce constat conduit à rechercher une représentation à la fois simple et générale. La formulation de système dynamique stochastique contrôlé de l'équation (1.3) est celle choisie dans cette thèse.

1.2.2 Une typologie des aléas

Dans sa version déterministe, le cadre viabiliste pour la résilience ne fait aucune hypothèse sur les aléas. Cette thèse introduit un aléa récurrent dans la formulation de système dynamique stochastique contrôlé. La thèse fait donc l'hypothèse d'une typologie simple des aléas. La typologie choisie repose sur l'information disponible sur cet aléa, par le biais d'une distinction opérée par Carpenter *et al.* (2008) entre aléas calculables et aléas incalculables. Un aléa calculable peut être exprimé par une distribution de probabilité. C'est la quantification de ce type d'aléa qui permet de calculer et d'élaborer des indicateurs. Dans le cas contraire, l'aléa est dit incalculable. Cette sous-section vise à décrire de manière plus précise ce dernier type d'aléa.

La terminologie d'aléa incalculable qualifie notamment des aléas dont la nature est connue, mais dont les observations passées ne permettent pas de tirer de conclusions sur l'avenir. De tels aléas sont résumés par ce qu'en hydrologie, Mandelbrot et Wallis (1968) ont appelé respectivement "l'effet Noé" et "l'effet Joseph". L'effet Noé prévoit que les événements extrêmes, par exemple une inondation, sont parfois plus extrêmes qu'anticipés. L'effet Joseph dit littéralement que les périodes anormalement humides – ou sèches – sont parfois très longues. Il traduit le fait que les tendances dans les changements des variables environnementales ne sont pas extrapolables dans le futur en l'absence d'un mécanisme explicatif adéquat (Koutsoyiannis, 2006).

Les interférences humaines, notamment par le biais des changements environnementaux en cours, rendent plus prégnante cette non-extrapolabilité des données historiques au futur. En supposant qu'un extrême climatique soit un aléa calculable lorsque le climat est stationnaire, Felici *et al.* (2007b) ont démontré que cela pouvait ne plus être du tout le cas lorsque l'hypothèse de stationnarité disparaît.

La dénomination d'aléa incalculable s'applique aussi à des aléas qui sont soit imprévus dans un modèle, soit imprévus tout court. Dans les deux cas, la survenue de l'événement peut être considérée comme une surprise (Folke *et al.*, 2002b; Janssen, 2002; Folke *et al.*, 2004), d'où son caractère incalculable. Le caractère surprenant de l'événement dans des cas où il est seulement imprévu dans le modèle a de quoi étonner, mais Carpenter *et al.* (2008) expliquent que par exemple, les mécanismes que les modèles ne savent pas reproduire sont parfois laissés de côté. Cela est le cas en hydrologie où Koutsoyiannis (2002) constate que les modèles statistiques n'intègrent pas l'effet Joseph à cause de sa difficulté conceptuelle et opérationnelle. De tels modèles ne sont pas capables de prévoir une sécheresse longue, qui arrive alors comme une surprise.

Partant de ces constatations sur la nature de l'aléa en fonction de l'information que l'on a sur lui, cette thèse distingue l'aléa usuel et l'aléa spécifique. L'aléa usuel modélise à chaque pas de temps l'aléa et l'inclut dans la dynamique. Un aléa spécifique se définit comme un aléa qui est considéré comme certain dans un scénario donné. Considérer un aléa incalculable comme aléa spécifique est le seul moyen d'évaluer et de quantifier son impact potentiel par des indicateurs, mais un aléa spécifique peut faire partie des aléas calculables. Par exemple, considérons une crue centennale : c'est par définition un aléa calculable puisque sa probabilité d'ocurrence est supposée connue. Mais on peut la prendre comme aléa spécifique pour évaluer son impact.

1.2.3 Clarification des aspects descriptif et normatif

Les deux premières parties de la Section 1.2 se sont focalisées surtout sur les aspects descriptifs de la représentation d'un SSE. Celle-ci s'intéresse au contraire aux aspects normatifs, en mettant en perspective les apports respectifs des recherches antérieures sur la résilience et sur la vulnérabilité.

En écologie, et aussi longtemps que l'on s'intéresse à un système dont on n'a pas à prendre en compte la dimension humaine, la résilience peut être vue comme un concept essentiellement descriptif. Le formalisme des attracteurs et des bassins d'attraction qui y est associé permet de parler de manière totalement neutre de la résilience d'un régime de fonctionnement, dont la frontière coïncide avec celle d'un bassin d'attraction (par exemple Anderies *et al.*, 2002; López *et al.*, 2013; Perz *et al.*, 2013). Cette neutralité est plus difficile à observer lorsque l'on considère un SSE, puisqu'en général, la composante humaine du système ne juge pas tous les états d'un bassin d'attraction de la même manière. La prise en compte de l'intérêt social dans la compréhension de la résilience a débouché sur une multiplicité de points de vue, et conduit Brand et Jax (2007) à constater qu'une dimension normative s'était introduite dans le concept. Cette thèse soutient l'idée que la dimension normative de la résilience doit être clarifiée et formalisée pour que l'aspect normatif soit contenu dans la description des propriétés du systèmes, et non dans les concepts.

A cet égard, la définition d'un sous-ensemble de l'espace d'état comme propriété d'intérêt est donc l'un des apports majeurs du cadre viabiliste pour la résilience (Martin, 2004; Deffuant et Gilbert, 2011). D'une part, il intègre le point de vue normatif dans la description des propriétés du SSE. Mais d'autre part, il permet à la résilience de rester un concept descriptif : la capacité à garder ou récupérer des propriétés après une perturbation. Mieux, l'aspect descriptif est renforcé par le fait que la possibilité de calculer la résilience est décorrélée de la présence d'attracteurs, grâce justement à la définition de propriétés.

Il convient ici de remarquer que le caractère normatif réfère seulement à la déclaration d'un intérêt pour des propriétés données. Cela ne signifie pas qu'un jugement de valeur leur soit porté, car il est tout à fait possible d'étudier la résilience de propriétés que l'on trouve indésirables. Par exemple, certaines structures institutionnelles qui favorisent une gestion catastrophique des ressources ont un mode de fonctionnement qui peut être vu comme une propriété résiliente (Young, 2010; Schlüter et Herrfahrdt-Pähle, 2011).

De son côté, la vulnérabilité, de par son origine, est explicitement rattachée aux aspects normatifs du système. Par exemple, elle a été reliée à des seuils de pauvreté (Adger, 2006), dont la fixation suppose une part d'objectivité. Néanmoins, le rapport exact des aspects normatifs au concept est longtemps resté flou. Il n'a été éclairé que récemment par la définition de la vulnérabilité de Hinkel (2011) comme une mesure des dommages futurs. Les dommages sont alors définis comme un jugement normatif associé à un état. Dans la thèse, cette idée sera utilisée en tandem avec la formulation de l'équation (1.3) qui permet de réfléchir sur un horizon donné, pour définir les dommages de manière plus générale comme associés plutôt à une trajectoire, c'est-à-dire une séquence d'états.

La vulnérabilité a aussi été définie par rapport à des seuils de dommages (Luers *et al.*, 2003; Luers, 2005; Béné *et al.*, 2011). Le rapport entre les notions de seuil et de dommages est ambigüe, et elle est un moyen de relier résilience et vulnérabilité. Elle sera donc explorée plus avant dans la Section 1.4 où les complémentarités entre les deux concepts seront explicitées.

1.3 Thème (B) : Des concepts aux indicateurs et à leur calcul

Les différents indicateurs sont liés aux différents types d'aléas et à leur impact. Une typologie en sera donc donnée (Section 1.3.1), avant de se pencher plus précisément sur le calcul des indicateurs par la PDS (Section 1.3.2).

1.3.1 Une typologie des indicateurs

La résilience étant la capacité à garder ou récupérer des propriétés, c'est un concept biface. D'un côté, la résilience est liée à la persistance des propriétés (Walker *et al.*, 2004). C'est une évolution depuis la notion de stabilité, qui définit la résilience à partir d'attracteurs. En écologie, de nombreux termes ont été utilisés dans un sens proche de ceux résilience et de stabilité (Grimm et Wissel, 1997). D'un autre côté, comme le cadre viabiliste le met en relief, la résilience est liée à la restauration des propriétés.

Martin (2004) montre que le cadre viabiliste prend en compte les deux faces de la résilience, et ce pour le même système. Si le système est encore dans le noyau de viabilité (équation (1.2)) après l'aléa, les propriétés persistent. S'il n'y est plus, se pose la question de la restauration. Tout dépend donc de l'état du système après un aléa ou une séquence d'aléas.

La viabilité stochastique s'intéresse au fait de contrôler le système pour que les propriétés ne soient pas perdues. Elle s'intéresse exclusivement aux aléas usuels, car il est nécessaire de pouvoir définir une densité de probabilité dans l'espace des séquences d'aléas aussi appelé espace des scénarios (De Lara et Doyen, 2008; Doyen et De Lara, 2010). Un indicateur de viabilité est attaché à l'état, mais aussi donné par un ensemble d'états. C'est ainsi que l'on parle, pour une formulation stochastique comme l'équation (1.3), de noyau de viabilité stochastique. Ce dernier terme renvoit par ailleurs à un objet, le noyau de viabilité, qui perd dans le cas stochastique certaines propriétés importantes pour son calcul (pour une revue de ces propriétés, voir Aubin *et al.*, 2011). Une recherche est donc effectuée au fil de la thèse pour proposer une terminologie plus représentative. En particulier, la fiabilité (voir la revue de Rackwitz, 2001) a pour objet d'évaluer la probabilité de défaillance d'un système, ce qui n'est pas sans rappeler la viabilité stochastique. La confrontation des deux théories dans le Chapitre 3 conduira ainsi à proposer le nom alternatif de noyau de fiabilité.

En revanche, lorsque l'on est en présence d'un aléa spécifique, ou d'aléas usuels qui entraînent néanmoins la perte des propriétés, il s'agit de comprendre comment en limiter l'impact. L'enjeu est de connaître les conséquences de l'occurrence de l'aléa

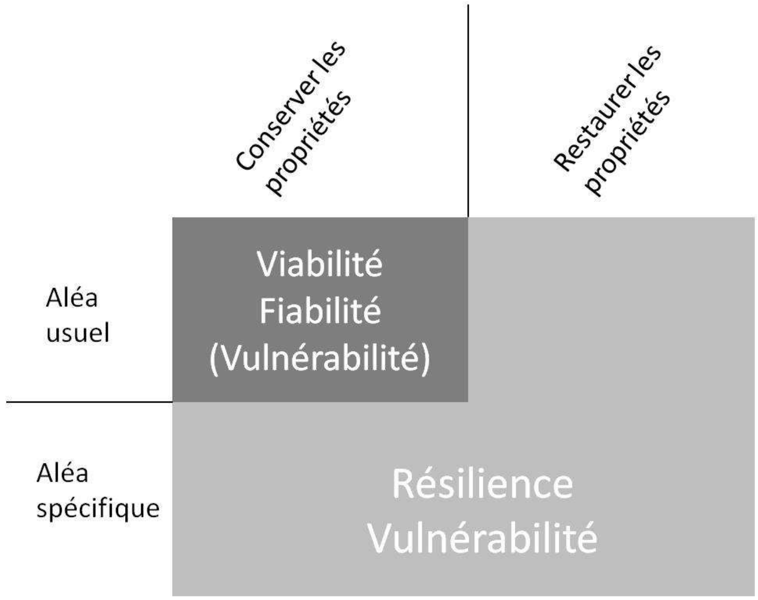

FIGURE 1.2 – Typologie des indicateurs en fonction de l'aléa et de la question de savoir si les propriétés ont été perdues ou non.

(Cobb et Thompson, 2012). C'est dans ce cadre-là que les indicateurs de résilience – comprise comme la restauration des propriétés – et de vulnérabilité développés dans la thèse pourront être utilisés. Les indicateurs de vulnérabilité pourront aussi s'appliquer aux aléas usuels qui n'entraînent pas la perte des propriétés, car la vulnérabilité est parfois définie par rapport au franchissement d'un seuil de dommage (Luers *et al.*, 2003). Il s'agit alors d'éviter ce franchissement, et ce problème est à rapprocher de ceux trouvés en théorie de la viabilité. La typologie des indicateurs est résumée par la Figure 1.2.

1.3.2 Utilisation de la programmation dynamique stochastique

Les concepts de la thèse sont rendus opérationnels par la PDS, un algorithme d'optimisation qui s'appuie sur une fonction valeur. Celle-ci est initialisée à la date T définie par la formulation de l'équation (1.3), puis optimisée itérativement par récurrence inversée. Dans certains cas, il est possible de relier la valeur de cette fonction à la date initiale à un objectif, et de prouver qu'elle l'optimise. L'algorithme donne aussi les contrôles qui permettent d'atteindre cet optimum.

La PDS est ainsi a même de minimiser l'espérance mathématique d'un coût défini sur une trajectoire comme la somme des coûts occasionnés à chaque date De Lara et Doyen (2008). Or la vulnérabilité est souvent interprétée par le biais d'un coût, qu'il soit humain, monétaire ou autre (Yohe et Tol, 2002; Adger, 2006;

Peduzzi *et al.*, 2009). La minimisation d'un coût peut donc être comprise comme un calcul de vulnérabilité. Dans les régions de l'espace d'état où ce coût est faible, il est possible de se retrouver dans une configuration de contrôle optimal dans certains cas.

Récemment, Doyen et De Lara (2010) ont montré que la PDS pouvait maximiser la probabilité pour un système d'être viable en respectant des contraintes sans interruption jusqu'à la date T. Cela mène à la définition et au calcul du noyau de viabilité dans le cas stochastique. Un tel algorithme était nécessaire car les propriétés du noyau de viabilité, utilisées pour le calcul de noyau (Saint-Pierre, 1994; Bonneuil, 2006; Deffuant *et al.*, 2007) ne sont en général plus vérifiées dans le cas stochastique. La maximisation de viabilité est omniprésente dans cette thèse. Par ailleurs, l'algorithme de Doyen et De Lara s'applique à ce que l'on appelle un problème de cible, dans lequel la contrainte est d'atteindre une zone donnée de l'espace d'état à une date donnée. Ce type de problème a été utilisé dans le cas déterministe pour définir la notion de temps de crise (Doyen et Saint-Pierre, 1997) qui entretient des liens étroits avec le concept de résilience (Martin, 2004; Hardy, 2013).

La vulnérabilité est parfois définie par rapport à un seuil. Un noyau de viabilité stochastique est un cas particulier d'ensemble pour lequel la vulnérabilité, comprise comme probabilité de franchissement du seuil, est faible. Ainsi, la vulnérabilité peut être – parfois simultanément – reliée aux deux algorithmes de PDS sus-mentionnés. On connaît alors les décisions qui maximisent une probabilité d'être viable ou de minimiser un coût. Celles-ci sont généralement différentes, indiquant alors qu'il existe un compromis à trouver entre les deux objectifs. Toutefois, aucun des deux algorithmes ne permet de trouver ce compromis. C'est pourquoi le Chapitre 5 s'intéresse au problème d'utiliser la PDS sur un indicateur qui agrège ces deux types d'objectifs.

Toutefois la PDS, comme les algorithmes de viabilité utilisés dans le cas déterministe, sont sujets à ce que l'on appelle la malédiction de la dimension. Cela signifie que le temps de calcul comme la mémoire requise augmentent exponentiellement avec le nombre de dimensions de l'espace d'état, rendant le calcul impossible. C'est dans cette optique-là qu'il convient de ne pas négliger l'apport potentiel d'autres disciplines. Par exemple, la fiabilité (Rackwitz, 2001) a développé des algorithmes de calcul de probabilité de défaillance dans des cas où l'espace à explorer est de dimension très élevée, grâce par exemple à des méthodes d'approximation de surfaces de réponse. Certaines de ces méthodes, comme FORM (First Order Reliability Method) ou SORM (Second Order Reliability Method) sont statiques, donc *a priori* loin de cas décrits par l'équation (1.3). Mais il est intéressant de noter qu'une étude récente a porté sur la viabilité de pêcheries côtière guyanaises en considérant treize espèces simultanément, et une évolution sur un seul pas de temps (Cissé *et al.*, 2013). Un tel problème ressemble à ceux que l'on peut traiter par FORM ou SORM.

1.4 Thème (C) : Complémentarité de résilience et vulnérabilité

Cette section constitue une mise en perspective des concepts de résilience et vulnérabilité, dans la démarche de la thèse qui est de progresser vers un cadre commun aux deux concepts.

La résilience et la vulnérabilité ont souvent été employées comme contraires l'un de l'autre, et la vulnérabilité a même été définie comme le “revers de la médaille” de la résilience (Folke *et al.*, 2002b). Cependant, une revue comparative des deux concepts a conclu qu'il existait des aires de convergence et de complémentarité entre les deux concepts (Miller *et al.*, 2010), et suggéré que des travaux supplémentaires étaient nécessaire pour réellement les mettre en lumière. Cela correspond au thème (C) de cette thèse : la mise en évidence de complémentarités entre ces deux concepts. Cette section a pour but de donner quelques perspectives.

1.4.1 Liens entre concepts et aspects normatifs du système

Une première piste concerne la manière dont les deux concepts sont reliés aux aspects normatifs du système. Par exemple, dans leur revue de la littérature de la vulnérabilité, Eakin et Luers (2006) évoquent “l'identification de seuils de dommages significatifs”, ce qui implique que les dommages potentiels sont évalués pour chaque état ou trajectoire avant de fixer un seuil. Mais ils dépeignent aussi un seuil comme “un point de référence à partir duquel mesurer” la vulnérabilité. Le dommage n'est alors défini qu'au-delà du seuil, et une fois que celui-ci est fixé.

En réalité, ce dernier cas correspond à définir une zone de dommage nul et à considérer que le “seuil de dommage significatif” délimite cette zone. Il est donc plus général d'estimer les dommages et de définir ensuite des seuils. Emergent alors deux manières alternatives de définir les propriétés selon que l'on définit ou non un seuil, et qui restent à explorer.

En définissant un seuil, on définit les propriétés du système par un sous-ensemble de l'espace d'état qui n'est autre que le noyau de faible vulnérabilité. Cela pose des problèmes évidents de définition, puisqu'un tel objet n'est défini qu'à la date initiale, alors que les contraintes sont définies pour chaque date de l'horizon temporel considéré. Toutefois, c'est une piste à explorer afin de définir des propriétés pour le calcul de résilience en s'appuyant sur la description normative du système donnée par l'association d'une fonction de dommage à l'espace des trajectoires.

Les propriétés d'un SSE ont systématiquement été modélisées par un prédicat dans le cadre viabiliste de la résilience. Par exemple dans le cas, abondamment repris dans la thèse, de l'eutrophisation d'un lac, l'approche viabiliste fait le choix de supposer que le lac est eutrophe quand sa concentration en phoshpore est supérieure à un seuil. L'eutrophisation se traduit par une flore essentiellement algale, qui se développe au détriment du reste des espèces initialement présentes dans le lac, et peut conduire à rendre l'eau impropre à la baignade ou à la consommation. Toutefois, on peut aussi faire correspondre à la propriété “lac oligotrophe” (le contraire d'eutrophe) plusieurs seuils dans l'espace d'état, selon le niveau d'eutrophisation et les désagréments associés. Le respect de la propriété “lac oligotrophe” peut alors être évalué pour chaque trajectoire à travers les dommages liés au franchissement de chaque seuil, et le temps passé au-delà de ces seuils. Si l'on se réfère à la terminologie du Chapitre 4, elle peut dans ce cas précis être évaluée à travers une fonction de dommage telle qu'établie pour calculer la vulnérabilité.

La question est alors de savoir si le noyau de faible vulnérabilité traduit alors la persistance de la propriété “lac oligotrophe”. En effet, le fait que l'état se trouve dans le noyau traduit le fait que les chances de franchir durablement un ou plusieurs des seuils d'eutrophisation à un horizon donné sont limitées. Toutefois, une telle notion de persistance est plus équivoque que celle proposée par le noyau de viabilité stochastique. Sa pertinence reste à explorer et le cas échéant, à démontrer.

1.4.2 Rapport à la persistance de propriétés

Dans le cadre viabiliste, la résilience est caractérisée par le retour dans le noyau de viabilité. Cet ensemble caractérise de manière non équivoque la persistance des propriétés, car aussi longtemps qu'aucun aléa ne vient perturber le système, celui-ci va toujours pouvoir être contrôlé de manière à ce que ses propriétés soient respectées. En particulier, il sera possible de garder l'état dans le noyau de viabilité.

Une telle stabilité des trajectoires dans le noyau n'est plus garantie dans le cas stochastique. En effet, seule la stabilité dans l'ensemble des contraintes est garantie avec un niveau de confiance et pour une période donnés. Une question pratique est de savoir comment traiter le cas (*) où l'état quitte le noyau de viabilité stochastique, mais que les propriétés sont conservées. L'idée du cadre de la résilience du Chapitre 2 est de restaurer la persistance des propriétés en optimisant la vitesse de retour dans le noyau de viabilité stochastique. La stratégie qui optimise la vitesse de restauration des propriétés et celle qui optimise la viabilité ne sont pas forcément les mêmes. Il peut donc exister dans le cas (*) un dilemne entre la probabilité d'être viable et la probabilité de résilience à certains horizons temporels. Si la non-viabilité est reliée à la vulnérabilité comme dans le Chapitre 4, cela peut se traduire par un dilemne entre résilience et vulnérabilité. Dans ce cas précis, les rôles des deux concepts de résilience et de vulnérabilité dans un cadre commun restent donc à établir.

Une autre question est la possibilité de décrire la persistance des propriétés en utilisant un noyau de faible vulnérabilité autre que le noyau de viabilité stochastique. *A priori*, utiliser un critère de viabilité traduit mieux la persistance qu'un critère de coût. Toutefois, la question mérite d'être posée, surtout à la lumière des noyaux de faible vulnérabilité basés sur des critères composite de coût et de viabilité définis dans le Chapitre 5. De tels noyaux sont peut-être à même de résoudre le dilemne exposé au paragraphe précédent.

1.4.3 Description des trajectoires de restauration

En définissant la résilience comme l'inverse d'un coût de restauration, Martin (2004) accrédite l'idée que la vulnérabilité et la résilience sont des contraires. Toutefois, cette idée a ensuite été nuancée (Deffuant et Gilbert, 2011) en dissociant la résilience – le fait de restaurer les propriétés – du coût associé. La vulnérabilité et la résilience donnent alors une information complémentaire.

Dans un cadre stochastique, le fait de rentrer n'est plus seulement certain ou impossible. Il devient en général une probabilité. Cela s'applique aussi au fait de rentrer dans un horizon temporel donné. Dans ces conditions, il existe de nombreuses statistiques associées à la distribution de probabilité de dates de rentrée. Si de nombreuses métriques peuvent décrire la résilience du système, alors la vulnérabilité devient d'autant plus nécessaire pour décrire les dommages occasionnés par la perte des propriétés.

Cependant, il faut garder à l'esprit que les politiques d'action qui maximisent une probabilité de résilience et celles qui minimisent une fonction de dommage ne sont pas en général les mêmes. Là encore, les indicateurs composites définis dans le Chapitre 5 semblent en mesure d'expliciter les compromis éventuels à faire entre résilience et vulnérabilité.

– 2 –

Extending the viability theory framework of resilience to uncertain dynamics, and application to lake eutrophication

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Resilience, the capacity for a system to recover from a perturbation so as to keep its properties and functions, is of growing concern to a wide range of environmental systems. The challenge is often to render this concept operational without betraying it, nor diluting its content. The focus here is on building on the viability theory framework of resilience to extend it to discrete-time stochastic dynamical systems. The viability framework describes properties of the system as a subset of its state space. This property is resilient to a perturbation if it can be recovered and kept by the system after a perturbation : its trajectory can come back and stay in the subset. This is shown to reflect a general definition of resilience. With stochastic dynamics, the stochastic viability kernel describes the robust states, in which the system has a high probability of staying in the subset for a long time. Then, probability of resilience is defined as the maximal probability that the system reaches a robust state within a time horizon. Management strategies that maximize the probability of resilience can be found through dynamic programming. It is then possible to compute a range of statistics on the time for restoring the property. The approach is illustrated on the example of lake eutrophication and shown to foster the use of different indicators that are adapted to distinct situations. Its relevance for the management of ecological systems is also discussed.

2.1 Introduction

Resilience has growingly been regarded as a central concept for many ecological systems, as well as for many human systems relying on ecosystem services. Indeed, resilience is related to the continued existence and sustainability of these systems in an era of upcoming widespread changes (e.g. Folke *et al.*, 2002b; Walker *et al.*, 2004). Since the original definition of resilience in ecology by Holling (1973), there has been a flurry of definitions of the concept in related fields (Brand et Jax, 2007), while it can be tackled at different levels of abstraction (Carpenter *et al.*, 2001). In this context, and in order not to lose what is meant by resilience, one central challenge is to introduce generic computational frameworks that can at the same time accurately reflect the concept and produce case-specific indicators.

This paper aims at tackling this issue for discrete-time stochastic dynamical systems. It builds on the viability theory framework of resilience, initially introduced in Martin (2004) for deterministic systems, and to which a book has recently been devoted (Deffuant et Gilbert, 2011). It focuses on extending this framework to the stochastic case through the use of computational techniques based on dynamic programming. The viability framework generalizes the current mathematical definitions of the concept of resilience without betraying its intuitive sense. In this respect, we consider that the viability approach responds to the call from Brand et Jax (2007) for a clear descriptive definition of the term of resilience, to avoid an eventual loss of its conceptual content as its use as a boundary object is spread around different research communities, for instance ecology and social sciences (Adger, 2000; Folke *et al.*, 2002b; Adger *et al.*, 2005). Besides, one major advantage of the viability approach is to provide with ways of computing relevant courses of action to ensure the resilience of the system. The extension presented here preserves this aspect.

The viability framework of resilience starts with the idea that it is essential to define the resilience of "what" to "what" (Carpenter *et al.*, 2001). It assumes that we are interested in the resilience of a property of the dynamical system, this property being called resilient to a perturbation if the system is able to restore it if it is lost, then keep it. In mathematical terms, a property can be described as a subset of the system state space, delimited by constraints. Resilience is related to the ability to come back and stay in this subset. Hence, this approach particularly emphasizes the possibility for restoring the property after losing it, which in our view is what distinguishes resilience from the concept of stability (Grimm et Wissel, 1997).

In this respect, the viability framework is distinct from the initial definition of resilience in ecology by Holling (1973) as the amount of perturbation a system can withstand while keeping its properties and functions. A direct implication of such a definition is the emphasis put on the considered properties and functions. The reason is that the considered resilience is of the system itself, hence of the properties and functions that define its identity. When considering the resilience of any property, possibly a minor one, we open the possibility for the system to lose the property for a while, and then to restore it. We claim that focusing on the aspects of recovery and restoration, which is put forward in the viability based framework, is pivotal to the resilience concept. There is indeed a variety of studies that relates resilience to measures of the recovery capacity of a system (e.g. Hashimoto *et al.*, 1982; MacGillivray et Grime, 1995; Johnson *et al.*, 1996; Lesnoff *et al.*, 2012).

As explained in Deffuant et Gilbert (2011), the viability framework generalizes the mathematical attractor-based definitions of resilience (e.g. Anderies *et al.*, 2002) which focus on regime shifts. Indeed, in these latter definitions, the property of interest is identified as a set of states located around selected attractors which represent a desirable regime. Resilience indicators are related to the size of the attractor basins of these attractors, namely the part of space where the system avoids to fall into bad attractor basins. If one defines the property of interest as a subset around the good attractors, then the viability framework provides similar results to the ones of the attractor based definition. However, the property of interest can be defined as any subset of the state space and therefore offers wider possibilities, especially in cases where no attractor would exist.

Moreover, the viability framework of resilience provides operational tools for computing policies of actions (or feedback rules) to keep or restore a property of interest, namely by driving the system back to the desirable subset and keeping it there. Alternative mathematical frameworks for computing resilience at best suppose that a policy of action is already defined. This is a very important practical advantage of the viability framework, as illustrated in several case studies in (Deffuant et Gilbert, 2011). As any other framework, viability has its limits and drawbacks, namely :

1) The current numerical approaches for computing the resilient states and the restoration policies are very demanding computationally because they require to discretize the state space (and also the action space). Therefore, in practice only the dynamical models with very few degrees of freedom are tractable (see Deffuant et Gilbert (2011) for further information).

2) The current viability framework only considers one-time perturbations in otherwise deterministic dynamical systems, thus not taking into account the potential impact of other uncertainty sources while computing resilience to these events.

This paper aims at overcoming the latter limit and proposes an extension of the viability framework of resilience to stochastic dynamical systems. It is based on the key assumption that uncertainty, whatever its source, can be mathematically described in two different ways. On the one hand, some of the uncertainty can be described by stochastic processes, which can be assessed under the form of probability distribution functions (pdf) and reduced by experience (Allen *et al.*, 2011; Williams, 2011). Since this part of the uncertainty can be embedded into the dynamics, the system can be managed so as to be made robust to it. On the other hand, there is invariably a part of unpredictability that escapes such assessments (Gunderson, 2000; Walker *et al.*, 2002), as emphasized by the expression "uncertainty and surprise" found in the literature (Folke *et al.*, 2002b, 2004; Adger *et al.*, 2005). This second uncertainty source generally refers to events that initiate potentially major disturbances, and to which the resilience of the system must be assessed.

From a technical point of view, the present work uses the stochastic extensions to viability theory in the discrete-time case (De Lara et Doyen, 2008; Doyen et De Lara, 2010) and builds on the robust and stochastic viability frameworks, both based on dynamic programming, and which have been successfully applied to fisheries management (Doyen et Béné, 2003; De Lara et Martinet, 2009; Doyen *et al.*, 2012; Péreau *et al.*, 2012). One can expect three main differences between the deterministic and stochastic viability frameworks :

1) The viability kernel is a subset of the state space of a deterministic system in which the system can keep the property if it remains undisturbed. Stochastic viability kernels only guarantee that the system will keep the property with a given probability by a given time horizon, but this is also a guarantee that the system is robust to the uncertainty sources described by the stochasticity of its dynamics.

2) The single optimal trajectory for reaching the deterministic viability kernel is replaced by a set of trajectories. This fosters the definition a probability of resilience, the probability of reaching the stochastic viability kernel by a given time horizon. Besides, the single resilience indicator that prevails in the deterministic case (e.g. the inverse of the restoration cost as in Martin (2004)) is replaced by a pdf for the times or costs of restoration. Resilience indicators are derived from statistics on this pdf.

3) Contrary to what happens with deterministic viability, the time horizon considered for the management of the system becomes of paramount importance for policy design. Policies become specifically designed to maximize the probability of being

resilient by the time horizon, but are not in general meant to maximize this probability at any shorter horizon.

The paper is organized as follows. First, we recall the main concepts of the viability-based framework of resilience in the deterministic case, and illustrate it on a simple lake eutrophication model (Section 2.2). In section 2.3, we introduce the extension of the viability based framework of resilience to stochastic dynamics, defining the concepts of probability of resilience and different measures of restoration time and cost. Section 2.4 illustrates the approaches on the lake model, first for a given control strategy, then when controls are being optimized as resilience is computed. The relevance for the study of ecological systems is discussed in Section 2.5, before conclusions are drawn in Section 2.6.

2.2 Background : resilience in the deterministic case

The general problem of achieving resilience in limited time after a perturbations, considering constrained dynamics and controlled strategies, is laid out in discrete time in the deterministic case. The viability framework of resilience from Martin (2004) and Deffuant et Gilbert (2011) is then put into perspective.

2.2.1 Problem statement

Controlled dynamics

In the viability framework for resilience presented by Martin (2004), an important innovation is to introduce controls to explicitly account for the possibility to act on the system. In this framework, the policy is not fixed beforehand. Instead, the goal is to find policies that will make the system resilient. In discrete time, this means that at each time step, there is a set of possible actions that one must choose from, and a known transition equation between two consecutive dates. Let us note X the state space and x_t the state of the system at date t. Noting also $U(x_t, t)$ the set of available controls and u_t the chosen control value, a typical controlled discrete-time dynamical system can be written as :

$$x_{t+1} = g(x_t, u_t) \tag{2.1}$$

Resilience to "what"?

In this framework we focus on resilience to a given perturbation. There is no assumption on the nature or amplitude of the perturbation. Indeed, we place ourselves in the post-perturbation state of the system. Resilience of any state to any perturbation can then be assessed by looking at the new state of the system after the perturbation.

Management objectives and constraints

The system has some properties that are deemed desirable. We assume that these properties can be mathematically translated into state constraints, which define a set of desirable states noted K. The general goal then becomes to control the system so it stays within K.

After a perturbation, the goal becomes to ensure that the system gets back to a state where its dynamics are going to keep it within K for as long as it is not disturbed again. Management also has a time frame, or time scale of interest, which we will note T. This is the time by which the system's properties ought to be restored.

Management strategies

The objectives are achieved through management strategies. A strategy can be represented by a function which associates a control to any date $0 \leq t \leq T-1$ and to any state x. A strategy can be described at each time step by a feedback map, an application from X into $U(x,t)$ (or from K if we only want to see whether the system leaves K). In this work, strategies will be determined depending on the time horizon. Consequently, we choose to reference a feedback maps are referenced by their time distance to the horizon. A strategy is given, in chronological order, by the following succession of feedback maps :

$$f = (f_T, f_{T-1}, \dots, f_1) \tag{2.2}$$

Controls at any date $t \leq T$ are deduced from equation (2.3) :

$$\begin{cases} u_t = f_{T-t}(x) \\ u_t \in U(x,t) \end{cases} \tag{2.3}$$

We can also introduce the notation $F(T)$ for the set of all strategies f with a time horizon T. For a given f, it is possible to recursively compute all the states from the initial state x_0 to the final state x_T. Indeed, equation (2.1) can be written anew using equation (2.3) through :

$$x_{t+1} = g(x_t, f_{T-t}(x_t)) \tag{2.4}$$

Thus, we can define the trajectory g_f starting from the initial state x_0 and using the strategy f as :

$$\forall t \in [0,T],\ x_t = g_f(t, x_0) \tag{2.5}$$

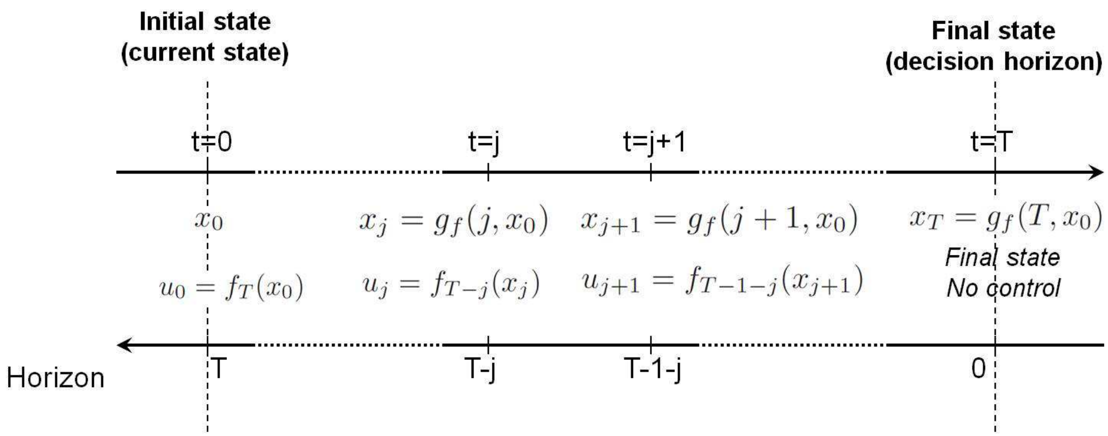

FIGURE 2.1 – Illustration of the notions of strategy and time horizon (equations (2.2) through (2.5)).

Figure 2.1 summarizes the notations introduced in this section for a horizon T and the associated strategy f. The choice of a strategy (equation (2.2)) allows for the computation of controls (equation (2.3)). Successive states can then be computed through equation (2.1), leading to the computation of trajectories (equation (2.5)).

2.2.2 The viability kernel

Prior to assessing resilience, one must determine which are the desirable states of K for which the dynamics can keep the system properties until another perturbation occurs, however long it may take for that to happen. In order to guarantee that, the properties need to be kept for a length of time much greater than the time scale of interest, T. Resilience will then be the ability to reach one of these states.

Under the framework of viability theory, the set of all the states for which there is a control strategy such that the system can be maintained inside the set of desirable states throughout a period of time $\tau >> T$ is called the viability kernel. In discrete time, it can be formally defined as the set of initial states for which there exists a trajectory that does not leave K during τ time steps :

$$\mathrm{Viab}(\tau) = \left\{ x_0 \in K \mid \exists f \in F(\tau),\ \forall t \leq \tau,\ x_t = g_f(t, x_0) \in K \right\} \tag{2.6}$$

and for simplicity, the notation Viab will be used instead of Viab(τ) in the remainder of Section 2.2.

In practice, several algorithms exist to determine which states belong to the viability kernel (Saint-Pierre, 1994; Deffuant *et al.*, 2007). Its computation also yields the set of controls which maintain the system, which are called the viable controls. Thus, it incorporates the impacts of management policies, and implicitly optimizes them.

2.2.3 Viability-based definition of resilience

This section puts into perspective the results from Martin (2004) and Deffuant et Gilbert (2011) about resilience to a single perturbation in an otherwise deterministic system. Resilience is associated to the possibility of getting back to the viability kernel in a relevant time frame, denoted by the horizon T.

Let us first consider a given strategy f at horizon T, with the only constraint that at any date and for any state within the viability kernel $f_{T-t}(x)$ is a viable control. One can then define what is called a resilience basin in Deffuant et Gilbert (2011) : the set of states for which the system is brought back to the viability kernel in a horizon $t \leq T$. This notion can first be written for a given strategy :

$$B^{\text{res}}(f,t) = \left\{\, x \in X \mid g_f(t,x) \in \text{Viab} \,\right\} \tag{2.7}$$

For $t_1 < t_2$, since any trajectories that reaches the viability kernel by t_1 can also reach it if given a greater time horizon, we have $B^{\text{res}}(f,t_1) \subset B^{\text{res}}(f,t_2)$. The union of all the resilience basins is $B^{\text{res}}(f,T)$, and it is the set of resilient states for the strategy f, from which the system can recover by getting back to the viability kernel in T time steps or less.

Yet, one has many strategies at her disposal, and may want to find the set of all the states for which it is possible to bring the system back to the viability kernel in a horizon $t \leq T$. This brings about a different definition of a resilience basin :

$$B^{\text{res}}(t) = \left\{\, x \in \mathbb{R}^n \mid \exists f \in F(t),\; g_f(t,x) \in \text{Viab}\right\} \tag{2.8}$$

The implicit difference between equations (2.7) and (2.8) is that in the latter, controls have been optimized to find trajectories that ensure the resilience of the system by the chosen time horizon. In this work, we choose to use dynamic programming to carry out this optimization. Dynamic programming is a recursive algorithm which enables to optimize a value function, noted $V_d(t,x)$ in this deterministic case, at each horizon t. It progresses backwards from date T (horizon 0) to the initial date (horizon T). It yields the same results in the deterministic case as algorithms like KAVIAR (Deffuant *et al.*, 2007) used by Deffuant et Gilbert (2011), but will be extended to the uncertain case in Section 2.3. Like any recursive algorithm, it works based on an initial equation and a transition equation. Here, initialization comes from the fact that resilience at a time horizon of 0 is the fact of belonging to the viability kernel :

$$V_d(0,x) = \begin{cases} 1 & \text{if} \quad x \in \text{Viab} \\ 0 & \text{if} \quad x \notin \text{Viab} \end{cases} \tag{2.9}$$

Then the algorithm progresses recursively by incrementing the horizon considered from 0 to T thanks to the following transition equation :

$$\forall t \in [1,T],\ V_d(t,x) = \max_{u \in U(x,T-t)} V_d(t-1, g(x,u)) \tag{2.10}$$

This value function can only take values 0 and 1. The work by Doyen et De Lara (2010) links the computation of V_d with the resilience basins $B^{\text{res}}(t)$ (see Appendix 2.7 for the proof) :

$$B^{\text{res}}(t) = \{\ x \in X \mid V_d(t,x) = 1\} \tag{2.11}$$

The set of resilient states is the resilience basin $B^{\text{res}}(T)$. The definitions introduced in this section are graphically summarized in Figure 2.2. A great strength of the viability framework is to dynamically compute the optimal controls and the resilience basins at the same time. One can thus the restoration time at $x \in B^{\text{res}}(T)$, which can be handily defined as the minimal horizon for which a trajectory starting at x can reach the viability kernel :

$$t^*(x) = \min_{t \leq T} \left\{ t | x \in B^{\text{res}}(t) \right\} \tag{2.12}$$

and the associated optimal policies are defined by recurrence. For x outside the viability kernel, and $t^*(x) > 0$, any control that enables the system to reach $B^{\text{res}}(t^*(x)-1)$ preserves the possibility for the properties to be restored by the horizon t^*. Besides, the controls only need to be set once, so that in the deterministic case, the same feedback map can be used regardless of the horizon. This will not be true in the stochastic case which we introduce now.

2.3 Resilience computations in uncertain discrete-time systems

This section presents the extension of the previous framework to stochastic systems, which is the main contribution of this paper. Uncertainty is first introduced into the modeling framework described by Section 2.2.1. A stochastic equivalent of the viability kernel is introduced to describe the safe states of the system. Since one can no longer

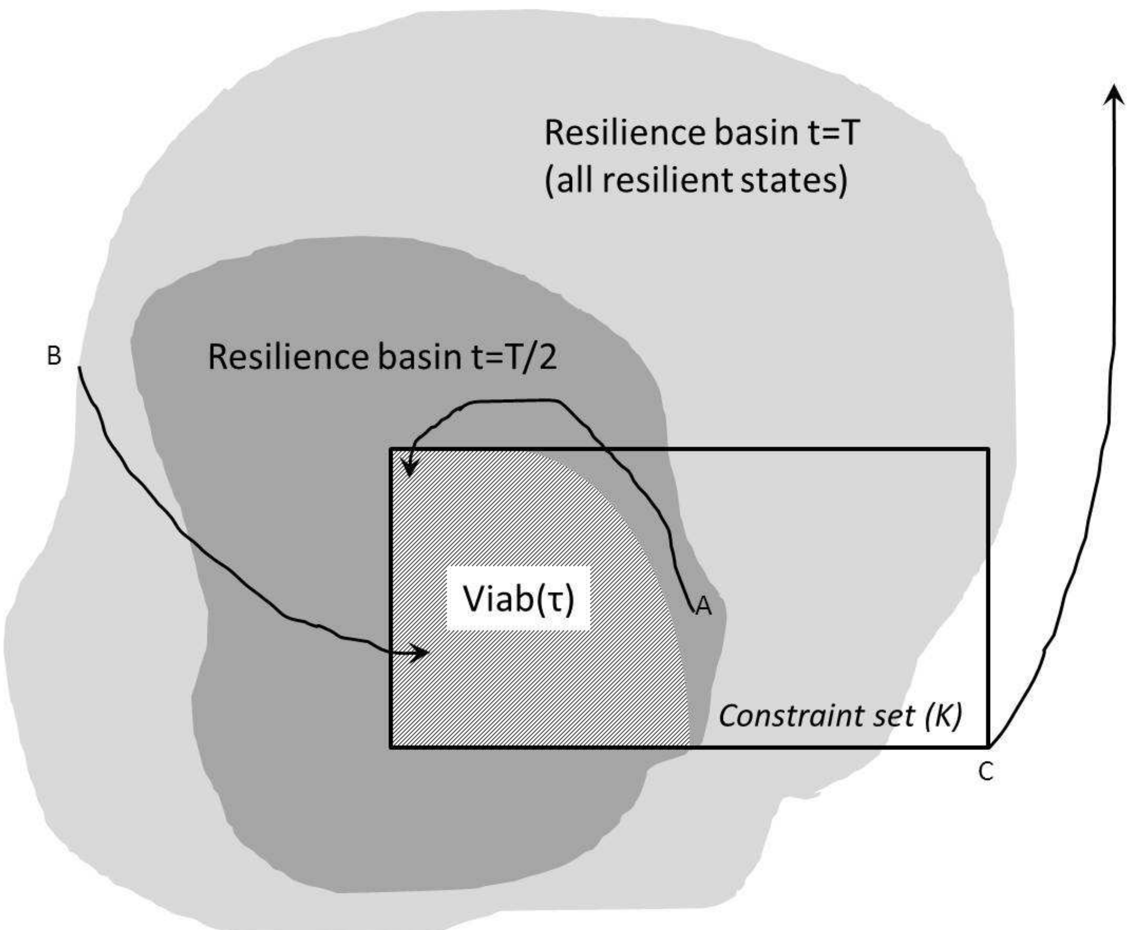

FIGURE 2.2 – Constraint set, viability kernel and some resilience basins in the original viability framework of Martin (2004) and Deffuant et Gilbert (2011). There is one possible trajectory for a given control strategy : post-perturbation states A and B are resilient while C is not.

guarantee that a set can be reached with probability one after a minimal amount of time, we choose to focus on maximizing the probability of being resilient by reaching the stochastic viability kernel. We thus introduce the notion of probability of resilience, before showing how quantities that can be related to a variety of possible resilience indicators can be handily computed thanks to this notion.

2.3.1 Incorporating uncertainty

Uncertainty in a modeling framework such as that of Section 2.2.1 can arise from numerous sources (Williams, 2011). Yet, here we are concerned with how uncertainty can be modeled, rather than where it comes from. Knowledge about uncertainty can be imperfect (Walker *et al.*, 2002), so that no pdf can capture all of the uncertainty, especially that related to large events. As mentioned in the introduction, the key assumption used here is to consider that aside from large uncertain perturbations which might happen from to time to time, the rest of the uncertainty can be modeled at each time step using random variables for which a pdf can be defined. Then, we compute resilience to a given large, unknown event while taking into account the sequential effects of smaller events which may be described in stochastic terms at each time step.

When considering uncertainty, the dynamic g of equation (2.1) has to be modified. Following Doyen et De Lara (2010), let us assume the existence of a known dynamic g which incorporates a vector ε of all uncertainties, whether they concern the dynamic itself or the state, control or parameter variables. We get :

$$x_{t+1} = g(x_t, u_t, \varepsilon_t) \tag{2.13}$$

Let us now suppose that the state space X has been discretized, which is the case in practice. Discretization methods are not within the scope of this paper. We note $\hat{X}$ the points of the discrete grid.

The rest of Section 2.2.1 is kept unchanged. In particular, the notion of strategy $f \in F(T)$ is kept ; unlike what happens in deterministic viability, one should not expect the feedback map to be the same for two different values of the horizon T. We also keep the notion of trajectory associated to a strategy, but we are now faced with a set of trajectories which depends on the events ε_0, ε_1, etc...Thus, $g_f(t, x_0)$ becomes a random variable.

The use of dynamic programming as an essential tool of stochastic viability theory (Doyen et De Lara, 2010) imposes the pdfs ε_t to be uncorrelated with one another. This is because dynamic programming essentially requires to travel backwards in time. What is more, in this framework, feedback optimization is only guaranteed when the

ε_t are independent and identically distributed (i.i.d.), which we are going to assume from now on. Uncertainty can then be made implicit in equation (2.13) :

$$x_{t+1} = g_\varepsilon(x_t, u_t) \tag{2.14}$$

For a given state x_t and decision u_t, the probability of the state value being y at date $t+1$ will be noted using the discrete probability $\mathbb{P}(g_\varepsilon(x_t, u_t) = y)$. Stochastic viability kernels will now be introduced to describe robustness to the uncertainty ε.

2.3.2 Robustness to uncertainty

With the introduction of uncertainty, it is not possible in general to find states that ensure with unit probability that a system will retain its properties if there is no major perturbation before a date τ. Instead, one can introduce the stochastic viability kernel (De Lara et Doyen, 2008), defined as the set of initial states for which the system has a probability β or higher of keeping its properties for a duration τ. Noted $\text{Viab}(\beta, \tau)$, it can be formally defined by the following equation :

$$\text{Viab}(\beta, \tau) = \left\{ x_0 \in K \mid \exists f \in F(\tau),\ \mathbb{P}\Big(\forall t \in [0, \tau], x_t = g_f(t, x_0) \in K\Big) \geq \beta \right\} \tag{2.15}$$

For instance, $\text{Viab}(0.99, 100)$ is the set of initial states such that the system has at least 99% chance of avoiding the loss of its properties for at least a hundred time steps. An interesting case arises when $\beta = 1$: indeed $\text{Viab}(1, \tau)$ is the set of initial states x_0 for which the system is kept in a desirable state with certainty during τ time steps. This set is called the *robust viability kernel* and is an analog of the viability kernel defined in Section 2.2.2. The stochastic viability kernel is computed thanks to the dynamic programming algorithm proposed by Doyen et De Lara (2010) (see Appendix 2.7 for details).

Like for the deterministic viability kernel, the feedbacks are being optimized, but optimization now occurs at each time horizon on the interval $[0, \tau - 1]$. Thus, dynamically computing the optimal management strategy while looking for the robust states of the system remains a major advantage of the viability approach.

Sets like $\text{Viab}(\beta, \tau)$ with β close or equal to 1 and $\tau \geq 100$ can be seen as the set of system states where the desirable properties of the system hold (almost) certainly despite the onset of quantifiable uncertainties or disturbances. Therefore stochastic viability kernels are sets where the system properties are robust to the uncertainty sources modeled by the process ε, and the parameters β and τ describe the extent of such robustness. The term ε-robustness is here used to designate the property of state within a stochastic viability kernel. However, the focus of this work is to compute resilience to a large unexpected perturbation, and a new notion is needed to extend to the stochastic case the viability framework, which precisely dealt with resilience to such events in the deterministic case.

2.3.3 Probability of resilience

Let us now assume that one has computed a stochastic (or robust) viability kernel $\mathrm{Viab}(\beta,\tau)$, the set of states for which the system's properties are guaranteed to be ε-robust. The conceptual notion of resilience can then be operationalized through the possibility of reaching this set in a given time frame T. This is very similar to what is done in the deterministic case, except that reaching $\mathrm{Viab}(\beta,\tau)$ now guarantees ε-robustness to quantifiable uncertainty. Besides, the considered horizon T has to be small compared to the time scale τ during which the system can be maintained with a high probability, so that the condition $T << \tau$ holds, like in Section 2.2.

For a given strategy $f \in F(T)$, the probability of getting into a given $\mathrm{Viab}(\beta,\tau)$ by a date $t \leq T$ is the capacity to recover from a perturbation under this strategy. It is noted $\mathbb{P}_{\mathrm{Res}}(f,t,x)$:

$$\mathbb{P}_{\mathrm{Res}}(f,t,x) = \mathbb{P}\left(\exists j \in [0,t], g_f(j,x) \in \mathrm{Viab}(\beta,\tau)\right) \tag{2.16}$$

and from equation (2.2), the controls that are applied between dates 1 and t are those of the successive feedback maps $(f_T, f_{T-1}, \ldots, f_{T-t})$.

Like in the deterministic case, the objective of viability is to find the feedbacks that maximize the probability of resilience, but now specifically as the probability of reaching $\mathrm{Viab}(\beta,\tau)$ by date T. This probability of resilience, noted $\mathbb{P}_{\mathrm{Res}}(T,x)$, is thus given by the value of $\mathbb{P}_{\mathrm{Res}}(f,T,x)$ when the strategy is optimal :

$$\begin{aligned} \mathbb{P}_{\mathrm{Res}}(T,x) &= \max_{f \in F(T)} \mathbb{P}_{\mathrm{Res}}(f,T,x) \\ &= \mathbb{P}_{\mathrm{Res}}(f^*,T,x) \end{aligned} \tag{2.17}$$

where the optimal strategy f^* is not necessarily unique. The probability of resilience $\mathbb{P}_{\mathrm{Res}}(T,x)$ represents the probability of being resilient within the considered time frame. If $\mathbb{P}_{\mathrm{Res}}(T,x) = 1$, the system can always be made ε-robust, and it is resilient.

To compute the probability of resilience in this stochastic case, we use a value function V_s depending on the horizon t and state x, which is the exact analog of the deterministic V_d :

$$\begin{cases} V_s(0,x) = \begin{cases} 1 & \text{if} \quad x \in \text{Viab}(\beta,\tau) \\ 0 & \text{if} \quad x \notin \text{Viab}(\beta,\tau) \end{cases} \\ \forall t \in [1,T], \\ V_s(t,x) = \begin{cases} \max\limits_{f_t(x)\in U(x,T-t)} \left(\sum\limits_{y\in\hat{X}} \mathbb{P}(g_\varepsilon(x,f_t(x)) = y)\ V_s(t-1,y) \right) \\ \quad \text{if } x \notin \text{Viab}(\beta,\tau) \\ 1 \text{ if } x \in \text{Viab}(\beta,\tau) \end{cases} \end{cases} \tag{2.18}$$

and the value function V_s yields the maximal value of the probability of resilience (proof to be found in Appendix 2.7) :

$$\mathbb{P}_{\text{Res}}(T,x) = V_s(T,x) \tag{2.19}$$

and the associated optimal strategy f^* is given by the feedback maps f^*_{T-t} : $x \to u_t$, computed through equation (2.18) at each time step. This strategy $f^* = (f^*_T, f^*_{T-1}, \ldots, f^*_1)$ yields the maximal probability for the system to be resilient at the horizon T. It can then used to compute the probabilities $\mathbb{P}_{\text{Res}}(f^*,t,x)$ of recovering the ε-robustness of the system's properties by a date $t \leq T$.

The feedback one has to apply to a given state at a given date depend on the distance to the horizon T, and thus, on the horizon itself. Indeed, the probabilities $\mathbb{P}_{\text{Res}}(f^*,t,x)$ are not necessarily the maximal probability of reaching the stochastic viability kernel by t. For that, one should use the optimized feedbacks $(f^*_t, f^*_{t-1}, \ldots, f^*_1)$ for the respective dates $(0,1,\ldots,t)$. These feedbacks would yield $\mathbb{P}_{\text{Res}}(t,x)$ which the maximal probability of being resilient with a time horizon t. Yet, one should keep in mind that between the initial date and t, the strategies $(f^*_t, f^*_{t-1}, \ldots, f^*_1)$ and $(f^*_T, f^*_{T-1}, \ldots, f^*_{T+1-t})$ are in general different, and that only the latter maximizes the probability of being resilient at date t.

The definitions introduced in this section are graphically summarized in Figure 2.3. One can notice differences with the resilience basins one can compute in the deterministic case (Section 2.2.3 and Figure 2.2). Due to the many possible trajectories for the optimal feedback strategy, resilience is not in general a certain property any more. As a consequence, minimizing the time to get back to the viability kernel does not make obvious sense like it does in the deterministic case. Instead, we choose to maximize the probability of getting back at a given horizon instead.

Besides, the probability of resilience is highly dependent on the prior choice of the β and τ. Indeed, the larger β and τ, the smaller the stochastic viability kernel, and the

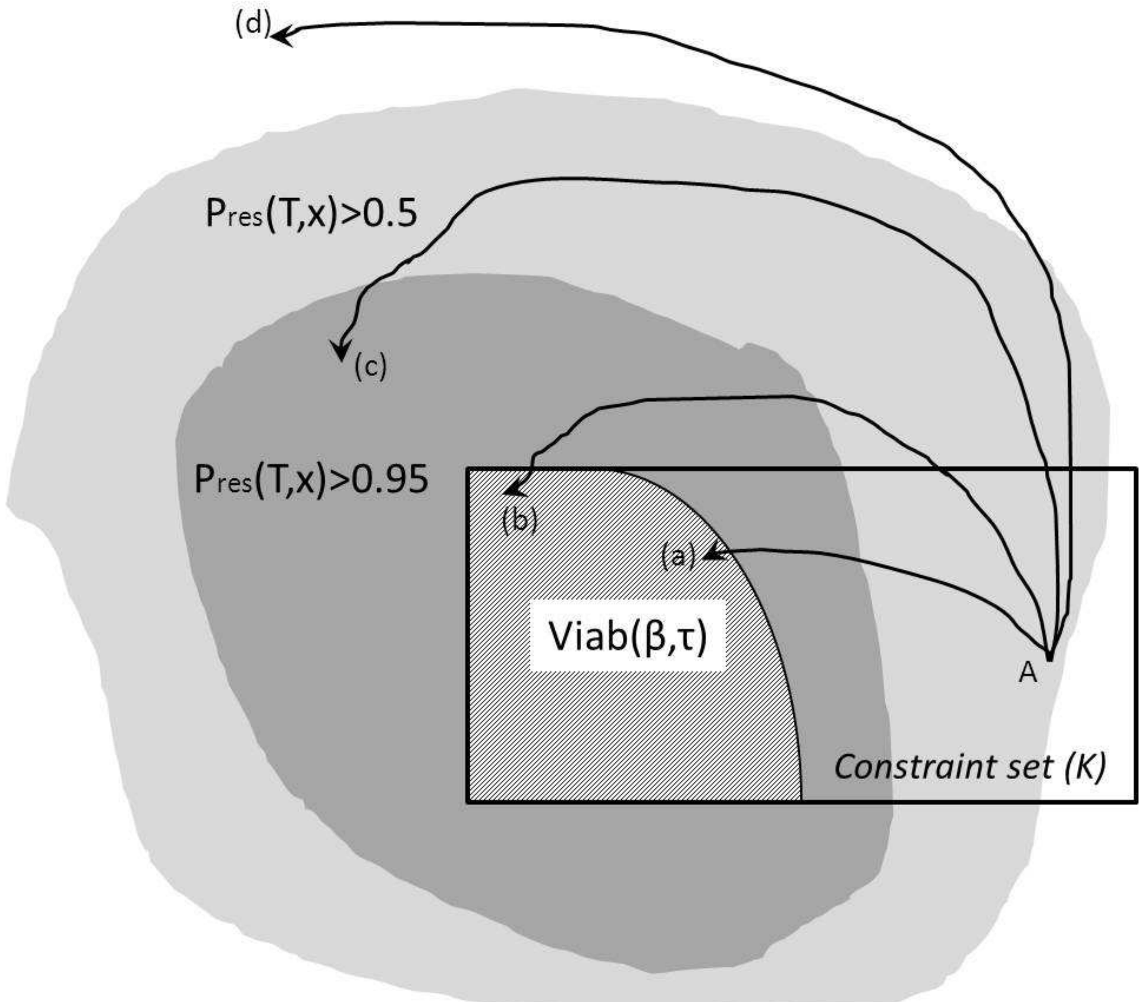

FIGURE 2.3 – Constraint set, stochastic viability kernel and level sets of the probability of resilience given a horizon T. Some possible trajectories starting from the post-perturbation state A are represented : (a) and (b) are resilient trajectories while (c) and (d) are not because they are outside Viab(β,τ) after T time steps.

longer it takes to reach it. Yet, the more draconian the robustness criteria, the better the robustness of the system when it reaches that set, and the more meaningful it is to see resilience as the probability of reaching it. Thus when setting the parameters β and τ, one has to consider a tradeoff between the quality of the resilience and robustness of the properties of the system, and how costly these are to achieve. This means resilience to a large, unpredictable perturbation is defined in tandem with ε-robustness.

Eventually, the optimal feedbacks used to compute the ε-robust states and the resilient ones are not the same in general. They correspond to two distinct problems : the first is to guarantee the properties of the system against "usual" uncertainty, while the second is to make the system robust again following a perturbation. In both cases, the feedbacks are dependent on the time horizon considered.

2.3.4 Resilience-related indicators

Computed for a given T, the quantities $\mathbb{P}_{\text{Res}}(f^*, t, x)$ form a set of indicators that gives the cumulative distribution function of the time needed to get to a set where the desirable properties of the system are guaranteed to a comfortable extent against uncertainty. Yet, measures of resilience are often given through a single performance indicator. Thus, while they represent well what resilience is at the conceptual level, these quantities are not necessarily relevant operational indicators. However, they are a basis for computing virtually any indicator that represents resilience based on the time taken to reach a situation that keeps the properties of interest of a system. Indeed, the pdf of the time taken to reach an ε-robust set $\text{Viab}(\beta,\tau)$ is given by the difference $\big(\mathbb{P}_{\text{Res}}(f^*, t, x) - \mathbb{P}_{\text{Res}}(f^*, t-1, x)\big)$, which is the probability to reach this set exactly at date t.

Index computation depends on whether or not the loss of ε-robustness is reversible with unit probability by the date T. If there are states such that $\mathbb{P}_{\text{Res}}(T, x) < 1$, one can directly relate this quantity to the resilience of the system. Yet further relevant indicators related to resilience exist, especially (but not only) when $\mathbb{P}_{\text{Res}}(T, x) = 1$. One such indicator can be the expected value of the time needed to acquire the resilience property using the strategy f^* :

$$E(t|f^*, x) = \sum_{t=0}^{T} (t+1)\big(\mathbb{P}_{\text{Res}}(f^*, t+1, x) - \mathbb{P}_{\text{Res}}(f^*, t, x)\big) \tag{2.20}$$

It should be noted that f^* is not necessarily the strategy that optimizes the expected entry time, and it is no equivalent to the entry time defined in the deterministic case by equation (2.12). Such an equivalent could be provided by tailoring the considered horizon to the state of the system, still keeping in mind that it means the feedback maps used will then change. For x, one would pick the shortest horizon t_α such that the probability of resilience $\mathbb{P}_{\text{Res}}(t,x)$ guarantees that the ε-robustness of the system at the horizon t_α with a confidence level α. When it exists, t_α can be defined by :

$$t_\alpha(x) = \min_{0\leq t\leq T} \left\{ t | \mathbb{P}_{\text{Res}}(t,x) \geq \alpha \right\} \tag{2.21}$$

and because of equation (2.19), it can be computed using yet again the value function V_s :

$$t_\alpha(x) = \min_{0\leq t\leq T} \{ t | V_s(t,x) \geq \alpha \} \tag{2.22}$$

Both indicators $E(t|f^*,x)$ and $t_\alpha(x)$ can be used in association with the notion of resilience, and a resilience indicator can even be defined as a decreasing function of one of them. Then, this decreasing function is often the inverse function (Hashimoto *et al.*, 1982; Martin, 2004). In a more general way, when resilience can be achieved with a nonzero probability, a useful indicator is the cost incurred while driving the system back to the viability kernel. One can choose whether to associate it to the notion of resilience or to define resilience as a decreasing function of this cost much like in Martin (2004) (e.g. its inverse).

The equations proposed here provide one with various ways to compute resilience and related indicators while explicitly incorporating the potential effects of uncertainty. They are also an illustration of the power of dynamic programming tools when it comes to carrying out resilience computations.

2.4 Application to an uncertain lake model

The lake eutrophication problem can be tackled through simple nonlinear dynamics, and has been well-studied (Carpenter *et al.*, 1999, 2001; Ludwig *et al.*, 2003). Here we choose, for illustrative purposes, to use the deterministic model from Martin (2004) and add uncertainty to it. Resilience computations introduced by Martin (2004) and Deffuant et Gilbert (2011) are outlined, and the added value of considering uncertainty is highlighted. The dynamics and resilience problem are laid out first, before being solved for a given strategy. Then, the interest of using stochastic viability to optimize the control strategy is emphasized.

2.4.1 The problem of lake eutrophication

Using the publications cited above, this section introduces the case of lake eutrophication under its simplest formulation, that is, not considering the mud dynamics. The problem is formulated much like in Section 2.2.1, but introduces uncertainty while introducing the system's dynamic.

Uncertain controlled dynamics

The state variables are the phosphorus concentration p in the lake, and the phosphorus input rate l. In continuous time θ, the evolution of phosphorus concentration reads as follows Martin (2004) :

$$\frac{dp}{d\theta} = -bp + l + r\frac{p^q}{p^q + m^q} \tag{2.23}$$

where the parameter b is the phosphorus sink rate (e.g. the quantity that flows out of the lake), r is the maximal recycling rate by algae, m is the value of p for which the recycling term $(rp^q)/(p^q + m^q)$ is half its maximal value, and q is a dimensionless parameter, set at $q = 8$ in several studies (e.g. Martin, 2004; Guttal et Jayaprakash, 2007).

Through dimensional analysis, the parameters b and m can be eliminated while p, l and r are turned into their respective dimensionless equivalents : the state variables P and L, and the only remaining parameter is R (see Appendix 2.8 for details). θ is also replaced by the dimensionless time t, and we get the new continuous-time equation :

$$\frac{dP}{dt} = -P + L + R\frac{P^8}{P^8 + 1} \tag{2.24}$$

Let us now tackle uncertainty. For the sake of simplicity, only phosphorus input uncertainty will be considered in this example. It represents, for instance, the uncertainty that is due to the soil storage of phosphorus. A discrete-time decomposition (for which the continuous-time equivalent is given in Appendix 2.9.1) between the mean input rate L^* and a deviation is introduced :

$$L_t = L_t^* + \sigma\varepsilon_t \tag{2.25}$$

Here ε is considered i.i.d., and it is modeled by a standard normal distribution $\mathcal{N}(0,1)$, due to the common use of such distributions. σ represents the standard deviation of the noise. Since the phosphorus input L is not updated by the dynamic of equation (2.23), choosing L^* instead of L as a state variable enables the modeler to project the uncertainty over one dimension (P) rather than two (L and P). It therefore reduces the computational complexity of the problem. From now on, the state x will be the couple (L^*, P).

The mean phosphorus input L^* can be modified due to modifications in farmers' behavior, changes in agricultural technology, a combination of both, or other factors. Such modifications take time to take full effect, so that the rate of change in L^* is bounded. One can write :

$$\begin{cases} L^*_{t+1} = L^*_t + u_t \\ u_t \in U(x_t) = [U_{\min}(x_t), U_{\max}(x_t)] \end{cases} \tag{2.26}$$

where for any x_t, $U_{\min}(x_t) \leq 0$ and $U_{\max}(x_t) \geq 0$. In this application, the set of possible controls depends only on the state, and not on time.

The discrete time equation for the evolution of the lake is, based on the derivations from Appendix 2.9.3 :

$$P_{t+1} = g(P_t, L^*_t, u_t, \varepsilon_t) \tag{2.27}$$

We discretize the state space with a resolution $\Delta P = \Delta L^* = 0.01$, so that for a given value of the control it is possible to directly compute the probability for the state to be closest to any given point of the grid at the end of the time step. One can thus describe the evolution of the lake state from any point $x_t = (L^*_t, P_t)$ of the discrete grid to another.

Resilience to "what"?

By considering the post-perturbation state, this framework can allow for computing the resilience to any given perturbation. In practice, a perturbation is any event not taken into account by equation (2.25). Later in this work, the focus will be on sudden, single increases in the value of L^* or P. The former represents an increase in the mean quantity to reach the lake, which can be due to the existence of a massive new phosphorus source, and its amplitude will be noted L^*_{per}. The latter rather represents a one-time arrival of phosphorus into the lake, which triggers an immediate increase in the phosphorus concentration P, and its amplitude will be noted P_{per}.

Management objectives and constraints

Phosphorus inputs are the by-product of the use of fertilizers, which benefit to farmers through an improved land productivity. However, a lake can have two regimes, and phosphorus concentration has been found to trigger a regime shift. Namely, the switch is from the oligotrophic or clear water regime, in which both ecologic and economic benefits from the lake are high, to the eutrophic or turbid water regime in which algae blooms cause oxygen depletion, leading in turn to a so-called dead lake. Therefore, this is an undesirable regime shift, and we can write the set of desirable states, which we note K, as :

$$K = [L^*_{\min}, L^*_{\max}] \times [0, P_{\max}] \tag{2.28}$$

where $L^*_{\min}$ corresponds to the minimum quantity of phosphorus needed for farming to remain economically viable ; $L^*_{\max}$ is the maximum amount of phosphorus that farmers are willing to use ; and $P_{\max}$ is the threshold above which the lake turns eutrophic.

For the rest of this work, bounds for L^* are set at $L^*_{\min} = 0.1$ and $L^*_{\max} = 1$ like in Martin (2004). We assume that the farmers will not accept to modify the value of L^* beyond the prescribed bounds, so that we can define $U_{\min}(x)$ and $U_{\max}(x)$ depending on those bounds and the maximal amplitude M at which L^* can be modified :

$$U_{\min}(x) = \max\{-M, L^*_{\min} - L^*\} \tag{2.29}$$

$$U_{\max}(x) = \min\{M, L^*_{\max} - L^*\} \tag{2.30}$$

Besides,the recycling term $(RP^q)/(P^q+1)$ increases sharply around $P = 1$, and this change in recycling rate characterizes the transition from the oligotrophic to the eutrophic regime, so that 1 is a plausible value for $P_{\max}$ in equation (2.28).

The goal of management is to keep the lake in a state in which it can be maintained in K for a long time, or alternatively, to reach such a state if a perturbation were to occur. Management considers a time horizon T to bring the system back to a state where it can be maintained for a long time. The certainty and uncontrolled case are respectively given by $\sigma = 0$ and by $M = 0$.

Management strategies

To achieve the management objectives, let us introduce management strategies as defined by equations (2.2) through (2.5). We note f the considered strategy, and T the time horizon at which we will consider managing the system :

$$\begin{cases} u_t = f_{T-t}(P_t, L^*_t) \\ u_t \in U(x) \end{cases} \tag{2.31}$$

In practice, one can expect strategies to mainly aim at reducing the mean phosphorus input L^*. This can be seen using the position of the equilibria of equation (2.27) in the classical certain and uncontrolled case ($\sigma = 0$, and $M = 0$, so that $L = L^*$) (Figure

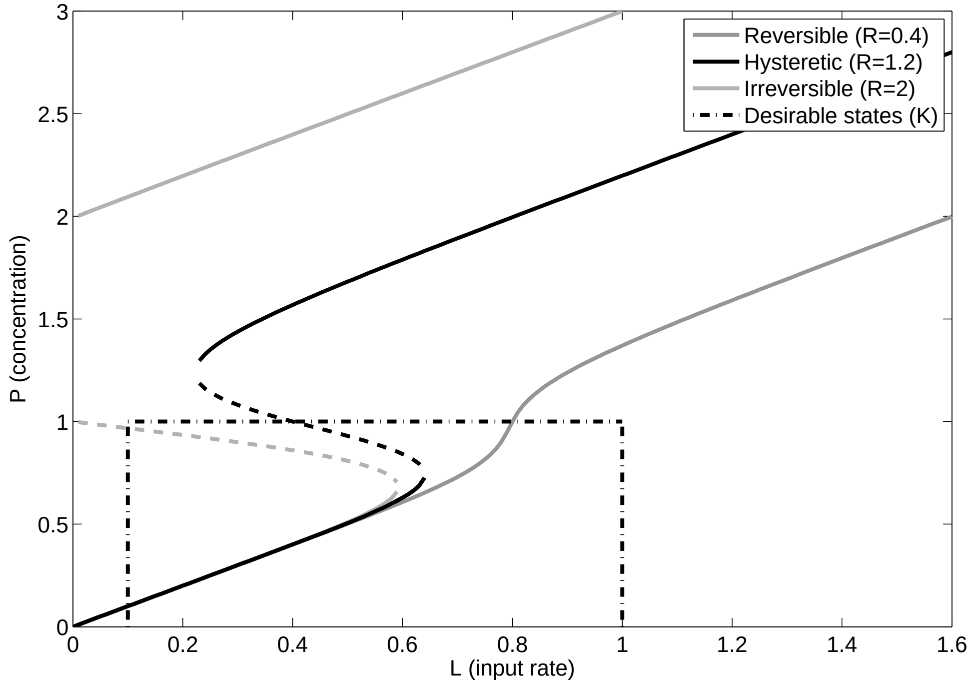

FIGURE 2.4 – Stable (continuous line) and unstable (dashed line) attractors for the lake eutrophication problem.

2.4). Depending on the value of the parameter R, there can be three types of behavior for the lake. These have been well studied in the literature. When R is low, for instance $R = 0.4$, there is only one value of P which is an equilibrium for each value of L, and the lake is called reversible. For higher values, such as $R = 1.2$, there can be alternate state states for the same value of L, making a regime shift much more difficult to reverse. The lake is then called hysteretic. Finally when R is even higher, for instance at 2, the lake can be called irreversible, because any transition towards the eutrophic regime cannot be reversed. The reversible case is only weakly nonlinear, so that the remainder of this paper will focus exclusively on the hysteretic and irreversible cases, with the same respective values of R (1.2 and 2) as in Figure 2.4.

2.4.2 Resilience for a given strategy

In this section, resilience is explored using predefined strategy. The focus will be on 1) strategy $f^{(1)} \equiv 0$ where no management strategy is implemented, and 2) a purely reactive strategy $f^{(2)}$ where an effort is made once the lake has become eutrophic, by reducing the mean phosphorus input as long as the constraint $L^* \geq L^*_{\min}$ is respected :

$$\forall t \in [0, T],\ f^{(2)}_{T-t}(x) = \begin{cases} 0 & \text{if} \quad x \in K \\ U_{\min}(x) & \text{if} \quad x \notin K \end{cases} \tag{2.32}$$

where we set $M = 0.05$. In this paragraph, we also set $R = 1.2$. We will first explore the advantage of using viability kernels rather than attractors to describe the states that are robust to uncertainty (still using the term of ε-uncertainty), then we will give resilience indicators as introduced in Section 2.3 for these two strategies.

Stochastic viability kernels and their advantages

The search for ε-robust states considers the trajectory of the system only as long as it does not exit the set K of desirable states. Since both the strategies considered here yield $f \equiv 0$ within K, we only need to consider strategy $f^{(1)} \equiv 0$, and $f^{(2)}$ will give exactly the same results.

Figure 2.5 shows how a system set at a stable desirable attractor of the (L, P) plane can switch towards an undesirable state in a short time span when uncertainty is present. That had already been noted by Guttal et Jayaprakash (2007). Thus, uncertainty causes a description of the dynamics of the system using stable equilibria and their basins of attraction to become precarious. Switching variables to (L^*, P) does not change this fundamental fact, even though they do allow for the projection of uncertainty over a single state variable instead of two (Figure 2.6). Indeed, while uncertainty does not influence the value of L^*, it does interfere strongly with the position of the equilibria in the (L^*, P) plane. For a system in the same initial state as in Figure 2.5, the 95% confidence interval (CI) associated with its state after 5 time steps (Figure 2.6) shows that the oligotrophic attractor only describes the dynamic very poorly. Using equilibria that change at every time step is very unpractical, and moreover, one can perceive that the system is very unlikely to be on a stable equilibrium at any date.

Since it situates a system with respect to a set rather than with respect to a point, viability theory is well-suited to describe out-of-equilibrium situations (Martin, 2004; Deffuant et Gilbert, 2011). One can easily represent different degrees of ε-robustness of the system by computing stochastic viability kernels $\mathrm{Viab}(\beta, \tau)$ for different values of β and τ (Figure 2.7). Equilibria for the mean case $\varepsilon_t = 0$ are still shown for an easier understanding. The limit case $\sigma = 0$ is also represented : it corresponds to the deterministic viability kernel.

As expected, for a given value of β and σ, the higher τ, the smaller the size of the kernel. Likewise, for (β, τ), the higher the scale σ of the uncertainty, the smaller the

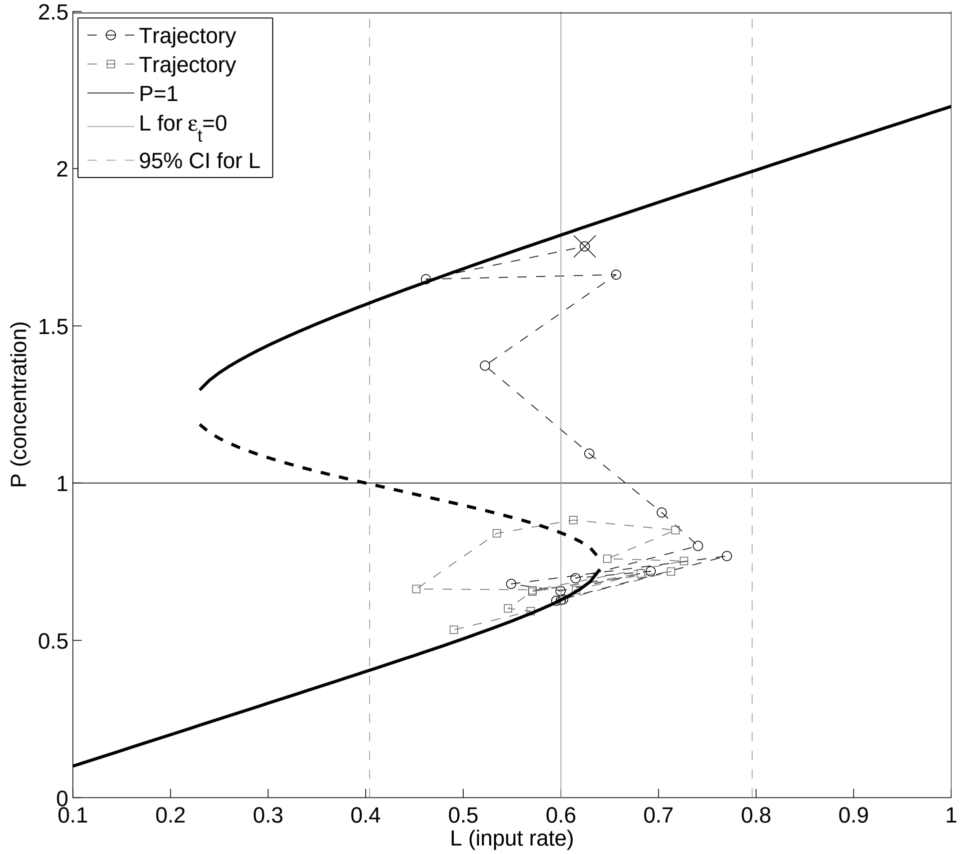

FIGURE 2.5 – Two trajectories simulated in the (L, P) plane using $\sigma = 0.1$ and with a constant value of $L^* = 0.63$. Both start at $x_0 = (0.6, 0.63)$, which is an oligotrophic stable equilibrium. The dates are $t = 0, \ldots, 15$, and the final date is signaled with a cross.

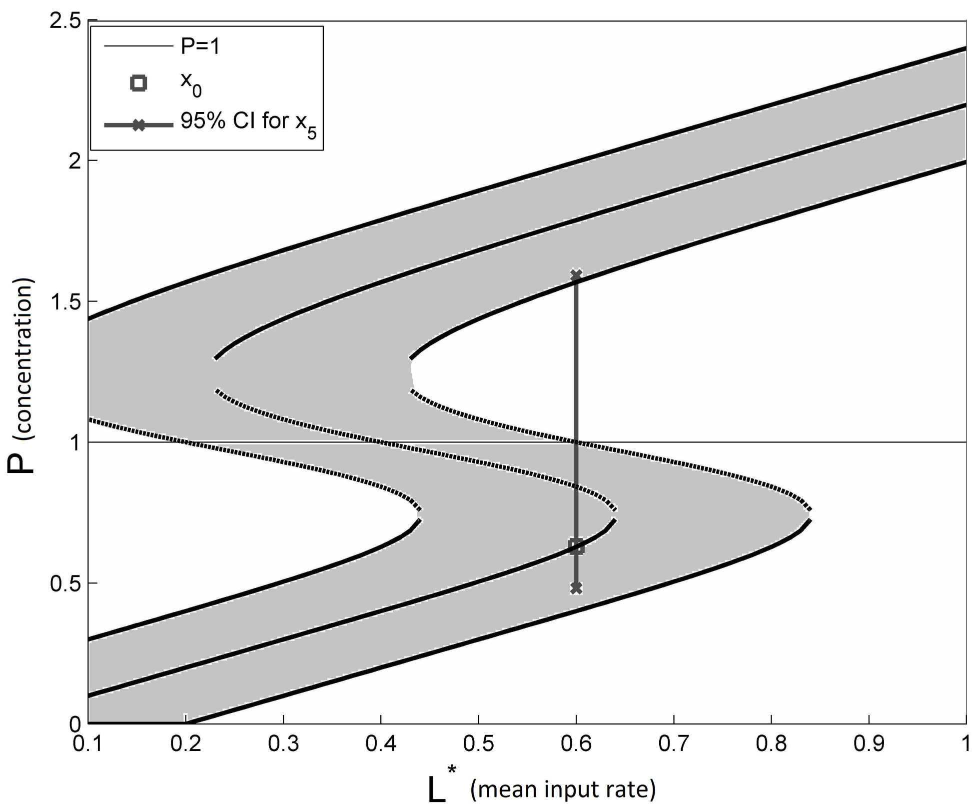

FIGURE 2.6 – The 95% confidence interval (CI) for the position of the attractors is represented in grey in the case $\sigma = 0.1$, as well as the 95% CI for the state x_5 after 5 times steps, if the initial state is $x_0 = (0.6, 0.63)$.

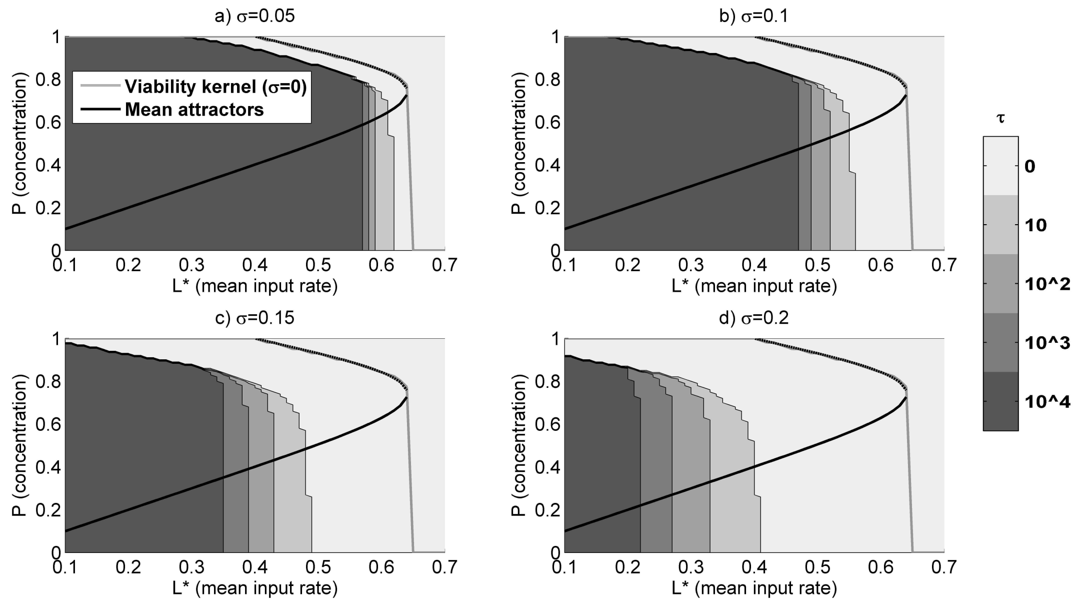

FIGURE 2.7 – Stochastic viability kernels for $\beta = 0.99$ and different values of σ and τ, reminding that for $\tau_1 < \tau_2$, $\mathrm{Viab}(\beta, \tau_2)$ is a subset of $\mathrm{Viab}(\beta, \tau_1)$. For reference the viability kernel ($\sigma = 0$) is on the left side of the grey line.

size of $\mathrm{Viab}(\beta, \tau)$. The same result holds when β increases for given values of β and σ. Besides, the size decrease caused by an increase in τ is magnified by higher values of σ. Figure 2.7 shows that for a hysteretic lake, one can get nonempty stochastic viability kernels $\mathrm{Viab}(\beta, \tau)$ with β close to one, e.g. $\beta = 0.99$, and a time horizon much more important than that of decision-making, e.g. $\tau = 10^3$. The set $\mathrm{Viab}(0.99, 10^3)$ will be used from now on to describe the ε-robust states. We also set $\sigma = 0.1$.

Resilience

Let us now assume a management horizon $T = 30$ ($T << 10^3$) and look at the possibility to reach a ε-robust state after a perturbation with either strategy $f^{(1)}$ or $f^{(2)}$. We solely look at the post-perturbation state of the system. The null strategy $f^{(1)}$ lets L^* stay constant over time, so that reaching $\mathrm{Viab}(0.99, 10^3)$ is only possible if for a given L^*, there are values of P which are ε-robust. Hence, outside the stochastic viability kernel, resilience is only possible for a small portion of the state space (Figure 2.8).

For strategy $f^{(2)}$ however, there is a possibility to control the system when the constraints are violated, and a possibility for the system to get to a ε-robust state once L^* gets very low. Hence, with $M = 0.05$, most post-perturbations states are resilient with unit probability at the horizon $T = 30$ (Figure 2.9), highlighting the importance of management actions. The only exception concerns states which are not ε-robust if

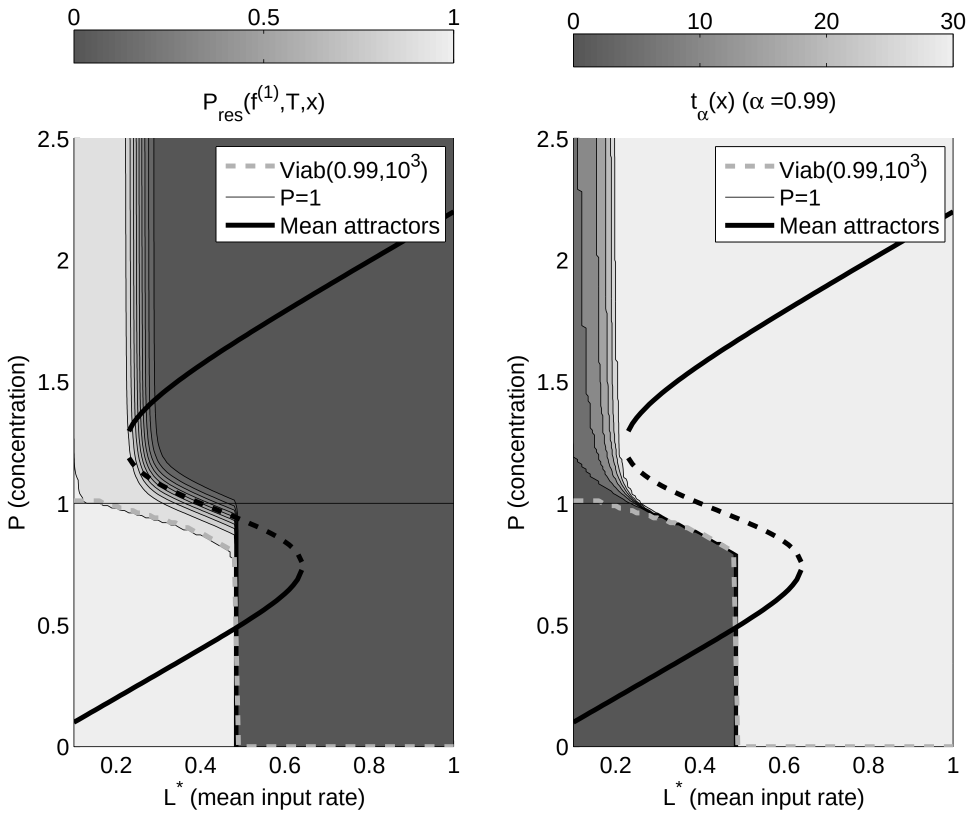

FIGURE 2.8 – Probability of resilience at $T = 30$ and entry time t_α for strategy $f^{(1)}$. By default, $t_\alpha(x) = 30$ when the probability of resilience is smaller than 0.99 over the management horizon.

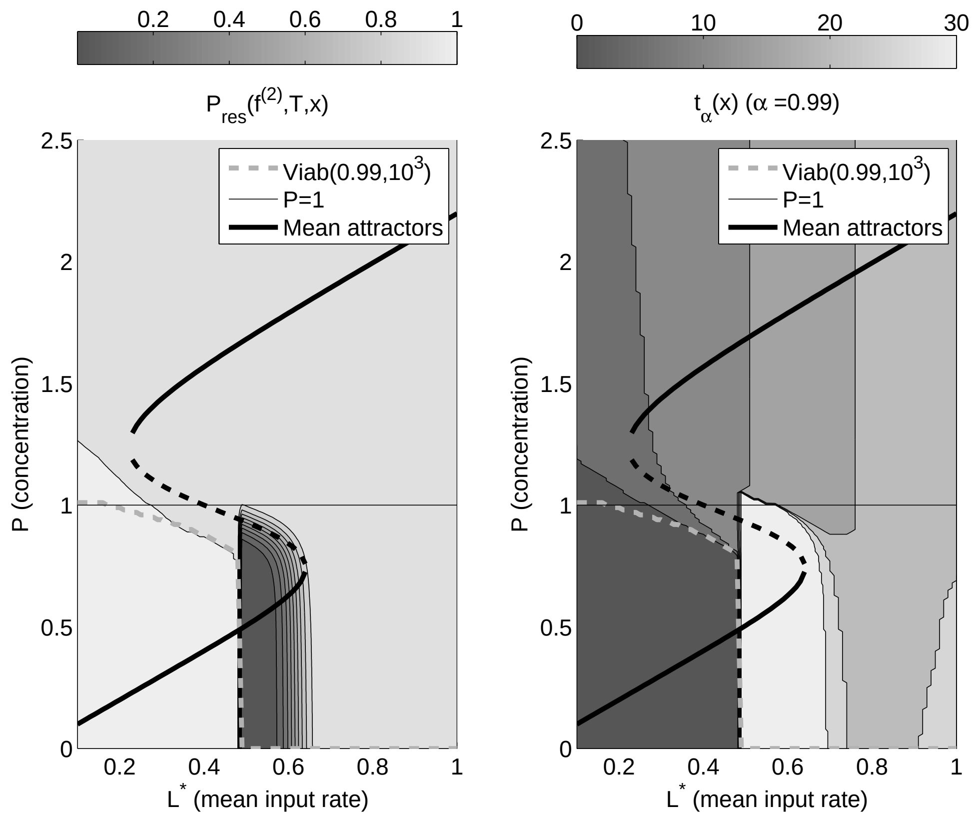

FIGURE 2.9 – Probability of resilience at $T = 30$ and entry time t_α for strategy $f^{(2)}$. By default, $t_\alpha(x) = 30$ when the probability of resilience is smaller than 0.99 over the management horizon.

we consider $\beta = 0.99$ and $\tau = 10^3$, but for which the probability of a switch towards a eutrophic lake is relatively slim. Strategy $f^{(2)}$ then leaves the system in a precarious state until that switch happens. One can intuitively sense that in such cases, a strategy that would take action and reduce L^* before the lake becomes eutrophic would be more environmentally efficient. In fact, as described in Section 2.3, viability theory allows for searching the best strategy available.

2.4.3 Resilience with an optimal management strategy

Let us now fully exploit a major advantage of viability theory : the possibility to select the policy choices that maximize the probability of resilience at the horizon T. Through this whole section, we will have $\sigma = 0.1$ and $M = 0.05$, and the lake will be considered hysteretic $R = 1.2$ unless mentioned otherwise.

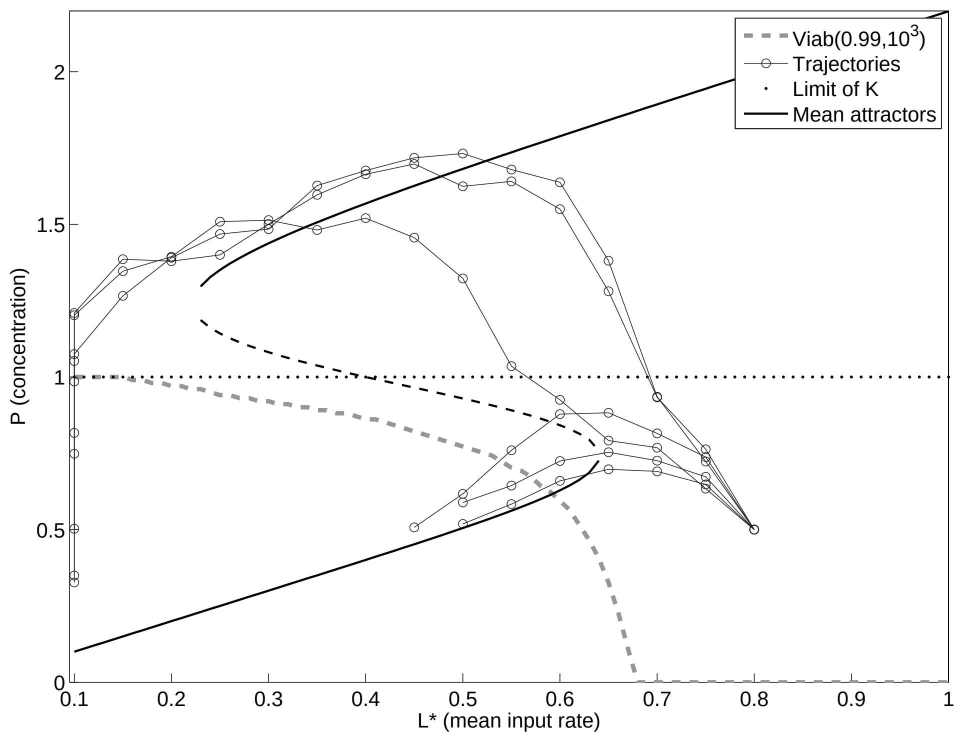

FIGURE 2.10 – Six trajectories towards the ε-robust set in the (L^*, P) plane, simulated with $\sigma = 0.1$, $M = 0.05$ and a post-perturbation state $(0.8, 0.5)$.

Implications for stochastic viability kernels

Thanks to the work by Doyen et De Lara (2010), feedbacks inside K can be computed by dynamic programming so as to maximize the probability of keeping the lake in a desirable state. Whatever the value of τ and for all three values of R considered, the method yields the same optimal feedback map at all dates, and it is noted f^* :

$$f^*(x) = U_{\min}(x) \tag{2.33}$$

Yet again, we choose to describe ε-robustness using $(\beta, \tau) = (0.99, 10^3)$. As expected, the associated viability kernel is much bigger than with strategies $f^{(1)}$ or $f^{(2)}$, as evidenced on Figure 2.10. It in fact encompasses states that we deemed "precarious" when using strategy $f^{(2)}$, so that the optimal strategy makes them "robust" to measurable uncertainty.

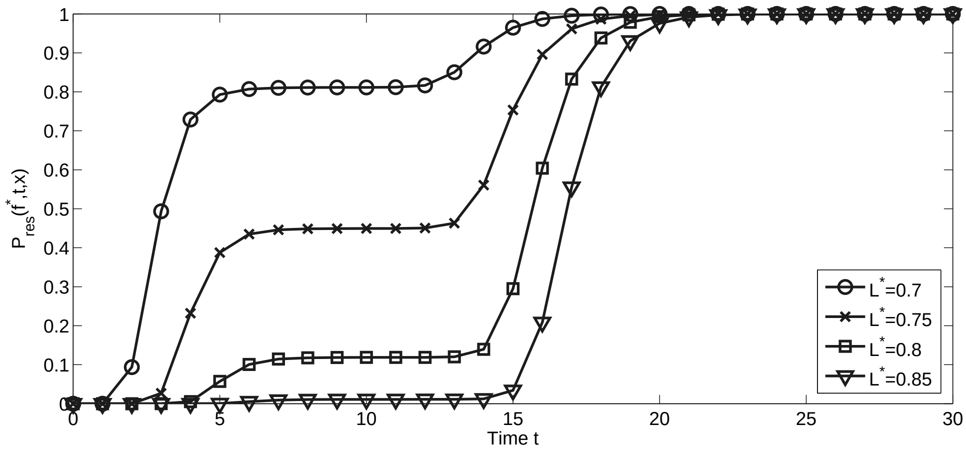

FIGURE 2.11 – Cumulative probability of reaching an ε-stable state in t time steps, for different post-perturbation values of the mean phosphorus input L^*, and a phosphorus concentration $P = 0.5$.

Implications for indicator computation

Now that we got the stochastic viability kernel $\text{Viab}(\beta, \tau)$ for the optimal strategy of 2.33, dynamic programming is used a second time to get the probability of resilience. This time it is carried out over a set that includes all the states trajectories can visit before getting back to $\text{Viab}(\beta, \tau)$, and a satisfactory set is $(L^*, P) = [0.1, 1] \times [0, 4]$. The proof from Appendix 2.7 ensures that we find the strategy that maximizes the probability of resilience, and yet again, the solution described through equation (2.33) is found to apply for all the considered states, and whatever the date and time horizon. Nevertheless, through this section, recall that only $T = 30$ will be used.

If the lake is a hysteretic one, its state is eventually going to become ε-robust after L^* has been brought down to L^*_{min}. This is suggested by Figure 2.10, which showcases two possibilities for a system that has been deprived of its ε-robustness by a perturbation. It can enter the stochastic viability kernel after a few time steps, or be attracted towards the eutrophic regime. In the latter case, the lake will remain eutrophic for several time steps, so that ε-robustness can only be restored after a much longer time. Thus, the cumulative probability of reaching $\text{Viab}(0.99, 10^3)$ shifts from 0 to 1 in two distinct phases, the first one taking place in the first few time steps and five to six time steps before the second starts (Figure 2.11). Besides, this figure showcases how the probability of reaching that set in a few time steps dramatically decreases when the post-perturbation state gets farther from its boundary.

Yet for all the considered states, ε-robustness is achieved with unit probability by the horizon $T = 30$, so that the system is resilient when this horizon is considered. One can then choose to use indicators such as those defined in Section 2.3.4. They can be

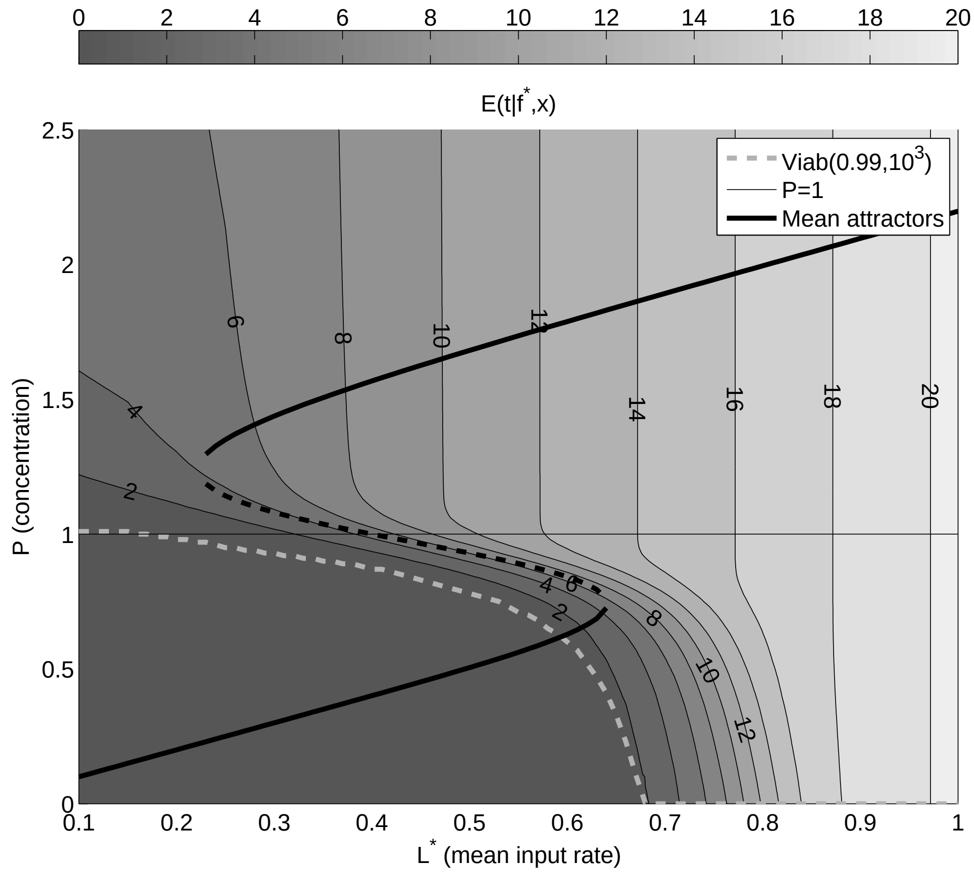

FIGURE 2.12 – The resilience-related indicator $E(t|f^*,x)$ for a hysteretic lake ($R=1.2$) with $M=0.05$, $\sigma=0.1$ and $T=30$.

for instance the mean duration as computed in (2.20) (Figure 2.12), or if a worst-case approach is taken, the maximal time needed to become ε-robust with 99% confidence like in equation (2.21) (Figure 2.13). Unsurprisingly, both indicators quickly increase as the starting state gets further from the boundary of the stochastic viability kernel, but the increase of the mean time $E(t|f^*,x)$ is less abrupt than that of the maximal $t_\alpha(x)$. This shows that the maximal time needed to get back to a ε-robust set can top 20 time steps when the trajectory involves leaving the set of desirable states and getting into the eutrophic regime. Thus, integrating potential management actions and uncertainty into the model confirms the fact that restoration of the lake's robustness can be a long and costly process in the hysteretic case.

From the maps of Figures 2.12 and 2.13, which give the values of indicators for post-perturbation states, one can deduce the impact of different perturbations for a given initial state. In this case study, the perturbation is applied to the system as a one-time jump in the value of one of its state variables. The impact of perturbation

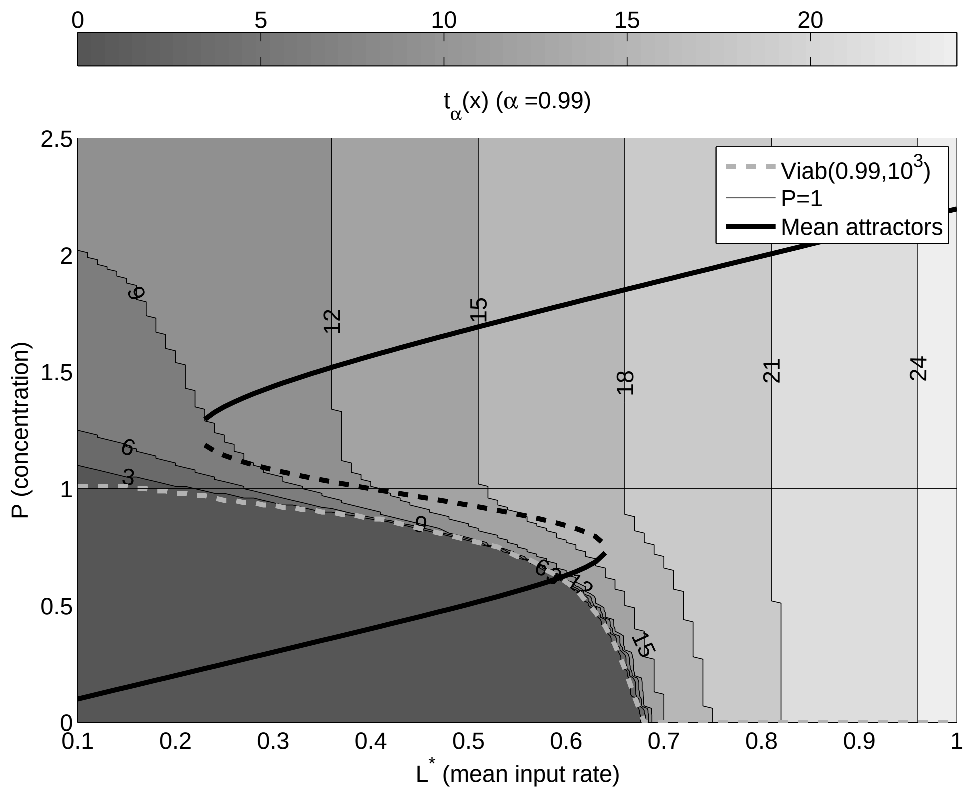

FIGURE 2.13 – The resilience-related indicator $t_\alpha(x)$ ($\alpha = 0.99$) for a hysteretic lake ($R = 1.2$) with $M = 0.05$, $\sigma = 0.1$ and $T = 30$.

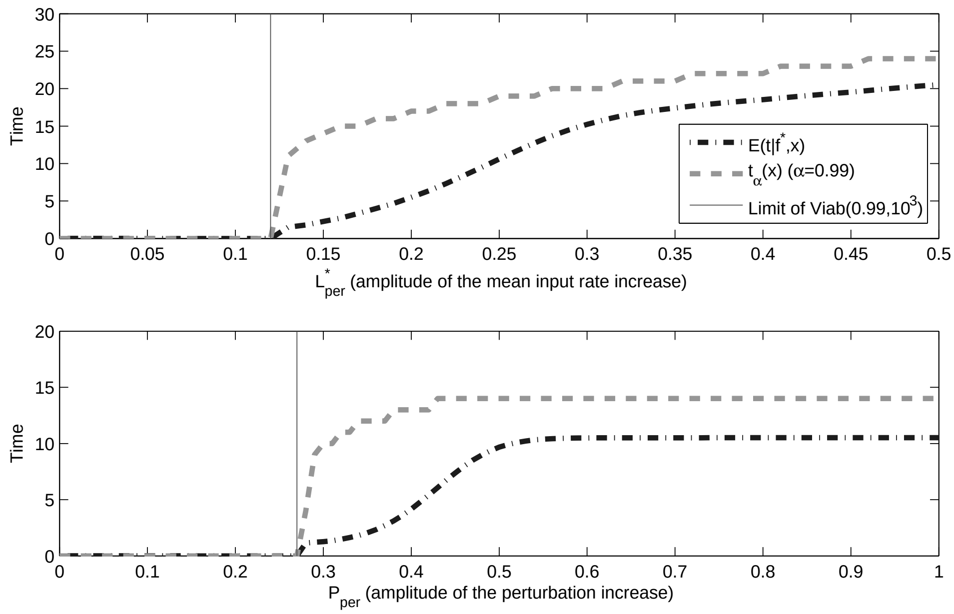

FIGURE 2.14 – Impact of the amplitude of a perturbation on resilience-related indicators for a hysteretic lake, if the system is at $(0.5, 0.5)$ before the perturbation.

amplitude on different indicators can then be observed through curves such as Figure 2.14, drawn for a system at $(0.5, 0.5)$ before the perturbation. The boundary of the stochastic viability kernel is a threshold beyond which the perturbation can affect the recovery time of the system, which illustrates the interest of keeping its state as far inside this boundary as possible.

Resilience for the irreversible case ($R = 2$) is worth investigating, all other variables keeping the same values. By definition, the system may get stuck in a eutrophic state with no possibility of switching back to the oligotrophic state. Recovery, and therefore resilience, can not guaranteed at any T, which hampers the relevance of using the same indicators as in the hysteretic case. Yet, the probability of resilience $\mathbb{P}_{max}$ keeps being relevant, and can be computed for $T = 30$ (Figure 2.15). The feedback map keeps being f^* at each time step, and the horizon is long enough for the system to either recover or switch for good towards an undesirable regime. Like in Figure 2.14, one can evaluate the impact of a given perturbation at any given pre-perturbation state, using this time the probability of resilience (Figure 2.16). This illustrates how the proposed framework can foster the use of different yet relevant indicators in distinct situations.

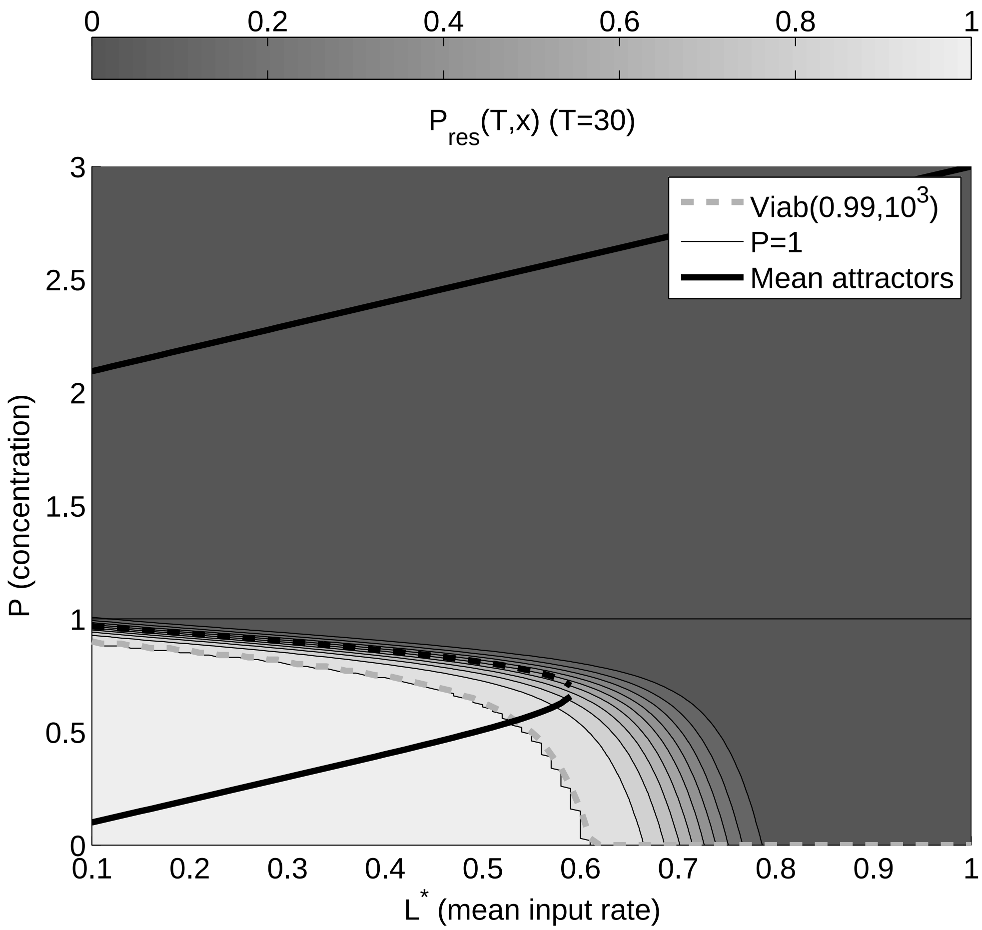

FIGURE 2.15 – $\mathbb{P}_{\text{Res}}(T, x)$ for an irreversible lake ($R = 2$), and $T = 30$.

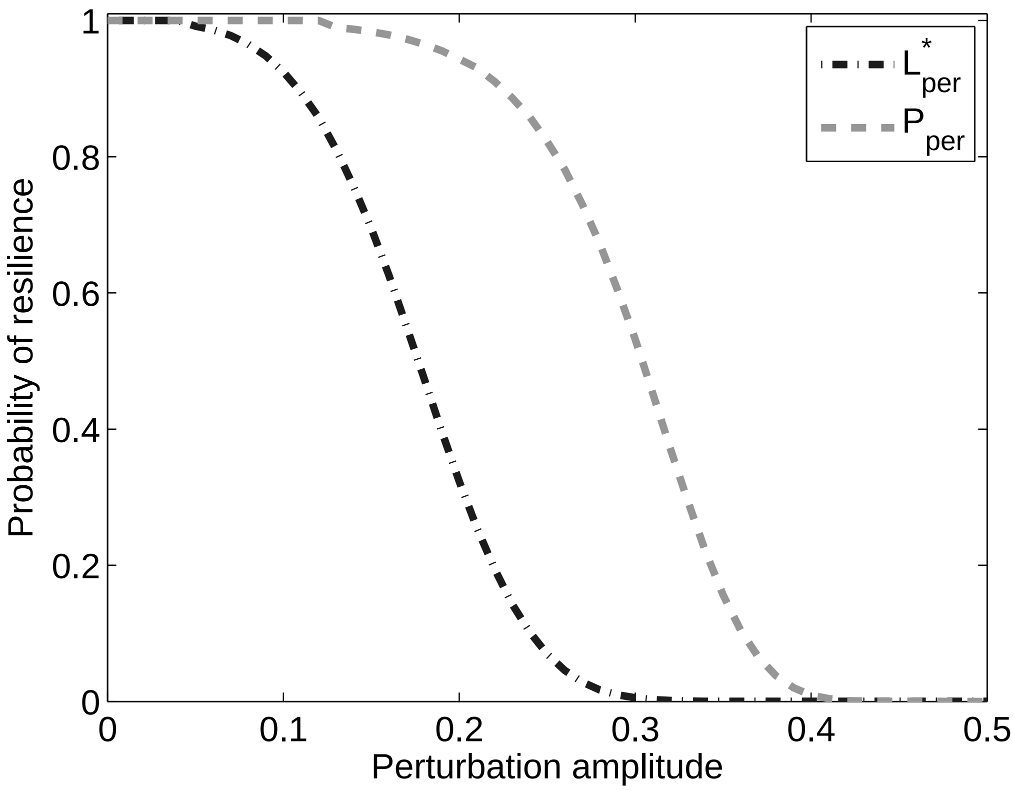

FIGURE 2.16 – For an irreversible lake which state is (0.5, 0.5) before the perturbation, possible impacts of a perturbation on the probability of resilience.

2.5 Discussion

In the stochastic viability framework, the policy is only optimal with respect to the horizon T. When the feedbacks f^* are computed for a given horizon T, the probabilities of resilience computed with f^* for $t < T$ are not necessarily the maximum probability of reaching $\text{Viab}(\beta, \tau)$ after t time steps. The policy is also only optimal for the objective of maximizing the probability of resilience. In the work by Martin (2004), the most resilient trajectory is the one that besides reaching the viability kernel in due time, also minimizes a cost function. In the stochastic case, there are many possible trajectories, some of which may not reach the viability kernel. Thus, minimizing the cost on the trajectory may sometimes conflict with maximizing the probability to reach the viability kernel, and one should define a trade-off between these objectives. Viability theory allows for the exploration of trade-offs in the latter case under the so-called framework of co-viability (Doyen *et al.*, 2012), but without addressing the issue of resilience. Exploring the integration of resilience computations in a co-viability framework could be a natural and useful continuation of this work.

However, in the case of lake eutrophication, the controls are independent on the horizon, so that the feedback map is the same at every date. This means that changing the management horizon will not affect the immediate decisions to be taken. Besides, the map is the same whether ones aims at optimizing the resilience of the system or its ε-robustness. Such a simplification is due to the fact that one always has interest in reducing L^*. Many other systems studied through viability theory in the deterministic case have the same characteristic : controls allow for changing a state variable which increase, or decrease, always leads to enhancing the resilience of the system (see examples throughout Deffuant et Gilbert (2011)). Ecological systems where the same feedback map applies whatever the horizon might in fact be common.

Yet, one should keep in mind that the model exposed in this paper is only a theoretical toy. It is a well-known benchmark model of a nonlinear ecological system, and can be used to explore some aspects of such systems. One of its advantages is that of being two-dimensional, which makes its dynamic easier to understand (Figures 2.5 and 2.10) and thus to showcase concepts and methodological developments. However, this model is not meant to be used for assessing resilience of actual lakes, for a number of reasons, some of which can be briefly described here. First, it accounts for mud dynamics only implicitly, through the recycling term $P^q/(1 + P^q)$ of equation (2.24). This term governs the transition from an oligotrophic to a eutrophic state, so that mud dynamics should be explicitly accounted for in any model aimed at being used for decision-making. Second, available policies generally have an unknown impact, so that uncertainty on the control should be included in the model, and they are also not constant in time. For instance, one can expect a policy response to not have immediate effects, and such delays are not taken into account in our hypothetical model. Third, the economic interests of farmers are only reflected by the existence of a bound $L^*_{\min} > 0$ to K, and the optimal policy involves reducing L^* until it reaches that value. Under such a formulation, $L^*_{\min}$ becomes the quantity of phosphorus farmers usually discharge into the lake, even though it may not be acceptable for them to maintain their use at that level for a prolonged period of time.

Last but not least, dynamic programming algorithms work in theory for any system described by equation (2.14), which would make this method very general. Its area of application, however, is limited in practice by the curse of dimensionality, which causes both the required computational time and memory to increase exponentially with the dimension of the state space. Yet, the notion of stochastic viability kernel, and that of resilience of a system based on a policy, are not defined through dynamic programming, which is merely a good way to compute them. It could be very interesting to explore in future work how they can be generalized to systems of higher dimensionality, even if such developments may render the computations more approximate.

2.6 Conclusions

This work presented a framework for defining and computing resilience of stochastic controlled dynamical systems, using stochastic viability kernels and dynamic programming. This allows for the incorporation of uncertainty so that computations can explicitly account for the capacity of a system to cope with adverse events. The two main advantages of the viability framework of resilience prove all their utility in the uncertain case. On the one hand, attractors cease to be fixed points when the dynamics is stochastic, yet viability tools such as stochastic viability kernels can still be defined. On the other hand, the possibility to use this framework to dynamically optimize the management policies while computing the resilience remains in the stochastic case.

The presented framework does not claim to give a single, standard formula for computing resilience, because what can be computed in practice is only a resilience indicator. Rather, it proposes a set of possibilities for deriving general indicators that can fit different applicative contexts. The case of the lake illustrates how this framework allows for the computation of different indicators for resilience depending on the value of the lake parameter R. In the end, the indicators given in this work are generic examples : they may not be the most relevant one in a given case, but they can be built upon to define such a relevant indicator.

2.7 Appendix : Relationship between V_d, V_s and resilience

In the work by Doyen et De Lara (2010), given a time horizon T, one is interested in keeping a system described by uncertain dynamics such as g of equation (2.14) in a constraint set $A(t)$ for all dates $t < T$. For simplicity, we also suppose that $A(t)$ represents a discretezed set of points. A dynamic programming algorithm is proposed and uses the following value function :

$$\begin{cases} V(T,x) = \begin{cases} 1 & \text{if} \quad x \in A(T) \\ 0 & \text{if} \quad x \notin A(T) \end{cases} \\ \forall t \in [0, T-1],\ V(t,x) = \max_{u_t \in U(x)} \left(\sum_{y \in A(t)} \mathbb{P}(g_\varepsilon(x,u_t) = y)\ V(t+1,y) \right) \end{cases} \tag{2.34}$$

The main result of their work is that this value function allows for finding the feedbacks that maximizes the odds of the system to remain in $A(t)$ at all dates $t < T$. In particular, setting $T = \tau$ and $A(t) = K$ at all dates yields the value function used to compute stochastic viability kernels. $x \in$Viab(β, τ) is then equivalent to $V(0,x) > \beta$.

Equation (2.34) can equivalently be written by considering the horizon $T - t$ instead of the date t :

$$\begin{cases} V(0,x) = \begin{cases} 1 & \text{if} \quad x \in A(0) \\ 0 & \text{if} \quad x \notin A(0) \end{cases} \\ \forall t \in [1, T],\ V(t,x) = \max_{f_t(x) \in U(x,T-t)} \left(\sum_{y \in A(t)} \mathbb{P}(g_\varepsilon(x,f_t(x)) = y)\ V(t-1,y) \right) \end{cases} \tag{2.35}$$

Setting $A(0)$ as the viability kernel and $A(t)$ as the entire state space for any horizon $t > 0$, and placing ourselves in the case where no uncertainty is present, V becomes the value function V_d. Then V_d is the optimal probability of the problem of reaching the viability kernel by T, and the set of states x such that $V_d(T,x) = 1$ is the set of states which are solution to this problem, which corresponds to the definition of the resilience basin Res(T). The same applies for any $t \leq T$.

Likewise for V_s, one can set $A(0) =$ Viab(β, τ) and $A(t)$ as the entire state space for $t < T$ in equation (2.35). Furthermore, to express that we are not interested in this computation by what might happen after the recovery is complete and the system reaches the viability kernel, we need to apply the results from Doyen et De Lara (2010) to modified dynamic in which all the points within Viab(β, τ) are fixed whatever the control and uncertainty ($\forall u, \varepsilon, g(x,u,\varepsilon) = x$). Then, V_s is the optimal probability of the problem of being in Viab(β, τ) at date T under the modified dynamics, or to reach it before T under the original dynamics. Hence, for any state x, $\mathbb{P}_{\text{Res}}(T,x) = V_s(T,x)$ because both entail optimizing the same probability over the same time frame.

2.8 Appendix : Dimensional analysis of the lake problem

Recall the continuous time equation (2.23) :

$$\frac{dp}{d\theta} = -b\,p + l + r\frac{p^8}{p^8 + m^8} \tag{2.36}$$

We want to reduce the number of parameters by introducing the following dimensionless variables and parameters :

$$P = \frac{p}{m} \qquad L = \frac{l}{b\,m} \qquad R = \frac{r}{b\,m} \qquad dt = b\,d\theta \tag{2.37}$$

so that dividing equation (2.36) by $(b\,m)$ yields equation (2.24), effectively reducing the number of parameters from three to one.

2.9 Appendix : Derivation of the discrete time equation (2.27)

We start from the continuous time equation (2.24), integrate uncertainty and controls before deriving the discrete time equivalent. The amount of time between dates t and $t+1$ is noted Δt, and is supposed constant between any pair of consecutive dates.

2.9.1 Uncertainty

The continuous time equivalent for equation (2.25) is :

$$L(t) = L^*(t) + \omega(t) \tag{2.38}$$

where ω is a brownian motion such that its pdf after a time interval Δt is $\mathcal{N}(0,\sigma)$.

2.9.2 Controls

In the continuous time formulation of Martin (2004), the introduction of a limited capacity of action in equation (2.26) is mathematically translated into the possibility to choose the temporal derivative of L^* within a bounded range, noted U_c in continuous time. This can be written as :

$$\begin{cases} \frac{dL^*}{dt} = L^* + u \\ u \in U_c(x) = \left[\frac{U_c^{\min}(x)}{\Delta t}, \frac{U_c^{\max}(x)}{\Delta t}\right] \end{cases} \tag{2.39}$$

2.9.3 Derivation of the discrete-time evolution equation

Integrating uncertainty, equation (2.24) becomes :

$$\frac{dP}{dt} = -P + L^* + \omega + R\frac{P^8}{p^8 + m^8} \tag{2.40}$$

Let us introduce δt such that $\Delta t = j\delta t$ with $j \in \mathbb{N}$, and use the smaller δt for the Euler approximation of equation (2.24) so as to minimize the computational error. For any $k \in [0, j-1]$ and a given value of u, we get :

$$\begin{cases} P_{t+(k+1)\delta t} = P_{t+k\delta t} + \delta t\left(-P_{t+k\delta t} + L^*_{t+k\delta t} + \frac{\sigma}{\Delta t}\varepsilon_t + R\frac{P^8_{t+k\delta t}}{P^8_{t+k\delta t} + 1}\right) \\ L^*_{t+(k+1)\delta t} = L^*_{t+k\delta t} + u\frac{\delta t}{\Delta t} \end{cases} \tag{2.41}$$

and the latter equation can be iterated j times over to get equation 2.27 :

$$P_{t+1} = g(P_t, L_t^*, u_t, \varepsilon_t) \tag{2.42}$$

where the parameters R and σ are made implicit. Throughout this paper we used $j = 10$.

– 3 –

Stochastic viability concepts and dynamic programming methods for controlled time-variant reliability problems

Soumis dans *Reliability Engineering and System Safety*.
Auteurs : Charles ROUGÉ, Jean-Denis MATHIAS et Guillaume DEFFUANT

The goal of this paper is twofold : (1) to show that stochastic viability and time-variant reliability address similar problems with different points of view, and (2) to demonstrate the relevance of concepts and methods from stochastic viability in reliability problems. On the one hand, reliability aims at evaluating the probability of failure of a system subjected to uncertainty and stochasticity. Most research done in the development of reliability concepts has been related to find good approximations of this probability in increasingly complex cases. On the other hand, viability aims at maintaining a controlled dynamical system within a survival set. When the dynamical system is stochastic, a viability problem is a controlled time-variant reliability problem. We show that dynamic programming, which is used for solving stochastic viability problems, can thus yield the control strategy which guarantees reliability at a significance level α and the set of states for which there exists such a strategy. Besides, it leads to a straightforward computation of the date of the first out-crossing, informing on when the system is most likely to fail. We illustrate this approach with a simple example of population dynamics, including a case where load increases with time.

3.1 Introduction

This paper connects two lines of research, viability and reliability, that have ignored each other up to now despite strong similarities. Both frameworks study whether a system will retain desirable properties over a given amount of time. They were developed in different contexts and sometimes tackle different specific technical or conceptual issues in relation with the same type of problems, which makes their confrontation promising. In particular, this work focuses on showing how concepts and methods coming from the stochastic viability framework can be applied to time-variant reliability so as to extend its applicability to a class of problems where reliability is dynamically dependent on the decisions taken over time to influence system performance. The rest of this introduction presents in more detail how both frameworks deal with systems that are either in a survival state (noted $S(t)$) or in a failure state (noted $F(t)$).

The viability approach (Aubin, 1991) deals with controlled dynamic systems under state constraints, which can still be noted $S(t)$. An emphasis is put on finding the viability kernel, the set of all states which can be controlled so that their trajectory stays in $S(t)$ at all times. This theory has first been developed in controlled deterministic systems and the algorithms generally yield both the viability kernel and the associated viable controls at once (e.g. Saint-Pierre, 1994). Viability tools have been successfully applied to a variety of fields such as finance, robotics, or ecology (Deffuant et Gilbert, 2011). Recent work has extended the framework of viability theory in discrete time by considering uncertainties in the dynamics, leading to the definition of the stochastic viability kernel (De Lara et Doyen, 2008), a set of states for which the respect of the constraints can be guaranteed with a desired minimal probability and for a desired time frame. Dynamic programming can compute stochastic viability kernels and determine the control strategy that maximizes the probability to stay in $S(t)$ during that period, as demonstrated in Doyen et De Lara (2010). This is the specific development which applicability to reliability we propose to demonstrate throughout this work.

Reliability theory initially comes from the field of mechanical and structural engineering (Rackwitz, 2001) and has a wide range of applications, from material science (Mathias et Lemaire, 2012) and industrial maintenance (Rausand, 1998) to ecology (Naeem, 1998), environmental management (Aliev et Kartvelishvili, 1993) and hydrology (Melching, 1992). In these applications, different numerical methods enable the estimation of the response surface and the associated probability of a system to reach the failure set $F(t)$. A large body of research has been done to provide ever-improving approximations of this probability in cases of growing complexity. For instance, while the Monte Carlo method is used when the complexity of the model does not lead to disproportionate computation times, methods such as the First (respectively Second) Order Reliability Method (FORM, respectively SORM) can be used in the case of more computationally demanding systems. These methods have been perfected and tailored to an increasing number of applications (Rackwitz, 2001). They approximate the failure surface, which separates $S(t)$ from $F(t)$. However, most of these studies deal with time-invariant systems, since they are carried out under a single definite period of time or while taking time at a parameter. When the system under consideration evolves in time, the reliability problem is referred to as time-variant.

Yet, even for explicitly time-variant systems, there exist many cases where the problem is treated exactly as if it were time-invariant. This is the case for instance when system performance decays monotonously in time (Andrieu-Renaud *et al.*, 2004), or if time is used as yet another random variable in a time-invariant problem when reliability is used to estimate pollutant transport in a heterogeneous aquifer (Skaggs et Barry, 1997). Other applications decompose a time-variant problem into a series of related time-invariant ones, by using bayesian analysis to update reliability estimates (Hsiao *et al.*, 2008) or by considering the different time steps as parts of a component reliability problem (Oviedo-Salcedo, 2012). In other problems where the above solutions are not suitable, methods are based on the computation and time integration of the out-crossing rate, i.e. the rate at which the state may go through the limit state surface (e.g. Li et Der Kiureghian, 1995). Even then, some algorithms exist that follow a decomposition of the time-variant problem into a series of time-invariant ones (Hagen et Tvedt, 1991), leading to methodologies such as PHI2 (Andrieu-Renaud *et al.*, 2004; Sudret, 2008). The challenge is then to propagate these time-invariant approximations through time. This work shall show how another decomposition into simpler problems, fostered by stochastic viability and dynamic programming, can help meet these challenges. Furthermore, it will be demonstrated to dynamically find the decisions which can optimize system reliability during its lifetime.

In fact, both reliability and stochastic viability deal with the question of whether the system performance is satisfactory, i.e. whether the reliability is guaranteed at a significance level α. It should be noted that these two methods, tackling very similar performance problems, have both been used in fields concerned with environmental and resources management. Thus, reliability theory has been used for more than three decades for water resources systems, following the pioneering work by Hashimoto *et al.* (1982) later completed by Moy *et al.* (1986) and Kundzewicz et Laski (1995). In ecology, the definition of ecosystem failure by Naeem (1998) has fostered discussions on the link between species redundancy and ecosystem reliability (Naeem et Li, 1997; Rastetter *et al.*, 1999). In both fields, the goal is to assess whether the performance of the system is consistent or satisfactory over a given time frame, which can be expressed in terms of whether and how reliability may reach or best a threshold value. The stochastic viability framework, which uses the word of viability instead of reliability, has tackled the same type of performance problem for fishery management (Doyen *et al.*, 2007; De Lara et Martinet, 2009; Doyen *et al.*, 2012), the latter publication using the concept of stochastic viability kernel which explicitly addresses the issue of whether staying in the survival set can be guaranteed with a desired probability.

The paper is organized as follows. In Section 3.2, the controlled time-variant reliability problem is introduced and its identity with a stochastic viability problem is discussed. Then in Section 3.3, the main concepts of reliability and viability are exposed and confronted so as to highlight what stochastic viability methods bring to time-variant reliability. An application is proposed in Section 3.4 in order to illustrate how dynamic programming can be applied to a reliability problem. These findings are further discussed in Section 3.5, and summarized in the concluding Section 3.6.

3.2 The controlled time-variant reliability problem

This section introduces the controlled time-variant reliability problem through classic reliability problems. Besides, it shows the challenges posed by this problem, which cannot be resolved by existing reliability methods.

3.2.1 Time-invariant reliability

Let us consider a system $\mathscr{S}$ and a vector of random variables $\mathbf{W}$ which represents the uncertainty and stochasticity of this system. The realizations of $\mathbf{W}$ are noted w and belong to $\mathbb{R}^q$. The state of the system depends on the vector $\mathbf{W}$, so that it is a random vector $\mathbf{X}$:

$$\mathbf{X} = f(\mathbf{W}, \pi) \tag{3.1}$$

where π is a vector describing the known deterministic parameters of the system. The parameter space is $\mathbb{R}^p$ and π is chosen in a subset which we note Π. The state space is noted A and is a subset of $\mathbb{R}^n$. A realization of $\mathbf{X}$ is x and there exists a realization w of $\mathbf{W}$ such that $x = f(w, \pi)$. Assuming that the performance of the system can be expressed in terms of its load l (demand) and resistance r (capacity), the performance function $g(\mathbf{X})$ is commonly written as :

$$g(\mathbf{X}) = r(\mathbf{X}) - l(\mathbf{X}). \tag{3.2}$$

The failure (or limit state) surface, $g(\mathbf{X}) = 0$, separates the states that lie in the failure domain F (where $g(\mathbf{X}) < 0$) from those in the survival domain S (where $g(\mathbf{X}) \geq 0$). The object of reliability is to determine the probability of failure p_f of the system :

$$p_f = \mathbb{P}(\mathbf{X} \in F) = \mathbb{P}(g(\mathbf{X}) < 0)). \tag{3.3}$$

A diversity of methods such that FORM or SORM have been developed in that purpose. Such computations prompt the use of an iso-probabilistic transformation of the random variable $\mathbf{X}$ into $\mathbf{Y}$ in the standard normal space, thus making the state space centered around a mean state equal to zero. The limit surface from Equation (3.2) is defined in accordance with this geometric transformation of the state space, and becomes $g(\mathbf{Y}) = 0$.

A classic problem is to find values of the parameter vector π that make the system reliable with a significance level α (i.e. a confidence level $1-\alpha$). From now on in this work we call reliable at the significance level α a system such that there exists such a value of the parameters, and we call a reliable parameter such a value. This kind of problem is close to reliability-based design optimization, often abbreviated RBDO, which balances the design and maintenance cost of a system with the expected cost of a failure (Rackwitz, 2001). Then, one wishes to achieve a certain reliability using a design that is under a given cost, and we can assume Π to give the possible options.

3.2.2 Time-variant reliability

We now place ourselves between an initial date $t_0 = 0$ and final date T, so that the problem is studied in discrete time within an interval $[0, T]$ called the planning period. Uncertainty and stochasticity are represented at each date by the random vector $\mathbf{W}(t)$. Following a convention from De Lara et Doyen (2008), the sequence $\mathbf{W} = (\mathbf{W}(0), \mathbf{W}(1), \ldots, \mathbf{W}(T-1))$ is called a scenario, and it is an element from the set of all scenarios $\mathbb{S}$. Instead of a representation of the time evolution of a system through its temporal correlation structure, usual within reliability theory, in this work time dependence is expressed explicitly using a dynamical system formulation. In discrete time this can be described using the following transition equation between two consecutive dates :

$$\mathbf{X}(t+1) = f(t, \mathbf{X}(t), \pi, \mathbf{W}(t)) \tag{3.4}$$

In the latter, the state $\mathbf{X}(t)$ can in fact be formulated as a function of the date t, scenario $\mathbf{W}$, initial state $x_0 = x(0)$ and parameter vector π :

$$\mathbf{X}(t) = f(t, x_0, \pi, \mathbf{W}) \tag{3.5}$$

The performance of the system described by equation (3.2) now evolves to take into account its possible time evolution :

$$g(t,\mathbf{X}(t)) = r(t,\mathbf{X}(t)) - l(t,\mathbf{X}(t)) \tag{3.6}$$

Likewise, the limit state surface $g(t,\mathbf{X}(t)) = 0$ is also dependent on time, and so are the failure domain $F(t)$ (where $g(t,\mathbf{X}(t)) < 0$) and the survival domain $S(t)$ (where $g(t,\mathbf{X}(t)) \geq 0$). The probability of failure $p_f(T, x_0, \pi)$ is the probability for a trajectory $(x_0, \mathbf{X}(1), \ldots, \mathbf{X}(T))$ to leave the survival set over the planning period :

$$p_f(T, x_0, \pi) = \mathbb{P}(\exists t \in [0, T], \mathbf{X}(t) \in F(t)) \tag{3.7}$$

where the dependence of p_f on x_0 and π is explicit through equation (3.5).

Existing time-variant reliability methods aim at finding the value of the probability of failure given the initial state x_0 and the parameter vector π. Yet, the dynamics and uncertainty now have to be propagated through T time steps, making the computation of the failure probability $p_f(T, x_0, \pi)$ more expensive than in the time-invariant case. This is the case for instance for out-crossing (or out-crossing-based) methods (Rackwitz, 2001; Andrieu-Renaud *et al.*, 2004). These methods are based on the out-crossing rate $\nu^+(t)$ defined in continuous time as the instantaneous rate at which the system leaves the survival set, providing an upper bound for the probability of failure :

$$p_f(T, x_0, \pi) \leq \int_0^T \nu^+(t)dt \tag{3.8}$$

and this inequality becomes an equality under the assumption that failure occurs only once. The aim is then to determine when the time-dependent system crosses the limit state. Like in the time-invariant state, the state space is transformed into the standard normal space at each time step so that the state vector is centered around its mean. The time evolution of limit state surface $G(t,\mathbf{X}(t)) = 0$ also needs to be accounted for as a result. Different methods are available in the literature enabling the estimation of the failure probability $p_f(T, x_0, \pi)$ from the out-crossing rate such as the $PHI2$ method (Andrieu-Renaud *et al.*, 2004) or the $EOLE$ method (Li et Der Kiureghian, 1995).

Much like in the time-invariant case, existing methods can deal with the problem of finding values for the initial state x_0 and parameter vector π such that the system is reliable over the planning period with a significance level α. A very similar problem consists in finding a reliable π for a given initial state x_0, which is then called reliable as well. We call reliability kernel the set of all the reliable initial states at a significance level α, and note :

$$\text{Rel}(\alpha, T) = \left\{x_0 \in S(0) | \exists \pi \in \Pi, p_f(T, x_0, \pi) \leq \alpha\right\} \tag{3.9}$$

Let us now explore the consequences of modifying this problem so that the parameters value can be changed to a certain extent at each time step, rather than just fixed beforehand. This is what we shall call the controlled time-variant reliability problem.

3.2.3 The controlled time-variant reliability problem

Still on a planning period $[0, T]$, let us now consider the possibility to decide the value of some parameters at each date t, depending on the state of the system. These parameters are now called *controls* and $u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_p(t,x))$ is the vector of the control variables that are applied at date t if the system is at state x. Controls are chosen among the set $U(t,x)$ of admissible controls and in particular, the set of available options at the initial date, noted Π previously in this work, can also be noted $U(0,x)$. A function $u(t,.)$ which associates a control to all states at date t is called a feedback map. The sequence of feedback maps between an initial date 0 and a final date T is called a strategy and noted $u(.)$. The set of all strategies that can be implemented during the time frame $[0, T]$ is noted $\mathscr{U}(T)$. Equation (3.4) can now be updated into :

$$\mathbf{X}(t+1) = f(t, \mathbf{X}(t), u(t), \mathbf{W}(t)) \tag{3.10}$$

or alternatively, like in equation (3.5), $\mathbf{X}(t)$ can be now expressed as a function of t, x_0, $u(.)$ and $\mathbf{W}$:

$$\mathbf{X}(t) = f(t, x_0, u(.), \mathbf{W}) \tag{3.11}$$

Trough the latter equation, the probability of failure within the planning period $[0, T]$ becomes a function of T, x_0 and $u(.)$:

$$p_f(T, x_0, u(.)) = \mathbb{P}(\exists t \in [0, T], \mathbf{X}(t) \in F(t)) \tag{3.12}$$

and this definition can also be extended to anterior dates :

$$p_f(t, x_0, u(.)) = \mathbb{P}(\exists \tau \in [0, t], \mathbf{X}(\tau) \in F(\tau)) \tag{3.13}$$

The controlled time-variant reliability problem is to find, given an initial state x_0, the control strategy $u(.)$ such that the system is reliable at the significance level α during the planning period. Its reliability kernel, analogous to equation (3.9), is the following set :

$$\mathrm{Rel}(\alpha, T) = \left\{x_0 \in S(0) | \exists u(.) \in \mathscr{U}(T), p_f(T, x_0, u(.)) \leq \alpha\right\} \tag{3.14}$$

We can call reliable at the significance level α a state of the reliability kernel $\mathrm{Rel}(\alpha, T)$. To call x_0 reliable, one needs to find an associated reliable strategy, but the search for its existence is very challenging because a strategy is described by one variable for each state and for each time step. Searching a strategy in $\mathscr{U}(T)$ thus means searching a space of very high dimensionality. Besides, for each state and time step, one must potentially consider many possible controls.

Reliability methods are not meant to tackle this problem of searching $\mathscr{U}(T)$. Yet, viability methods and concepts have been devised to search for suitable controls.

3.3 Stochastic viability as controlled time-variant reliability

The identity between the stochastic viability and controlled time-variant reliability problems is demonstrated in this section. The relevance of stochastic viability concepts and methods to reliability is a consequence of this, and is detailed henceforth. Before that, however, the original deterministic version of viability is exposed to elicit how the reliability and viability framework deal with the same type of problems with different point of view.

3.3.1 Deterministic viability

In its original deterministic version (Aubin, 1991), viability theory deals with controlled systems such that $\mathbf{W}(t) \equiv 0$. Equation (3.10) can be simplified into :

$$x(t+1) = f(t, x(t), u(t)) \tag{3.15}$$

where the random variable $\mathbf{X}$ has been replaced by a deterministic variable x, all else being the same. f is called the dynamic of the system. In this framework, for a given initial state x_0 and strategy $u(.)$, there is only one trajectory, so the state can be noted $x(t, x_0, u(.))$. The central question of viability is whether that trajectory leaves $S(t)$, given by equation (3.6) as previously, at any given date within the time frame $[0, T]$. An answer to this question is brought about by a central object, the viability kernel, which is the set of all initial states for which the system can be controlled so its trajectory does not leave the survival set. We can write :

$$\text{Viab}(T) = \{x_0 \in S(0) | \exists u(.) \in \mathscr{U}(T), \forall t \in [0,T], x(t,x_0,u(.)) \in S(t)\} \quad (3.16)$$

Thus, an initial state can either be viable or not, which we can translate into reliability terms by stating that in a deterministic context, the probability of failure is either 0 or 1. Figure 3.1 summarizes this. Properties of the viability kernel have provided the foundation of viability algorithms. This is for instance the case for algorithms that use the binary nature of a state under deterministic viability (e.g. Saint-Pierre, 1994; Deffuant *et al.*, 2007), or the fact that viable trajectories are tangent to the surface of the viability kernel (Bonneuil, 2006). An interest of these algorithms is that they find both the viable initial states and the associated viable controls, notions from which we derived those of reliable states and controls for the controlled time-variant reliability problem.

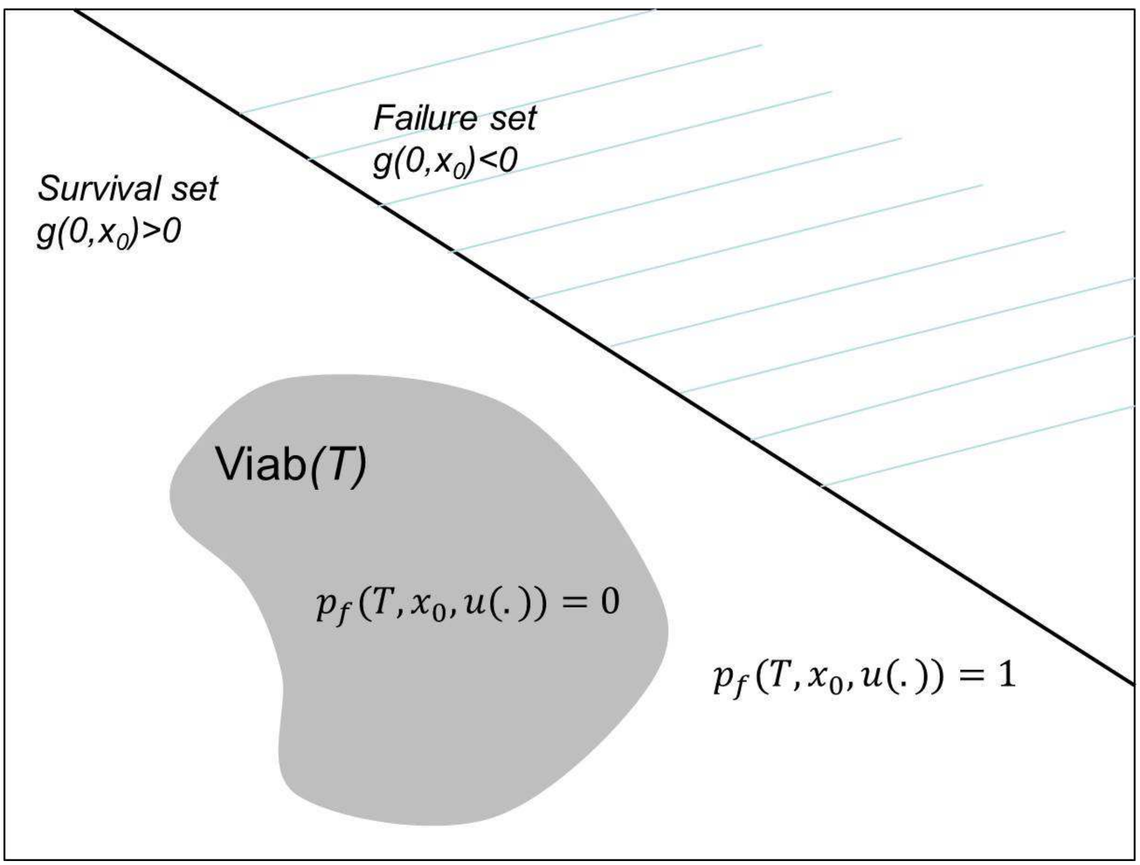

FIGURE 3.1 – Link between the viability kernel Viab(T) and the probability of failure $p_f(T, x_0, u(.))$ with deterministic controlled dynamics. All sets are represented at the initial date $t = 0$.

3.3.2 The stochastic viability problem

In the discrete time stochastic viability framework of De Lara et Doyen (2008), uncertainty is introduced into viability problems using the same concepts and notations as in Section 3.2.3. The dynamic becomes, still following the notations from De Lara et Doyen :

$$x(t+1) = f(t, x(t), u(t), w(t)) \tag{3.17}$$

where one can recognize equation (3.17) where the stochastic process **X** is replaced at all dates by its realization. Notations nonwithstanding, both equations are thus the same. Thus, equation (3.11) to describe the state $\mathbf{X}(t)$ still applies.

In fact, the goal is also the same : reliability focuses on the probability of failure which is the probability of reaching the failure set during a given time frame, while the concern of stochastic viability is the probability of staying in the survival set during that time frame. Much like in the original viability framework, a central concept is the stochastic viability kernel, defined as the set of all states for which the system has a probability ρ or higher of staying in the survival set $S(t)$ for a given time horizon T. It can be formally defined by the following equation in which it is noted $\text{Viab}(\rho, T)$:

$$\text{Viab}(\rho, T) = \{x_0 \in S(0) | \exists u(.) \in \mathscr{U}(T), \mathbb{P}(\forall t \in [0, T], \mathbf{X}(t) \in S(t)) \geq \rho\} \tag{3.18}$$

For instance, $\text{Viab}(0.99, 100)$ is the set of initial states such that the system has a 99% chance of staying in the survival set $S(t)$ for at least a hundred time steps. The stochastic viability problem can be related to the controlled time-variant reliability problem by the remark that :

$$p_f(T, x_0, u(.)) = 1 - \mathbb{P}(\forall t \in [0, T], \mathbf{X}(t) \in S(t)) \tag{3.19}$$

and thus, using equations (3.14) and (3.18) we have :

$$\text{Rel}(\alpha, T) = \text{Viab}(1 - \alpha, T) \tag{3.20}$$

and both concepts are interchangeable. One can argue that the concept of reliability kernel is more intuitive because the analogy between the stochastic viability kernel and its deterministic counterpart defined in equation (3.16) is limited. It is adapted for states inside the stochastic viability kernel, for which the property of being viable is replaced by one of being viable with probability $1 - \alpha$. However, all the properties of the viability kernel which were used for its computations are lost when uncertainty is introduced. For instance, a initial state x_0 which is not in $\text{Viab}(0.95, T)$ may have a

94% chance of staying in the survival set under the right strategy, while its reliability is zero if it is outside the viability kernel Viab (T). Viability and reliability in the stochastic case are summarized by Figure 3.2.

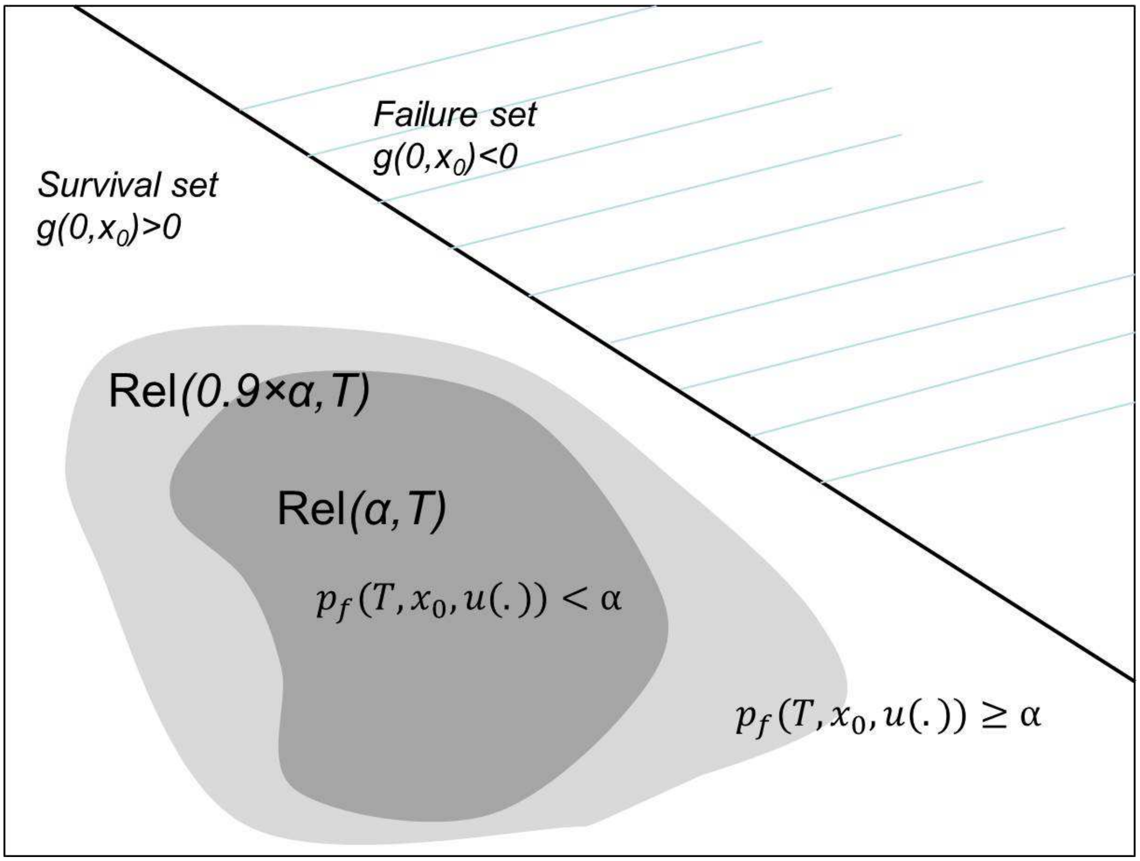

FIGURE 3.2 – Reliability kernels Rel(α, T) and Rel$(0.9 \times \alpha, T)$ with stochastic controlled dynamics. All sets are represented at the initial date $t = 0$.

Yet, equation (3.20) does not allow for the computation of the reliability kernel. Its interest comes from the fact that there exists a dynamic programming to compute the stochastic viability kernel.

3.3.3 A dynamic programming solution

Provided that at all dates, the $w(t)$ are independent from each other, Doyen et De Lara (2010) establish that the problem of finding the stochastic viability kernel can be solved by dynamic programming, a widespread category of recursive algorithms, designed to solve the problem backwards from date T to the initial date. Let us assume that the continuous bounded sets that form the state and control spaces have been discretized, as is the case in practice. The state is then represented by a finite set of points x_i of the discrete space A_d, of which the survival set $S(t)$ is a subset. Likewise, the discrete control space U_d is represented by points noted u_j, and each control space $\mathcal{U}_d(t,x_i)$ is a subset of U_d. Then the transition equation (3.10) (or equivalently, equation (3.17)) is given by the probabilities $\mathbb{P}(f(t,x_i,u_j,w(t)) = x_k)$, which we assume to be handily computable.

Then, Doyen et De Lara (2010) link Viab$(1-\alpha,T)$ to a value function $V(t,x_i)$ that is defined by an initial equation at date T and a recursive equation. The initial equation reads :

$$V(T,x_i) = \begin{cases} 1 & \text{if} \quad x_i \in S(T) \\ 0 & \text{if} \quad x_i \in F(T) \end{cases} \tag{3.21}$$

while the latter reads for all t between dates 0 and $T-1$:

$$V(t,x_i) = \begin{cases} \max\limits_{u_j \in \mathcal{U}_d(t,x_i)} \sum\limits_{x_k \in A_d} V(t+1,x_k).\mathbb{P}(f(t,x_i,u_j,w(t)) = x_k) & \text{if} \quad x_i \in S(t) \\ 0 & \text{if} \quad x_i \in F(t) \end{cases} \tag{3.22}$$

Doyen and De Lara demonstrate that Viab$(1-\alpha,T)$ is the set of all states such that $V(0,x_i) \geq (1-\alpha)$. Thus, the states for which there exists a reliable strategy $u(.)$ at the significance level α are also given by $V(0,x_i) \geq 1-\alpha$. Besides, another result from Doyen et De Lara (2010) is that the computation of the value function yields the control strategy $u^*(.)$ that minimizes the probability of failure for a trajectory starting at a state x_0. In fact one has :

$$V(0,x_0) = 1 - p_f(T,x_0,u^*(.)) \tag{3.23}$$

A major advantage of the stochastic viability approach is that the controls are not fixed beforehand. It dynamically and simultaneously computes both the optimal management strategy and the associated probability of viability (ore reliability) associated to that strategy.

In fact, what the dynamic programming approach does is find for each date t and state x_i the optimal control $u_j^* \in \mathcal{U}_d(t,x_i)$ for a problem that is none other than a time-invariant reliability problem such as the one introduced in Section 3.2.1 (see Appendix 3.7). It decomposes a time-variant reliability problem where uncertainty has to be propagated through space and time into separate time-invariant problems.

3.3.4 Approximating the date of first out-crossing

While the out-crossing rate is usually integrated over a period of time to yield the probability of failure, viability methods allow for finding the reliability kernel without computing it. Nevertheless, given an initial state x_0 and a strategy $u(.) \in \mathcal{U}(T)$, it is useful to know around which dates the system is most likely to leave the survival set. Noting t_{out} the date at which the system first leaves the survival set for the first time, we have :

$$\begin{aligned}\mathbb{P}(t_{out}=t) &= \mathbb{P}([\mathbf{X}(t)\in F(t)]\cap[\forall\tau<t,\mathbf{X}(\tau)\in S(\tau)]) \\ &= \mathbb{P}([\mathbf{X}(t)\in F(t)]\,|\,[\forall\tau<t,\mathbf{X}(\tau)\in S(\tau)]).\left(1-p_f(t,x_0,u(.))\right)\end{aligned} \tag{3.24}$$

where $p_f(t,x_0,u(.))$ is defined in equation (3.13), and using the identity $\mathbb{P}(\forall\tau<t,\mathbf{X}(\tau)\in S(\tau))=1-p_f(t,x_0,u(.))$. Besides, $\mathbb{P}(t_{out}=t)/\Delta t$ is an approximation of the rate of the first out-crossing. Contrary to what happens in usual out-crossing approaches (equation (3.8)), we no longer need to assume that the system crosses the failure surface only once. The probability $\mathbb{P}(t_{out}=t)$ can be computed by noting that :

$$\mathbb{P}(t_{out}\leq t)=p_f(t,x_0,u(.)) \tag{3.25}$$

so that the probability distribution for the date of the first crossing t_{out} is given by :

$$\mathbb{P}(t_{out}=t)=p_f(t,x_0,u(.))-p_f(t-1,x_0,u(.)) \tag{3.26}$$

Computation of the probability of failure $p_f(t,x_0,u(.))$ for $t<T$ can be achieved both by backward and forward programming, and both methods are presented in Appendix 3.8.

3.4 Application

In this section we apply the stochastic viability techniques from Section 3.3 to a simple dynamical model of controlled population growth. Despite the apparent simplicity of the equations, complex control strategies can and may have to be devised. Then we consider a performance function that decreases with time, like in a number of reliability applications, and use the results from stochastic viability to compute the out-crossing rate.

3.4.1 A simple population model

We consider a modified version of a simple model of population growth introduced by Aubin et Saint-Pierre (2002). It is discretized and uncertainty is integrated as an additive term to the population variable at each time step. The evolution of the state

$x = (a, b)$ reads :

$$\begin{cases} a(t+1) &= a(t) + (a(t)b(t) + w(t))\Delta t \\ b(t+1) &= b(t) + u(t) \end{cases} \tag{3.27}$$

where the state variables are the population $a(t)$ and its growth coefficient $b(t)$; this coefficient is controlled by $u(t)$. The control is bounded by $[U_{\min}, U_{\max}]$, which represents the inertia in the evolution of the population. These bounds are taken to be $U_{\min} = -0.5$ and $U_{\max} = 0.5$. Besides, throughout this section $\Delta t = 1$ and the uncertainty term $w(t)$ is a realization of $W(t) \sim \mathcal{N}(0, 0.25)$. In what follows, all quantities are assumed to be non dimensional for simplicity.

The size of the population is constrained, so that the survival set is represented by the following performance function :

$$g(t, x(t)) = g(t, a(t), b(t)) = (a(t) - 0.2)(c(t) - a(t)) \tag{3.28}$$

so that the survival set is defined by $a(t) \in [0.2, c(t)]$ where $c(t)$ is the carrying capacity of the system and is assumed to be greater than 0.2. In ecology, the carrying capacity is the maximal size of the population that can be sustained by the environment it lives in.

In this study, the state space has been discretized, with resolutions $\Delta a = 0.01$ and $\Delta b = 0.05$, and the control space is likewise discretized with a resolution $\Delta u = 0.05$. In this discrete space, the transition function between two time steps was obtained by interpolating from equation (3.27). The goal is to assess reliability at a time horizon T for a given initial state $x_0 = (a_0, b_0)$, first with a constant carrying capacity, then with a decreasing one. In what follows the relevant range for y was found to be $[-1.5, 2.5]$.

3.4.2 Constant carrying capacity

Let us assume that the carrying capacity is constant $c = 3$. Then, the performance function from equation (3.28) is under the following form g_1 :

$$g_1(t, x(t)) = (a(t) - 0.2)(3 - a(t)) \tag{3.29}$$

Dynamic programming leads to the strategy $u(.)$ that optimizes reliability at any horizon, and the only approximation is that of the discretization. This is showcased for $T = 100$ by Figure 3.3, which shows that only initial states grouped around $b_0 = 0$ have a good reliability. There is, however, a sizable reliability kernel Rel(0.95, 100). Such reliability kernels can also be computed at any horizon, so that one can for instance observe the evolution of Rel$(0.95, T)$ as T increases, and observe that its size decreases very little until it abruptly ceases to exist when T tops 254 (Figure 3.4).

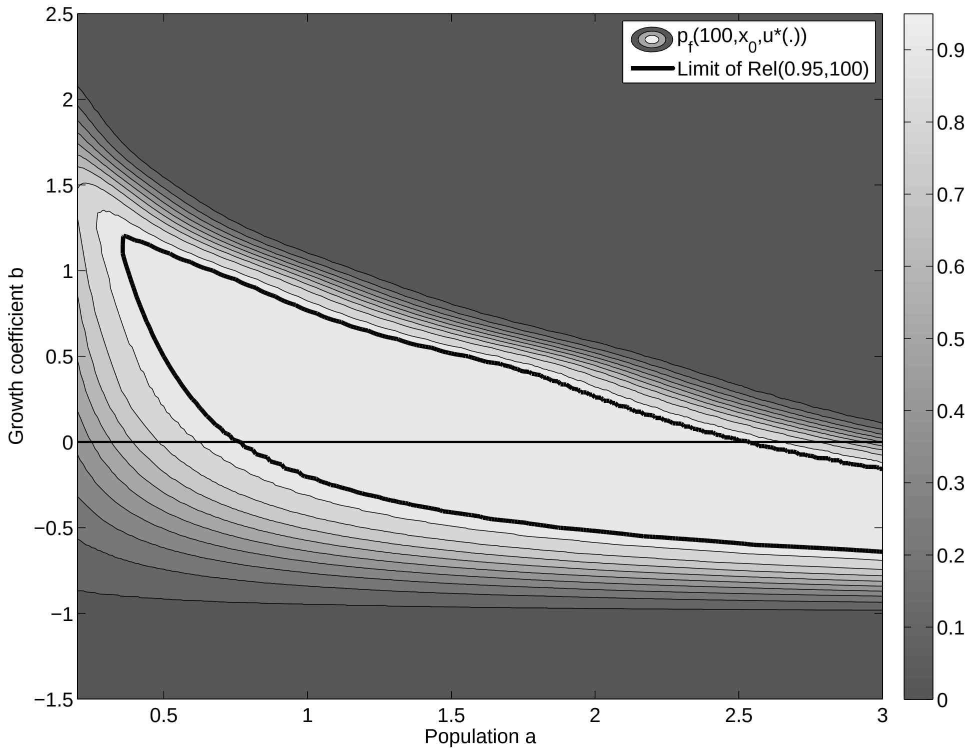

FIGURE 3.3 – Reliability with the performance function g_1 and a time horizon of a hundred time steps.

This stability of the reliability kernel Rel$(0.95, T)$ as the horizon increases is matched by that of the optimal strategy. Whatever the horizon, the backward sequence of feedback maps $u(t, .)$ from the final date T to the initial date is the same, and what is more, the map $u(t, .)$ becomes constant for $t \leq T - 10$. It is noted u^* and represented on Figure 3.5. One can see that for a given value of a, the value of u^* increases as b increases, but the relationship between u^* and b is different for each single value of a, making the map is very complex.

Out-crossing rates as approximated by $\mathbb{P}(t_{out} = t)/\Delta t$ can be estimated using this feedback map. Since $\Delta t = 1$, using equation (3.24) the out-crossing rate takes the approximate value of $(\lambda(1 - p_f(t, x_0, u^*(.)))$ after less than 10 time steps, where $\lambda \approx 2 \times 10^{-4}$ is probability of leaving at t conditional on staying in the survival set up to $t - 1$. It is independent on the initial states, so that the differences in reliability displayed in Figure 3.3 account for the probability of leaving the survival set within these first ten time steps. After $t = 10$, the probability of failure only increases very

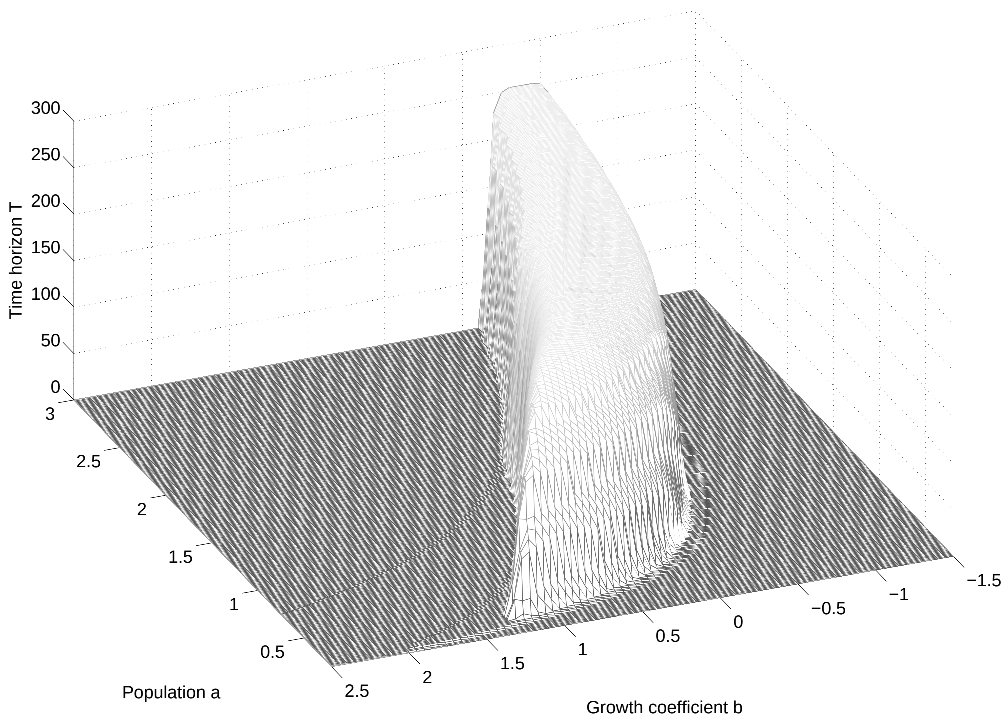

FIGURE 3.4 – Initial states x_0 belonging to $\mathrm{Rel}(0.95, T)$, for different values of the horizon T and the performance function g_1.

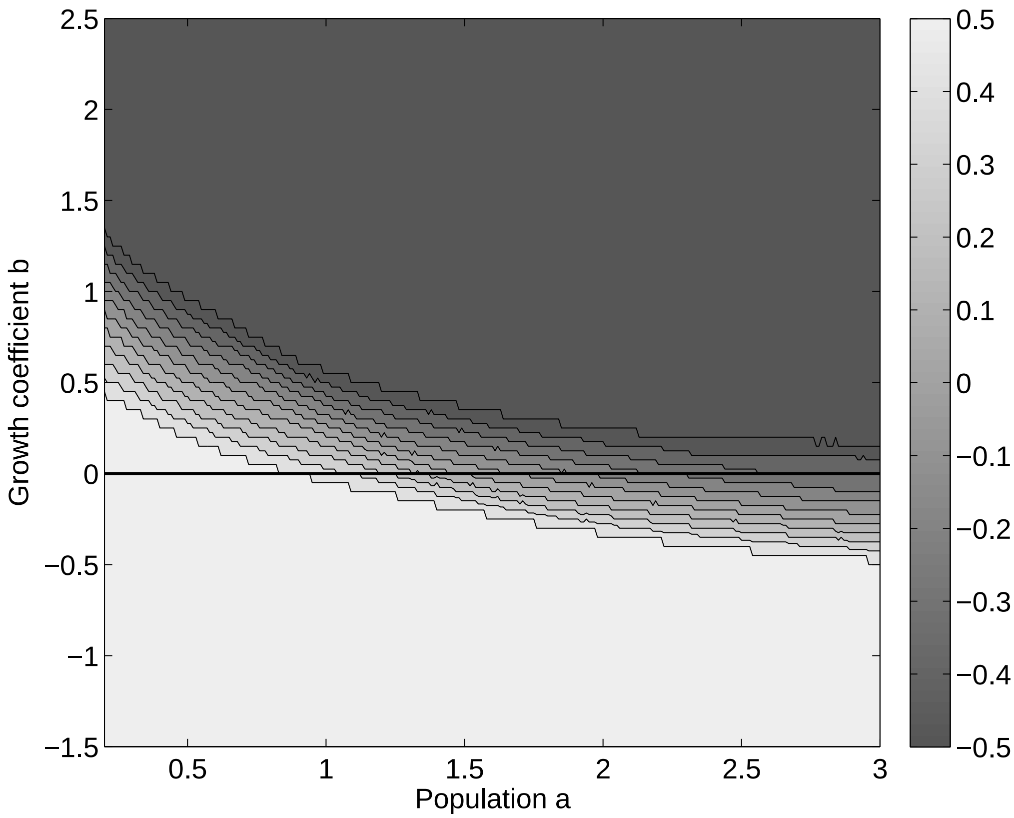

FIGURE 3.5 – Map of the optimal controls $u^{*}(x)$ for the performance function g_1, 10 time steps or more from the horizon.

slowly.

3.4.3 Decreasing carrying capacity

Let us now suppose that the system performance decreases linearly over time, with a diminishing adaptive capacity $c(t) = 3 - 0.01t$. Let us note this function g_2, equation (3.28) becomes :

$$g_2(t, x(t)) = (a(t) - 0.2)(3 - 0.01t - a(t)) \tag{3.30}$$

As expected, this linear decrease in performance affects reliability, so that Rel$(0.95, T)$, even though it assumes a similar shape as for g_1, vanishes for $T > 54$ (Figure 3.6). Besides, unlike for the case of a constant carrying capacity, the optimal control maps change at each time step. This would make them very difficult to find if it were not for dynamic programming.

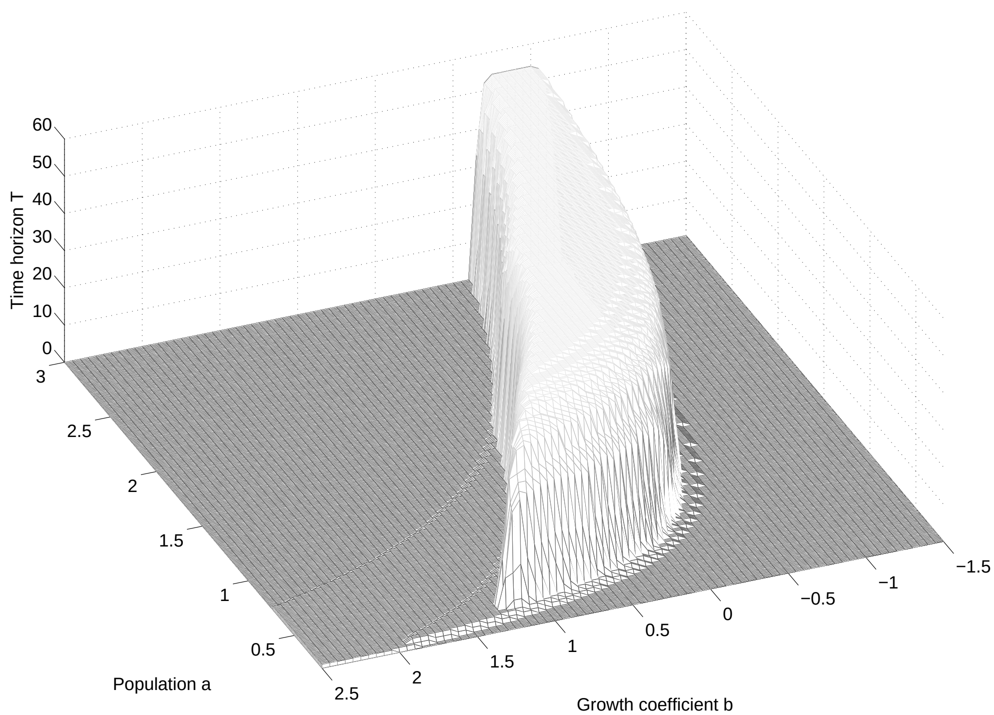

FIGURE 3.6 – Initial states x_0 belonging to Rel$(0.95, T)$, for different values of the horizon T and the performance function G_2.

Yet, for $t \leq T - 10$ it seems that the map $u^*(t, .)$ does not depend on the horizon T. Thus, the maps for $T = 100$ and $T = 200$ are identical until the date $t = 92$, while those for $T = 150$ and $T = 200$ are identical until $t = 145$. This makes the computation of the out-crossing rates values computed with the optimal strategy for $T = 200$ applicable to lower time horizons. No matter the initial state, the out-crossing rate is low after the first ten time steps then gradually increases to peak at $t = 124$ (Figure 3.7). Then, it decreases because the decreasing quantity $(1-p_f(t,x_0,u^*(.))$ in equation (3.24) compensates the growth of the probability of leaving the survival set at t conditional on staying in it until $t-1$. Like for the previous case, the amplitude of the out-crossing rate after $t = 10$ depends on the odds of leaving the survival set within the first few time steps. The cumulative probability of failure through time can be computed alongside the out-crossing rate (Figure 3.8).

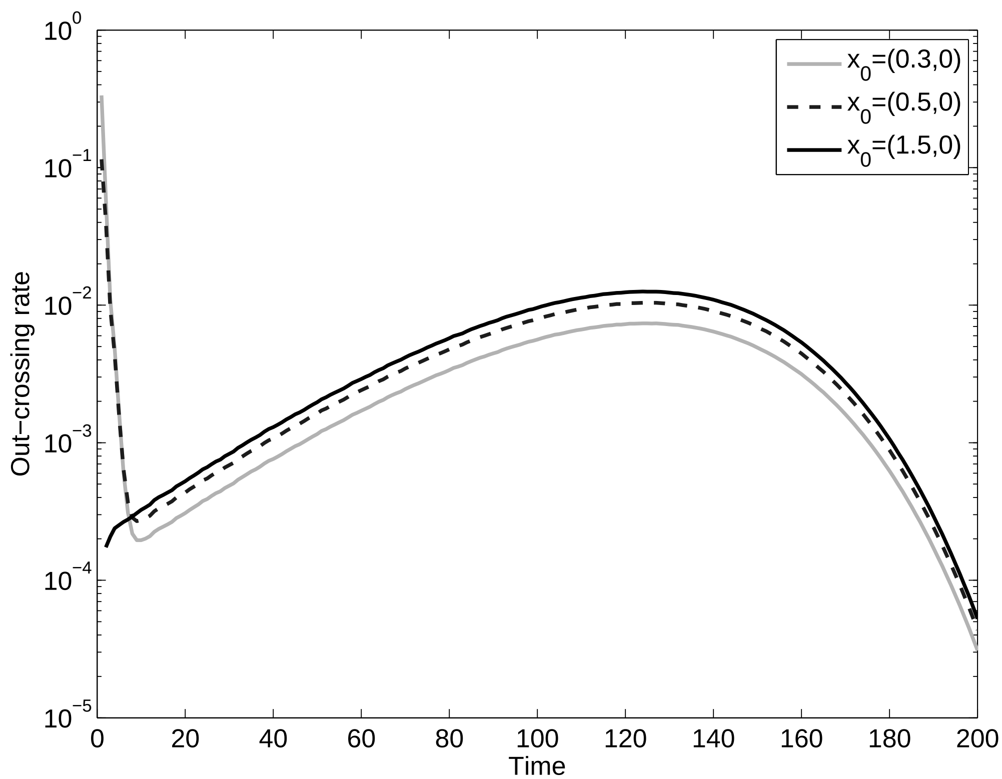

FIGURE 3.7 – Evolution of the out-crossing rate under the performance function G_2.

3.5 Discussion

Both viability and reliability have been used to explore other concepts related to the performance of a system. On the one hand, resilience has been defined as the possibility for the system to recover and get to a set of states robust to uncertainty after a major event dragged it into the failure set, this robust set being the stochastic viability kernel (Rougé *et al.*, 2013). In the same work, stochastic viability methods such

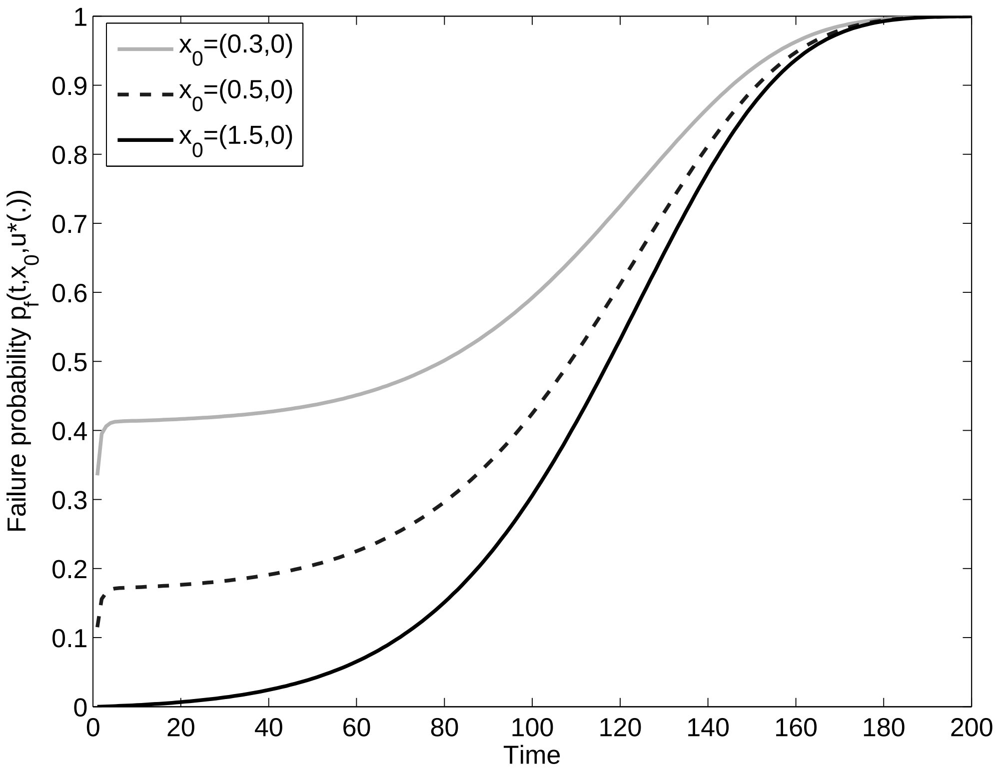

FIGURE 3.8 – Evolution of the probability of failure under g_2.

as dynamic programming are used to compute the probability of reaching a given stochastic viability kernel within a given time frame after an event. On the other hand, resilience but also vulnerability have been defined alongside reliability as performance indicators for water resources systems (Hashimoto *et al.*, 1982) and further, a method computing all three concepts using FORM also exists (Maier *et al.*, 2001). Thus, one can hope that bringing viability and reliability methods together may improve the definition and computation of other related concepts such as resilience and vulnerability.

As mentioned in Section 3.3, dynamic programming decomposes a time-variant reliability problem into many simpler time-invariant reliability problems. Thus, while this work focused on showing how stochastic viability methods could foster a better computation of reliability in time-variant cases, it is possible to have time-invariant reliability methods help solve stochastic viability problems. For instance, if the transitions from the state x_t to the state x_{t+1} along a trajectory are difficult to compute, then equation (3.22) may be approximated through methods such as FORM or SORM.

Further, one must keep in mind the limitations of dynamic programming algorithms. They only work for systems where the state space has a low dimension. Too high a dimension leads to the so-called "curse of dimensionality" which designates the exponential increase of the needed computational time and memory. This leads to problems that are unpractical to solve. Besides, this method requires that the random variables $\mathbf{W}(t)$ be independent with each other. In practice, temporal dependence can be solved at a given date t by artificially adding as state variable the values of the uncertain vectors that are needed to compute $\mathbf{W}(t)$. However, such a strategy leads to an explosion of the dimension of the problem. There exist decomposition algorithms that have been used to deal with the dimension problem in dynamic programming, such as the Benders decomposition (Perreira et Pinto, 1985), but their applicability is outside the scope of this work.

3.6 Conclusion

Stochastic viability and reliability have the same broad goal of computing the probability for a system to not violate its constraints. Thus, the search for feedback strategies that guarantee the reliability of a system with a probability $1 - \alpha$ can be addressed through the dynamic programming methods introduced in stochastic viability theory. The set of states for which such feedbacks exist is the stochastic viability kernel, which can also appropriately be named reliability kernel. The proposed alternative name is also motivated by the fact that the properties of the viability kernel in the original deterministic framework of viability theory no longer hold when stochasticity and uncertainty are introduced. For a given adequate strategy, one can then also compute the out-crossing rate to understand when failure may occur.

Application to a simple system shows that this method is well-suited to tackle cases where the performance function decreases over time. Further, it showcases how the optimal strategy that dynamic programming finds to avoid failure may be very complex and have hundreds of degrees of freedom, even in seemingly simple cases. There is to our knowledge no other case of a method that can yield the most reliable policies in an uncertain dynamical system.

Acknowledgements

This work would not have been possible without funding from Région Auvergne.

3.7 Appendix : Equation (3.22) as a time-invariant problem

We are in a situation where the date t and discrete state x_i are fixed parameters, and we set the parameter vector $\pi = (t, x_i, u_j)$. Then equation (3.10) can be written over as $\mathbf{X}(t+1) = f(\mathbf{W}(t), \pi)$, where one can recognize a time-invariant problem since time is a parameter. Let us introduce the Bernoulli variable $\mathbf{B}$ dependent on the value of the realization of $\mathbf{X}(t+1)$ defined by :

$$\mathbb{P}(\mathbf{B} = 1 | \mathbf{X}(t+1) = x_k) = V(t+1, x_k) \tag{3.31}$$

We can also introduce the random variable $\mathbf{Z}$ defined by :

$$\mathbf{Z} = (\mathbf{X}(t+1), \mathbf{B}(\mathbf{X}(t+1))) \tag{3.32}$$

The latter equation corresponds to $\mathbf{Z} = g(\mathbf{W}(t), \pi)$ where $g(\mathbf{W}(t), \pi) = (f(\mathbf{W}(t), \pi), \mathbf{B}(f(\mathbf{W}(t), \pi))$, and one can have a time-invariant type problem by setting the survival set $S = (A_d \times \{1\})$ for $\mathbf{Z}$, where A_d represents the discrete state space of $\mathbf{X}$. Then, a design problem is to find the value of u_j that maximizes reliability. The law of total probability yields :

$$\mathbb{P}(\mathbf{Z} \in S) = \sum_k \mathbb{P}(\mathbf{B} = 1 | \mathbf{X}(t+1) = x_k).\mathbb{P}(\mathbf{X}(t+1) = x_k) \tag{3.33}$$

and replacing with equations (3.31) and (3.17) leads to :

$$\mathbb{P}(\mathbf{Z} \in S) = \sum_k V(t+1, x_k).\mathbb{P}(f(\pi, w(t)) = x_k) \tag{3.34}$$

The value of u_j, hence that of π, that maximizes the latter probability is exactly the one yielded by the maximization problem of equation (3.22).

3.8 Appendix : Computation of $p_f(t, x_0, u(.))$ for $t < T$

3.8.1 Forward

This is done through the direct computation of the possible trajectories, as long as they do not leave the survival set. We recursively compute $V_1(1, x_0, u(.), x_k) = \mathbb{P}(\mathbf{X}(t) = x_k | \forall \tau < t, \mathbf{X}(\tau) \in S(\tau))$. Initialization reads :

$$V_1(1, x_0, u(.), x_k) = \begin{cases} \mathbb{P}(f(0, x_0, u(0, x_0), w(0)) = x_k) & \text{if} \quad x_k \in S(0) \\ 0 & \text{if} \quad x_k \in F(0) \end{cases} \tag{3.35}$$

then the function V_1 recursively updated at each date $1 \leq t \leq T$:

$$V_1(t, x_0, u(.), x_k) = \sum_{x_i \in S(t-1)} V_1(t-1, x_0, u(.), x_i).\mathbb{P}(f(t-1, x_i, u(t-1, x_i), w(t-1)) = x_k) \tag{3.36}$$

Then we have :

$$p_f(t, x_0, u(.)) = \sum_{x_k \in S(t)} V_1(t, x_0, u(.), x_k) \tag{3.37}$$

The advantage of using the above approach is that it yields the failure probabilities at all dates recursively, in a single run. The inconvenient lies with the large amount of computational memory it requires, since it connects all the points of the successive survival sets with each other.

3.8.2 Backward

One uses a value function V_2 which is initialized through :

$$V_2(t, x_i) = \begin{cases} 1 & \text{if} \quad x_i \in S \\ 0 & \text{if} \quad x_i \in F \end{cases} \tag{3.38}$$

and then progresses backward from t to 0 :

$$V_2(\tau, x_i) = \begin{cases} \sum_{x_k \in A_d} \mathbb{P}(f(t, x_i, u(\tau, x_i), w(\tau)) = x_k).V_2(\tau+1, x_k) & \text{if} \quad x_i \in S \\ 0 & \text{if} \quad x_i \in F \end{cases} \tag{3.39}$$

These equations are exact analogous to equations (3.21) and (3.22) for the value function V, where there is only one possible control $u(\tau, x_i)$ at each date τ and state x_i. Thus, equation (3.23) becomes $V_2(0, x_i) = 1 - p_f(t, x_0, u(.))$. This is less expensive than the algorithm for V since there is no need to solve an optimization problem at each date and state to get the feedbacks. However, one will need to run this algorithm for each date separately so as to get the probability of failure at multiple dates.

– 4 –

Vulnerability : from the conceptual to the operational using a dynamical system perspective

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This work proposes to address a lack of conceptual consensus surrounding the term vulnerability, by using a minimal definition of vulnerability as a measure of potential future harm, and by basing this definition on a stochastic controlled dynamical system framework. Harm is defined as a normative judgment on the undesirability of a trajectory, the sequence of the system states between two dates. Given an initial state, action policy and time horizon, considering all possible trajectories leads to obtaining the probability distribution of harm values. Vulnerability is then defined as a statistic derived from this distribution. This framework (1) promotes a dynamic view of vulnerability by highlighting its temporal dimension and (2) fosters the development of operational indicators in a context which clarifies the descriptive and normative aspects in the system's representation. An object of interest is the low-vulnerability kernel, which regroups all the system states such that vulnerability is kept below a threshold value pending enforcement of an appropriate policy. For some indicators, this set can be found using the mathematical framework of stochastic viability theory, and its associated stochastic dynamic programming tools. Regarding the hazards that may affect the system, we propose to distinguish between the computable uncertainty, embedded in the stochastic dynamics, and extreme or surprising events whose impacts can be explored through specific scenarios. The relevance of this work to situations where a dynamical formulation is not available is stressed. A simple nonlinear model of lake eutrophication illustrates the framework, and showcases its link with concepts related to coping and adaptation, which it allows to effectively discuss.

4.1 Introduction

This work proposes an operational definition of vulnerability, based on a stochastic dynamical system framework which accounts for its uncertain evolution and for the decisions that can be taken to impact it. Vulnerability is defined in a most general way as a measure of potential future harm. It is an oft-used concept in the literature dealing with the potential negative impacts of natural hazards and social and environmental change. However, vulnerability concepts and tools originate from several different communities (Adger, 2006; Eakin et Luers, 2006; Miller *et al.*, 2010). Consequently, there is a lack of consensus about the conceptual definitions of the term and this breeds vagueness (Hinkel, 2011). Thus, despite the existence of similar operational protocols, unified frameworks in or across research fields are largely missing (Costa et Kropp, 2012).

The minimal definition we propose to use comes from a formal analysis of the term (Wolf *et al.*, 2013), and it is the lowest common denominator in most vulnerability definitions in the literature (Hinkel, 2011). To our knowledge, our framework constitutes the second attempt at mathematically formalizing the concept of vulnerability after that by Ionescu *et al.* (2009), who argue that such a formalization is warranted for several reasons, namely making vulnerability assessments systematic, clarifying the concepts and their communication, avoiding analytical inconsistencies and practical omissions, and enabling the development of computational approaches. These reasons stress that formalization is useful regardless of the presence of a dynamical system formulation in a given case.

We propose to start with the most general possible conceptual and mathematical formulation, and then to interpret elements of the framework that are relevant to the vulnerability literature. We argue that this approach provides both a formal basis and a great flexibility for the discussion of vulnerability concepts. Indeed, the mathematics remain a non ambiguous reference for the discussion of the concepts, for which several interpretations may be possible and relevant depending on their application. In that sense, flexibility appears as a prerequisite to bridging the gap between conceptual definitions and their many possible operational translations.

Besides, we propose to explicitly consider evolutions over several time steps, extending the work by Ionescu *et al.* (2009) who mainly base their vulnerability discussions on evolutions over a single time step. The fact that the vulnerability of an entity depends on uncertain dynamics over time (Wolf *et al.*, 2013) is often overlooked or kept implicit (Liu *et al.*, 2008) because such dynamics often cannot be expressed explicitly by a relevant model. Hence the need for a framework centered on the notion of possible future trajectories to which harm values are associated. Uncertainty in the value of future harm is considered through the occurrence probabilities of these possible trajectories, leading to obtaining a probability distribution function (pdf) of these harm values. Then, a relevant statistic derived from this pdf is a measure of potential future harm, in other words vulnerability.

This necessity for taking temporal dynamics into account has been highlighted in the literature (Liu *et al.*, 2008), and illustrated through the concept of path dependence (Preston, 2013), and through the idea that a natural hazard can impact a system long after hitting it (Menoni *et al.*, 2002; Lesnoff *et al.*, 2012). Moreover, a dynamic vision of a system, that includes normative judgments on its state and the underlying notions of trajectories and thresholds, can be successful in a participatory assessment of vulnerability (Béné *et al.*, 2011), with no need for the system to be modeled by explicit equations. Thus, regardless of the existence of a dynamical system formulation, a dynamic vulnerability framework can be helpful both for understanding and assessing the concepts, and for communicating with stakeholders.

However conceptually general, the proposed framework also aims at participating in a clarification of the operational meaning of the term by helping to specify vulnerability of "what" to "what". The former – what is the focus of vulnerability – is to be rendered explicit through a dual representation of the studied system. On one hand, the stochastic controlled dynamical system formulation describes the possible trajectories. On the other hand, the normative side of the assessment is clarified by associating a harm value to each trajectory.

When it comes to the to "what" part of vulnerability, i.e. the knowledge of the future events that may affect the system, the candidates abound. This is true for instance in the climate change literature (see e.g. Turner II *et al.*, 2003; Adger, 2006; Parry *et al.*, 2007) and in the natural hazards literature (e.g. Birkmann, 2006; Fuchs, 2009; Peduzzi *et al.*, 2009). We propose a simpler typology of events, using the notions of computable and uncomputable uncertainty from Carpenter *et al.* (2008). Computable uncertainty encompasses the known hazards events to which the system is exposed on a regular basis, and can be integrated in the stochastic dynamics. In contrast, uncomputable uncertainty generally refers to extreme or unexpected events, a change in the system, or a combination of the above. Vulnerability is then assessed with respect to one specific hazard, and both types of vulnerability assessments shall be distinguished in our framework.

Also of interest is the identification of the initial system configurations and associated action policies that lead to a low vulnerability. The set of all initial states such that vulnerability is below a threshold value will be introduced as the low-vulnerability kernel (to be noted LVK thereafter). This object also explicitly makes a connection between harm and the notion of threshold that is present in the vulnerability literature (e.g. Luers *et al.*, 2003; Luers, 2005; Béné *et al.*, 2011).

In particular, quoting Eakin et Luers (2006), the focus can be on "the identification of critical thresholds of significant damage", and the objective is then to find initial states which may respect a safety criterion if the system is properly managed. This is also the aim of viability theory (Aubin, 1991), a mathematical framework where a central object is the viability kernel, the set of initial states such that the system can be controlled so as to not cross thresholds of harms. This theory has been extended to uncertain controlled dynamics under the name of stochastic viability theory (De Lara et Doyen, 2008), and has been used to design sustainable policies under uncertainty in social-ecological systems such as fisheries (Doyen et Béné, 2003; De Lara et Martinet, 2009; Doyen *et al.*, 2012) or grassland agro-ecosystems (Sabatier *et al.*, 2010). It has been related to stochastic dynamic programming (SDP), an algorithm that helps compute the stochastic version of viability kernel (Doyen et De Lara, 2010). Drawing from this analogy, we shall show how SDP can help compute the LVK in some cases, and point out in which cases it coincides with a stochastic viability kernel.

A controlled stochastic dynamical system perspective also provides a connection, often explicit in definitions, between vulnerability and the capacity to act. The relationship between ability to act and vulnerability is often highlighted (Turner II *et al.*, 2003; Gallopín, 2006; Smit et Wandel, 2006), and the latter is often associated with limitations of the former (McCarthy *et al.*, 2001; Adger, 2006; Parry *et al.*, 2007). As shall be highlighted through the application to a simple model in which human activities lead to environmental degradation, the case of lake eutrophication (Carpenter *et al.*, 1999), this framework is relevant to discussions around coping and adaptation.

The rest of this work is organized as follows. Section 4.2 presents the dynamical system framework for vulnerability. Section 4.3 then proposes examples of vulnerability indicators, along with a practical computation of the LVK for some of them. Section 4.4 illustrates these concepts using a simple dynamical system model of lake eutrophication, while Section 4.5 discusses them. Section 4.6 provides concluding remarks.

4.2 The dynamical system framework for vulnerability

This section presents a general discrete-time stochastic controlled dynamical system formulation. We introduce, in this order, the dynamics, harm along a trajectory, vulnerability to computable uncertainty, vulnerability to a specific hazard, and the notion of LVK.

4.2.1 System dynamics

We consider a system and its uncertain and controlled dynamics. This system can for instance be a social-ecological system (SES), but the framework is applicable to any system that evolves with time. We are interested in this evolution during a given period spanning from an initial date 0 and a final date τ. In discrete time, the transition between two consecutive dates is given by (De Lara et Doyen, 2008) :

$$x_{t+1} = f_t(x_t, u_t, w_t) \tag{4.1}$$

The time-dependent dynamic f_t is the transition function between two dates. x_t is the vector of state variables describing the system at a date t. Meanwhile, u_t is the vector representing the decisions that are taken at date t by stakeholders to influence the state. The set of decisions that are available depends both on the date and the state, and this decision space is noted $U_t(x)$. w_t is the vector representing the uncertainty and variability that affect the system at date t.

A possible sequence of events can be called a scenario (De Lara et Doyen, 2008), and be noted $\omega = (w_0, w_1, \dots, w_{\tau-1})$. The space of all the scenarios is noted Ω, and for the computation of vulnerability indicators, one needs to assume the existence of a probability $\mathbb{P}$ defined over Ω[1]. One should keep in mind that any dynamic representation of a system may not take into account all the possible scenarios, nor evaluate correctly their probability of occurrence, so that such a probability is only partly computable (Carpenter *et al.*, 2008).

A strategy σ associates to any date t and state x a decision $u_t(x)$ chosen among the set of possible decisions $U_t(x)$. The set of all the strategies σ available within a horizon of τ is noted $\Sigma(\tau)$. In a way, σ and ω are the respective dynamic equivalents of u and w.

The initial state x_0 at $t = 0$, the strategy σ and the scenario ω define only one possible sequence of states according to equation (4.1). This is what we call a trajectory and note $\theta(x_0, \sigma, \omega, \tau)$. Each successive state x_t belonging to a given trajectory may also be noted $\theta_t(x_0, \sigma, \omega)$, hence equation (4.1) leads to the following recursive

[1] In mathematical terms, the probability $\mathbb{P}$ is in reality defined over $\mathscr{P}(\Omega)$, the set of all the subsets of Ω, and the triplet $(\Omega, \mathscr{P}(\Omega), \mathbb{P})$ is called a probability space. We omit to mention $\mathscr{P}(\Omega)$ in the main text so as not to overload it with mathematical notions.

sequence :

$$\begin{cases} \theta_0(x_0,\sigma,\omega) = x_0 \\ \forall t \in [0,\tau-1], x_{t+1} = \theta_{t+1}(x_0,\sigma,\omega) = f_t(x_t,u_t(x_t),w_t) \end{cases} \tag{4.2}$$

so that a trajectory can be seen as the dynamic equivalent of a state.

4.2.2 Harm

According to Wolf *et al.* (2013), harm is defined by associating a harm value to an evolution. Therefore, noting $H(x_0,\sigma,\omega,\tau)$ the harm value which corresponds to a trajectory $\theta(x_0,\sigma,\omega,\tau)$, and $\mathscr{F}$ the function that associates them, we have :

$$H(x_0,\sigma,\omega,\tau) = \mathscr{F}(\theta(x_0,\sigma,\omega,\tau)) \tag{4.3}$$

This notion of harm contains the idea that two trajectories can be compared, and it is intrinsically a subjective notion (Ionescu *et al.*, 2009; Wolf *et al.*, 2013). Importantly, the set of harm values is not necessarily continuous. For instance, one can simply distinguish between "harmed" and (comparatively) "unharmed" trajectories : harm is then a boolean and takes respective values 1 and 0, a binary classification sometimes used to define vulnerability (e.g. Mendoza *et al.*, 1997; Rockström *et al.*, 2012; Kasprzyk *et al.*, 2013).

In practice, harm $H(x_0,\sigma,\omega,\tau)$ along a trajectory is often a very abstract notion. It can be made more practical thanks to a two-step process in which harm is first introduced as a normative judgment on the "badness" of a state (Hinkel, 2011). A value $h_t(x)$, static equivalent of harm $H(x_0,\sigma,\omega,\tau)$ and called static harm to avoid any confusion, is given to each state x and date t. We note :

$$h_t(\theta_t(x_0,\sigma,\omega)) = h_t(x_0,\sigma,\omega) \tag{4.4}$$

Although it is not a general necessity, it may often be convenient to assume that static harm is a quantity that is never negative, but can be equal to zero if the state is deemed harmless. Naturally, like harm along a whole trajectory, static harm values $h_t(x_0,\sigma,\omega)$ can take discrete values.

Then, we can define harm $H(x_0,\sigma,\omega,\tau)$ along a trajectory directly as a function of the $\tau+1$ static harm values $h_t(x_0,\sigma,\omega)$ associated to the successive states. One can imagine a virtually infinite number of ways to build the function H. Yet, it seems natural to assume that this harm value $H(x_0,\sigma,\omega,\tau)$ is a growing function of each of the $\tau+1$ static harms. A few straightforward and common examples are given in

Section 4.3.1.

4.2.3 Vulnerability to computable uncertainty

Assessing vulnerability supposes the aggregation into one value of the harm values H associated to different scenarios. Given the density function $p(\omega)$ for the probability $\mathbb{P}$ defined over Ω, a vulnerability indicator can be defined as a statistic on the distribution of harm values. It is only dependent on the initial state x_0, the strategy σ and the horizon τ. It is a measure of possible future harm inflicted during the time frame $[0, \tau]$.

The most straightforward of these statistics is vulnerability as the expected value of future harm, expected vulnerability V^E :

$$V^E(x_0, \sigma, \tau) = \int_{\omega \in \Omega} H(x_0, \sigma, \omega, \tau) p(\omega) d\omega \tag{4.5}$$

Another approach to vulnerability is to take into account the worst case scenario, and vulnerability is then expressed through high quantiles of the distribution of harm values. This can be equated with finding the harm value that is not exceeded with a confidence level β such as 90%, 95%, 99% or even higher. This is also called the the value-at-risk V^β for harm, it is the minimal value that future harm will not exceed with a probability of at least β :

$$V^\beta(x_0, \sigma, \tau) = \min\left\{\eta \in \mathbb{R}^+ | \mathbb{P}(H(x_0, \sigma, \omega, \tau) \leq \eta) \geq \beta\right\} \tag{4.6}$$

One can certainly think about many other vulnerability indicators, but enumerating them is both virtually impossible and outside the scope of this work. Practical examples of vulnerability indicators using V^E and V^β are given in Section 4.3.2.

4.2.4 Specific vulnerability

Specific vulnerability is defined relative to a well-identified hazard Z, which can be any event which impact upon occurrence is investigated, whether it is uncomputable or not. It can in general be modeled by a modification of the pdf defined for the scenario space Ω. The density $p(\omega)$ becomes $p_Z(\omega)$, and this affects the vulnerability function $V(x_0, \sigma, \tau)$ which becomes $V_Z(x_0, \sigma, \tau)$. For instance let us assume we want to investigate the consequences of an extreme rainfall event at the initial date, so as to know whether it can lead to a flood and which impacts this flood may cause. Then, a scenario will be run in the hydrological model of the floodplain using a modified probability distribution of the rainfall amount, so that it is extreme with unit probability. Another example can an environmental change leading to a shift in the expected value of a parameter, which can be reflected by a shift in its pdf, expressed in w_t, between the initial date and the final date.

Since $V_Z(x_0, \sigma, \tau)$ represents the vulnerability to computable uncertainty conditional on the occurrence of the hazard Z, it represents vulnerability to both Z and computable uncertainty sources. Instead, specific vulnerability to the hazard Z at the state x_0 is given by the difference between vulnerability V_Z and vulnerability V which is the vulnerability to all the known hazards in its absence. For any two strategies σ_1 and σ_2 chosen among the set $\Sigma(\tau)$ of available strategies we get :

$$V(x_0, \sigma_1, \sigma_2, \tau; Z) = V_Z(x_0, \sigma_2, \tau) - V(x_0, \sigma_1, \tau) \tag{4.7}$$

where the result is more relevant if these two strategies are efficient, or better yet, if σ_1 and σ_2 respectively minimize V and V_Z.

4.2.5 Low-vulnerability kernels (LVKs)

From a static point of view, low-harm states are defined as states such that static harm is below a threshold value :

$$K_t(l) = \{x | h_t(x) \leq l\} \tag{4.8}$$

This delimits a constraint set, that generalizes the notion of threshold because it can have any shape. In a way, by separating the state space between two zones – low harm or not – it provides a coarser judgment on the system states than the original harm function does. A particular case of constraint set is $K_t(0)$, which corresponds to the limit case where no harm at all is incurred to the system (assuming harm is never negative).

Yet, one can imagine that the dynamics of system in a currently desirable state may drag it into a much less desirable one. This is the case for a healthy-looking economy or ecosystem right before it crashes, when the “healthy” property defined by a low unemployment rate or a high biodiversity still holds. Thus, the vulnerability of the properties is not only related to the states for which these properties are met, but also to the possible evolutions of the system.

The equivalent of the constraint set when considering a dynamic harm – related to a trajectory – is the LVK, noted $LVK(V; v, \tau)$. This is the set of initial states such that there exists an action strategy σ such that the value of the vulnerability indicator V

Static (at date t)		Dynamic (over the period $[0,\tau]$)	
Name	Notation	Name	Notation
Decision	u_t	(Action) Strategy	σ
Uncertainty	w_t	Scenario	ω
State	x_t	Trajectory	$\theta(x_0,\sigma,\omega,\tau)$
Static harm	$h_t(x)$	Harm	$H(x_0,\sigma,\omega,\tau)$
Constraint set	$K_t(l)$	Low-vulnerability kernel	$LVK(V;\nu,\tau)$

TABLE 4.1 – Summary of static notions, taken at date t, and their dynamic conceptual equivalents, spanning the entire period $[0,\tau]$.

remains below a threshold value ν with the time horizon τ :

$$LVK(V;\nu,\tau)=\{x_0\in\mathbb{X},\exists\sigma\in\Sigma(\tau),V(x_0,\sigma,\tau)\leq\nu\} \tag{4.9}$$

Some trajectories originating from a state belonging to a LVK may have very high harm values, yet vulnerability, understood as a relevant statistic over the distribution of harm values, is low under an appropriate strategy. Equation (4.9) effectively defines the states for which future harm is low : $LVK(V;\nu,\tau)$ is a dynamic equivalent to $K_t(l)$. Static notions and their dynamic equivalents are summarized in Table 4.1.

4.3 Examples of vulnerability indicators

We give a few examples of harm and vulnerability functions, then focus on how SDP can help find action strategies that minimize vulnerability in some situations, on how this impacts the computation vulnerability, whether specific to a hazard or not. From now on we assume the existence of static harm functions $h_t(x)$ which associate a harm value to each state and date, from the initial date to the final date τ.

4.3.1 Example of harm functions

The harm indicators given in this Section are common in the field of water resources planning and management. This is to our knowledge the only field where vulnerability statistics have been systematically evaluated from values taken along a trajectory, since the original definition of vulnerability for a water supply system by Hashimoto *et al.* (1982). Consequently, their relevance has been illustrated by over thirty years of practice (Moy *et al.*, 1986; Loucks, 1997; Kjeldsen et Rosbjerg, 2004; Sandoval-Solis *et al.*, 2011), the main difference being that vulnerability indicators are evaluated over a so-called failure period in the water resources literature, while we consider a fixed time frame $[0, \tau]$.

Maybe the most basic way to assess harm along a trajectory is by using a boolean H_l which value depends on whether static harm is kept below a threshold level at all times :

$$H_l(x_0, \sigma, \omega, \tau) = \begin{cases} 1 & \text{if} \quad \exists t \in [0, \tau], h_t(x_0, \sigma, \omega) > l \\ 0 & \text{if} \quad \forall t \in [0, \tau], h_t(x_0, \sigma, \omega) \leq l \end{cases} \tag{4.10}$$

so that H_l is in fact a binary (or Bernoulli) random variable. Then, another straightforward measure of harm is H_S, the sum of the static harm values h_t :

$$H_S(x_0, \sigma, \omega, \tau) = \sum_{t=0}^{\tau} h_t(x_0, \sigma, \omega)) \tag{4.11}$$

while an alternative may be to measure harm severity as the maximal value of static harm (Hashimoto *et al.*, 1982) along the trajectory, thus defining H_M :

$$H_M(x_0, \sigma, \omega, \tau) = \max_{t \in [0, \tau]} h_t(x_0, \sigma, \omega) \tag{4.12}$$

4.3.2 Examples of vulnerability indicators

Sandoval-Solis *et al.* (2011) enumerate possible vulnerability indicators as the expected value of the three above harm indicators. Using equation (4.5) we can set the expected vulnerabilities V_S^E and V_M^E :

$$V_S^E(x_0, \sigma, \tau) = \int_{\omega \in \Omega} H_S(x_0, \sigma, \omega, \tau) p(\omega) d\omega \tag{4.13}$$

$$V_M^E(x_0, \sigma, \tau) = \int_{\omega \in \Omega} H_M(x_0, \sigma, \omega, \tau) p(\omega) d\omega \tag{4.14}$$

while V_l^E is the expected value of the binary variable H_l. Thus, it is also the probability that $H_l(x_0, \sigma, \omega, \tau) = 1$:

$$V_l^E(x_0, \sigma, \tau) = \mathbb{P}(H_l(x_0, \sigma, \omega, \tau) = 1) \tag{4.15}$$

In this work V_l^E is called probability of being harmed, and it is sufficient to fully describe the distribution of the harm values $H_l(x_0, \sigma, \omega)$ along all the trajectories. Vulnerability as the probability for the system to evolve into a less desirable state has also been used outside of the field of water resources (e.g. Peterson, 2002).

Besides, Loucks (1997) gives two more formulas to evaluate worst-case scenarios, using the value-at-risk vulnerability indicator V^β from equation (4.6) with the harm

Type of vulnerability statistic	Binary harm H_l : crossing of a threshold	Sum of harm values H_S	Harm severity H_M
Expected value V^E	V_l^E	V_S^E	V_M^E
Value-at-risk V^β	–	V_S^β	V_M^β

TABLE 4.2 – Some possible vulnerability indicators.

functions H_S and H_M, so we can set V_S^β and V_M^β :

$$V_S^\beta(x_0,\sigma,\tau) = \min\left\{\eta \in \mathbb{R}^+ | \mathbb{P}(H_S(x_0,\sigma,\omega,\tau) \leq \eta) \geq \beta\right\} \tag{4.16}$$
$$V_M^\beta(x_0,\sigma,\tau) = \min\left\{\eta \in \mathbb{R}^+ | \mathbb{P}(H_M(x_0,\sigma,\omega,\tau) \leq \eta) \geq \beta\right\} \tag{4.17}$$

These possible indicators are summarized in Table 4.2. We are now to explore how in the cases of the vulnerability statistics V_l^E, V_M^β and V_S^E, SDP may be used to find the optimal strategy.

4.3.3 Searching for the optimal action strategy

SDP can be a way to determine the strategy that minimizes V for all the initial states, if it exists. Then, all the x_0 for which $V < v$ belong to the LVK. SDP needs the assumption, made from now on, that for any couple of different dates t_1 and t_2, the uncertainty vectors w_{t_1} and w_{t_2} are statistically independent from each other. SDP is a recursive algorithm carried out backwards from a final date to an initial date. Yet, only some of the vulnerability statistics introduced in Section 4.3.2 can be minimized, namely the probability V_l^E of being harmed, the value-at-risk V_M^β of the harm severity function H_M, and the expected value V_T^E of the sum H_S of harm values along a trajectory.

Probability V_l^E of being harmed

For a given initial state x_0, we are concerned with finding strategies that keep V_l^E defined by equation (4.15) below a threshold level. In other words, we are concerned with finding the initial states such that there exists strategies that allow for maintaining a low level of static harm during $[0,\tau]$ with a confidence level β. Such a set is the stochastic viability kernel (Doyen et De Lara, 2010) defined as :

$$\text{Viab}(\beta,\tau) = \{x_0 \in K_0(l) | \exists \sigma \in \Sigma(\tau), \mathbb{P}[\forall t \in [0,\tau], \theta_t(x_0,\sigma,\omega) \in K_t(l)] \geq \beta\} \tag{4.18}$$

where τ is the decision horizon. Using equations (4.8), (4.10) and (4.15), the stochastic viability kernel can also be written as (see Appendix 4.7.1) :

$$\text{Viab}(\beta,\tau) = \left\{x_0 \in K_0(l) | \exists \sigma \in \Sigma(\tau), V_l^E(x_0,\sigma,\tau) \leq 1-\beta\right\} \tag{4.19}$$

so that using equation (4.9) we have :

$$LVK(V_l^E; 1-\beta,\tau) = \text{Viab}(\beta,\tau) \tag{4.20}$$

Thus, the LVK can be interpreted as a stochastic viability kernel, and stochastic viability methods can be used to find it.

From a computational perspective, Doyen et De Lara (2010) establish the link between stochastic viability and SDP. They propose a SDP algorithm that finds the strategy that maximizes the probability of keeping the properties of a system (see Appendix 4.8.1 for details). Therefore, this strategy σ^* can therefore minimize V_l^E through the same algorithm. One can then use σ^* to define the probability V_l^E of being harmed as a function of the initial state x_0 and final date τ alone :

$$V_l^E(x_0,\tau) = V_l^E(x_0,\sigma^*,\tau) \tag{4.21}$$

Value-at-risk V_M^β of harm severity

Here we are concerned with finding the kernel $LVK(V_M^\beta; v,\tau)$ for given v and τ. In fact, like for the probability V_l^E of being harmed, one can show (see Appendix 4.7.2) that the computation of this LVK is that of the stochastic viability kernel using the constraint sets $K_t(v)$ instead of $K_t(l)$ at each date t. In other words, the initial states x_0 such that there exists σ such that $V_M^\beta(x_0,\sigma,\tau) \leq v$ are those for which such a strategy makes the system viable, or lowly vulnerable (in the sense of V_l^E) if one uses $K_t(v)$ instead of $K_t(l)$.

Therefore, $LVK(V_M^\beta; v,\tau)$ can also be computed through SDP (see Appendix 4.8.1), and the algorithm will both yield the lowly vulnerable state in the sense of V_M^β and the strategy σ^* which minimizes vulnerability. Like for equation (4.21), the value-at-risk V_M^β of the harm severity function H_M can be defined as a function of x_0 alone :

$$V_M^\beta(x_0,\tau) = V_M^\beta(x_0,\sigma^*,\tau) \tag{4.22}$$

Expected value V_S^E of the sum of static harm values

$LVK(V_S^E; v, \tau)$ is the set of initial states x_0 such that there is a strategy σ that can maintain the expected sum of the damage below a threshold value v. Contrary to the previous LVKs, this one cannot be expressed as a stochastic viability kernel, yet it can be easily computed through SDP. This amounts to a classical example of cost minimization (e.g. Loucks et van Beek, 2005; De Lara et Doyen, 2008), detailed in Appendix 4.8.2, and we can yet again define :

$$V_T^E(x_0, \tau) = V_T^E(x_0, \sigma^*, \tau) \tag{4.23}$$

4.3.4 Consequences for vulnerability computations

As illustrated through examples in Section 4.3.3, any vulnerability indicator V can be expressed as a function of the initial state alone when the action strategy σ^* that minimizes V is known :

$$V(x_0, \tau) = V(x_0; \sigma^*, \tau) \tag{4.24}$$

This also has consequences for expressing vulnerability to a specific hazard Z. If the strategies σ_1^* and σ_2^* that respectively minimize V and V_Z are known, then using equation (4.24) one can write equation (4.7) to express specific vulnerability as a function of the initial state and hazard alone :

$$V(x_0, \tau; Z) = V_Z(x_0, \tau) - V(x_0, \tau) \tag{4.25}$$

Arguably, having vulnerability to a hazard not depending on two distinct action strategies makes the computation of indicators more relevant and less prone to debate.

4.4 Application

4.4.1 A simple lake eutrophication problem

We illustrate the proposed framework with the discrete-time lake eutrophication model by Carpenter *et al.* (1999) (C99 thereafter), and we use the following discrete-time system, where all the variables are dimensionless :

$$\begin{cases} P_{t+1} = P_t + (L_t + u_t)e^{W_t} - b_t P_t + \dfrac{P_t^q}{1 + P_t^q} \\ L_{t+1} = L_t + u_t \end{cases} \tag{4.26}$$

This corresponds to the formulation $x_{t+1} = f_t(x_t, u_t, w_t)$ from equation (4.1), where the state is $x_t = (P_t, L_t)$, the control is u_t and the uncertainty vector is $w(t) = (\mathbf{W}_t, b_t)$.

The first part of equation (4.26) is the same as Equation (8) from C99 and describes the dynamical evolution of the state variable P which is the quantity of phosphorus (Ph) in the lake. Its second part describes the evolution of the other state variable L, the excess Ph from human activities. L is bounded by $L_{\max}$, a value for which all the needs in Ph of these activities are met. It is controlled by u_t chosen within the set $U_t(x)$ of available decisions. Since the presence of lags in the decision-making is common and can result in a slow evolution of L_t (C99), Martin Martin (2004) uses a bound $U > 0$ on the value of u_t, so we have $U_t(x) = [-U, U]$.

The Ph input into the lake is $L_t e^{\mathbf{W}_t}$, it is stochastic because the soil stores Ph and acts as a buffer. The random variable $\mathbf{W}_t$ has a Student distribution with standard deviation σ_d and df degrees of freedom. b_t determines how fast Ph is eliminated in from the lake, for example as outflow.

Ph has been proved to be the main inducer of lake eutrophication (Schindler, 2006; Schindler *et al.*, 2008), so that Ph inputs must be controlled to avoid the ecological degradation of lakes (Carpenter, 2008). Eutrophication lowers water quality in the lakes and leads to ecosystem changes that tend to lower the value of ecosystem services from the lake, such as fishing or recreation. Thus, one has to balance Ph-producing economic activities with the ecological preservation of the lake. Vulnerability is assessed from harm over the period $[0, \tau]$.

Harm values can be associated to 1) being limited in the quantity of Ph that can be used for economic activities and 2) the presence of Ph in the lake. We choose to reuse the utility functions from Equations (4) and (5) from C99, since harm functions can be seen as the opposite of utility functions. Static harm can be expressed as the sum of 1) economic and 2) ecological static harm functions. Since these are stationary, static harm is noted $h(x)$ and under its simplest form, it is given by :

$$h(x) = \alpha P^2 + (L_{\max} - L) \tag{4.27}$$

so that α compares the ecological harm, which increases with P_t, and the economic harm, which decreases as L_t increases. Using static harm functions, sets $K_0(l) = K_1(l) = \cdots = K_\tau(l) = K(l)$ can be defined in the same way as in equation (4.8). We also derive the sum of static harm values $H_S(x_0, \sigma, \omega, \tau)$ (equation (4.11)) and the binary harm $H_{0.25}(x_0, \sigma, \omega, \tau)$ (equation (4.10)), which assesses whether the boundary of $K(0.25)$ is crossed. Both harm functions are computed for trajectories that are defined over $[0, \tau]$, with $\tau = 100$ time steps. A time step can be thought of as a year in the model.

In what follows, the parameter values are from Figures 11 and 12 from C99, with $q = 2$, $\alpha = 0.2$, $df = 10$ and $\sigma_d = 0.25$, while b_t is picked to be certain and constant in a given case. We will use $b = 0.60$ and $b = 0.51$. P and L are discretized over a grid, with resolutions $\Delta P = 0.01$ and $\Delta L = 0.001$. The range for L is $[0, L_{\max}]$ where we set $L_{\max} = 0.2$, a reasonable value according to the results from C99 (see for instance Figures 11 and 14 from that publication). For such values of L, P ranging over $[0, 3]$ enables to cover all the dynamics from equation (4.26).

Static harm is pictured in Figure 4.1. The lines of constant harm are almost parallel with the horizontal, which could suggest this choice of α leads to valuing ecological harm over economic harm. Yet, the attractors of the dynamic that are in the zone of lowest harm (for $b = 0.60$) correspond to low ecological health and some restrictions on excess Ph input L_t. Lowering the value of α, for instance to 0.05, would make $L_t = L_{\max}$ the attractor with lowest static harm, which is the best-case scenario for economic actors using Ph and the worst-case scenario for the lake ecology and the ecosystem services attached to it. In that respect $\alpha = 0.2$ better illustrates a compromise between ecological and economic requirements. $h(x) = 0.25$ is chosen as a threshold of harm.

4.4.2 Vulnerability to computable uncertainty

In this section we assume that $b = 0.60$ and $\tau = 100$, and we assess vulnerability to the variable input of Ph into the lake under this assumption. Unless stated otherwise, we use $U = 5 \times 10^{-3}$. We note $V^1 = V_S^E$ the expected value of future harm, and $V^2 = V_{0.25}^E$ the probability of crossing the threshold $h(x) = 0.25$.

One can compute (Figure 4.2) $V^1(x_0, \tau)$ using the strategy that minimizes the expected value of total harm along a trajectory, which we get using SDP and note σ_1^*. $V^1(x_0, \tau)$ ranges from 12.3 to 22.7, which reflects the fact that there is no way to simultaneously minimize ecological and economic harm at the same time. Hence, the LVK can be taken to be the starting states for which $V^1(x_0)$ is within a certain percentage of its minimal value in the state space. In Figure 4.2, $LVK(V^1; 13.5, \tau)$ is represented : it is the zone for which $V^1(x_0)$ is within 10% of its minimal value. One can notice the extension of the LVK in the state space. In reality trajectories are converging towards an approximately rectangular zone, with L comprised between 0.08 and 0.1, and P between 0.1 and 0.6, as illustrated by two of the trajectories of Figure 4.3. This is were the attractors of the system where static harm is the lowest are situated.

One can also compute $V^2(x_0, \tau)$ with SDP, and thus minimize the probability of crossing the threshold $h(x) = 0.25$ (Figure 4.4). The strategy which minimizes $V^2(x_0, \sigma, \tau)$ is not the same as σ_1^*, so that it is relevant to understand how σ_1^* fares with respect of the objective of not crossing the threshold $h(x) = 0.25$. Results show

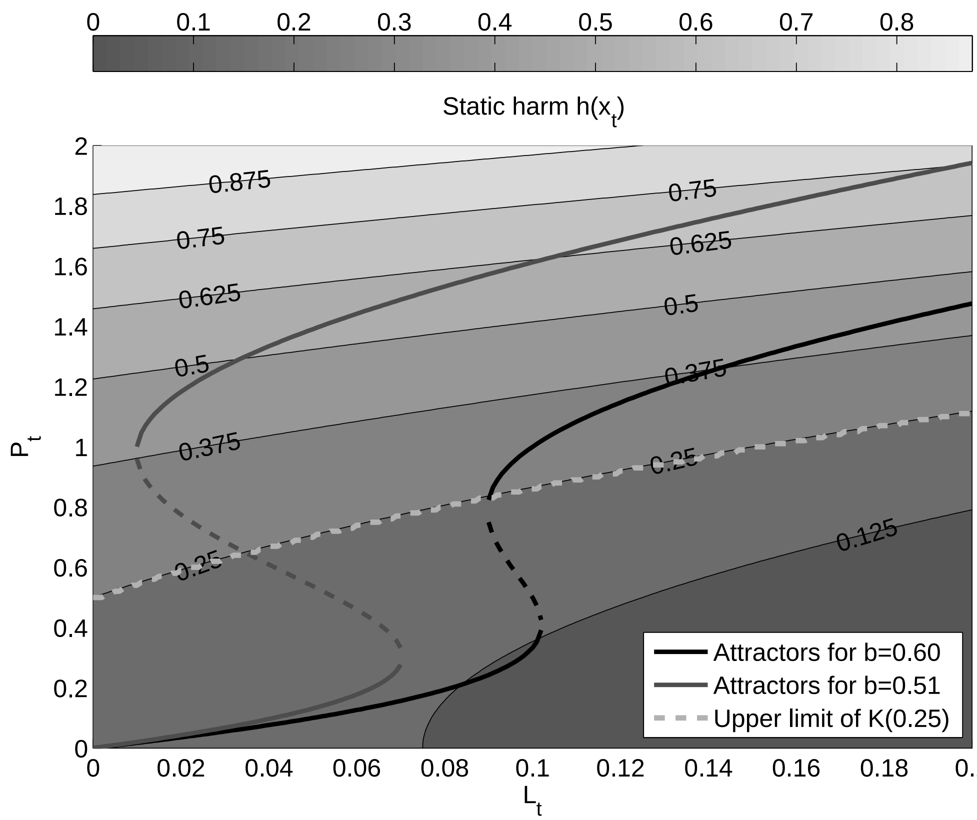

FIGURE 4.1 – Static harm, set $K(0.25)$ and attractors of the dynamic for $b = 0.60$ and $b = 0.51$. They represent the states P_t oscillates around for a given constant value of L_t.

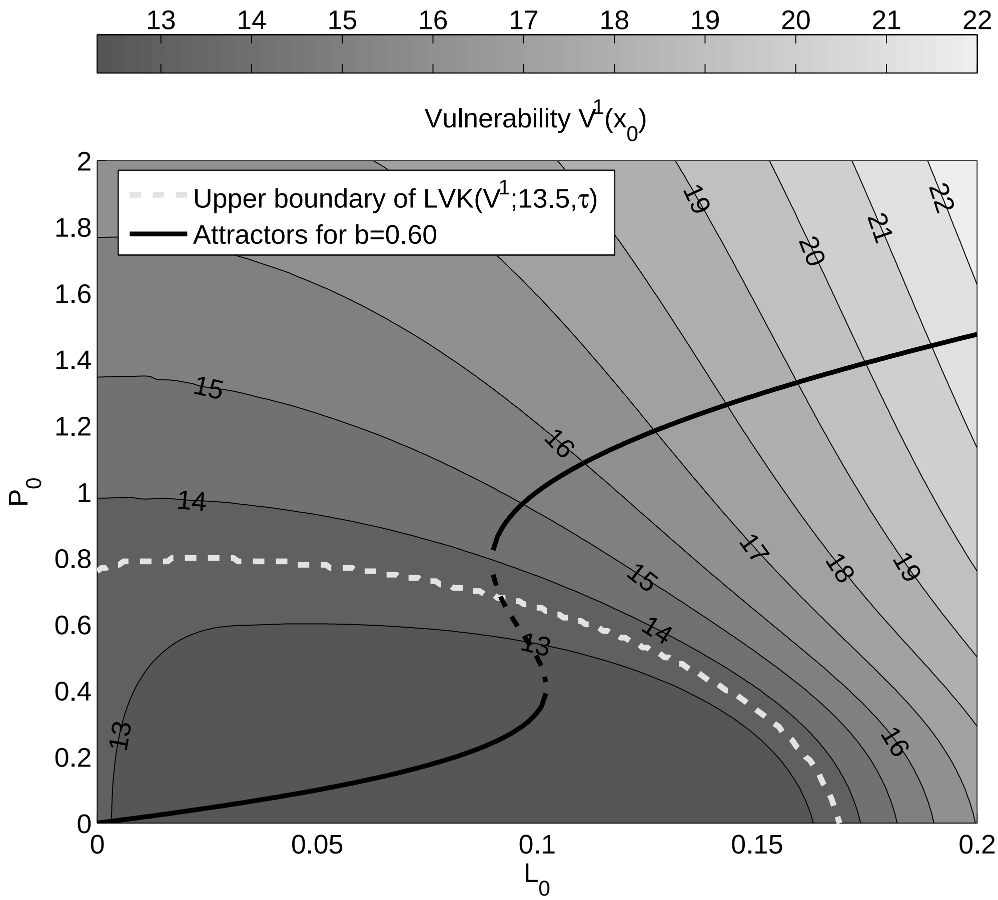

FIGURE 4.2 – Vulnerability as the expected value of total harm, $V^1(x_0,\tau) = V_T^E(x_0,\tau)$. Here $\tau = 100$.

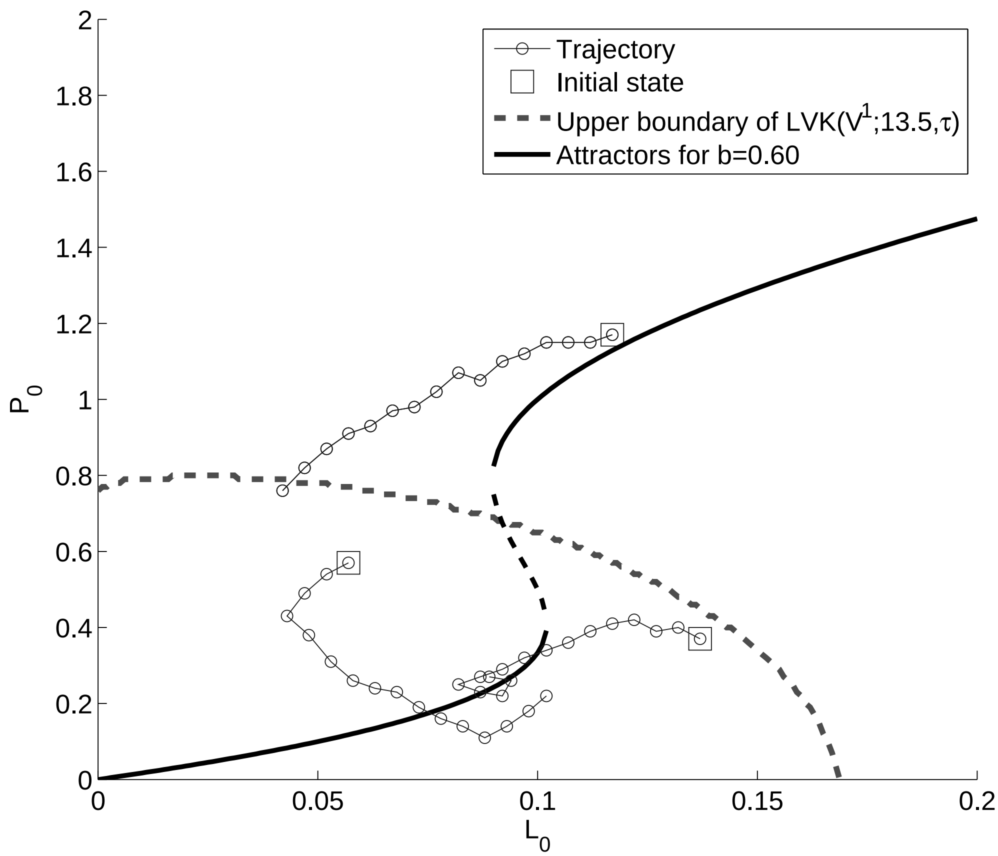

FIGURE 4.3 – Three examples of trajectories between $t = 0$ and $t = 15$, using the strategy σ_1^* that minimizes $V^1(x_0, \sigma, \tau)$. The state at each time step is represented by circles.

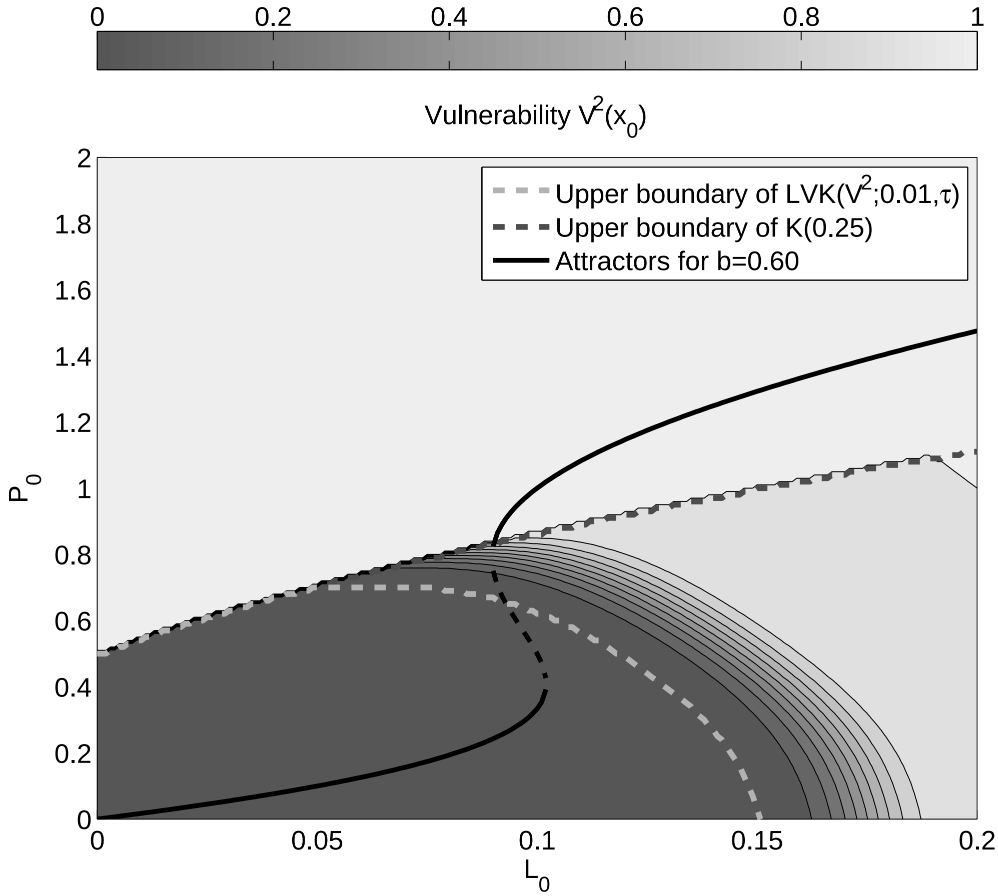

FIGURE 4.4 – Vulnerability as the probability of leaving $K(0.25)$, or equivalently, as the probability that the maximal value of static harm along the trajectory (static harm) exceeds 0.25. By definition $V^2(x_0, \tau) = 1$ beyond the threshold ($\tau = 100$).

(Figure 4.5) that the extent of the initial states such that $V^2(x_0, \sigma_1^*, \tau) \leq 0.01$ is almost that of $LVK(V^2; 0.01, \tau)$. Thus, σ_1^* can be used to achieve both objectives of minimizing expected future harm, and of not crossing a threshold.

One can also look at how the value of U, which determines how much L can be modified at each time step (Figure 4.6), affects vulnerability. The lower U, the more important the benefits associated with increasing U by 1×10^{-3} are in term of vulnerability reduction. Conversely, when U is above 5×10^{-3}, the benefits associated with increasing its value to 10×10^{-3} are marginal. If one considers that increasing U is a form of adaptation, then its effectiveness hinges on the value of U before that increase.

4.4.3 Specific vulnerability to a change in b

Global change can potentially affect a lake in a number of ways (e.g. Beklioglu *et al.*, 2007; Schindler *et al.*, 2008; Jeppesen *et al.*, 2009). For the purpose of this illustra-

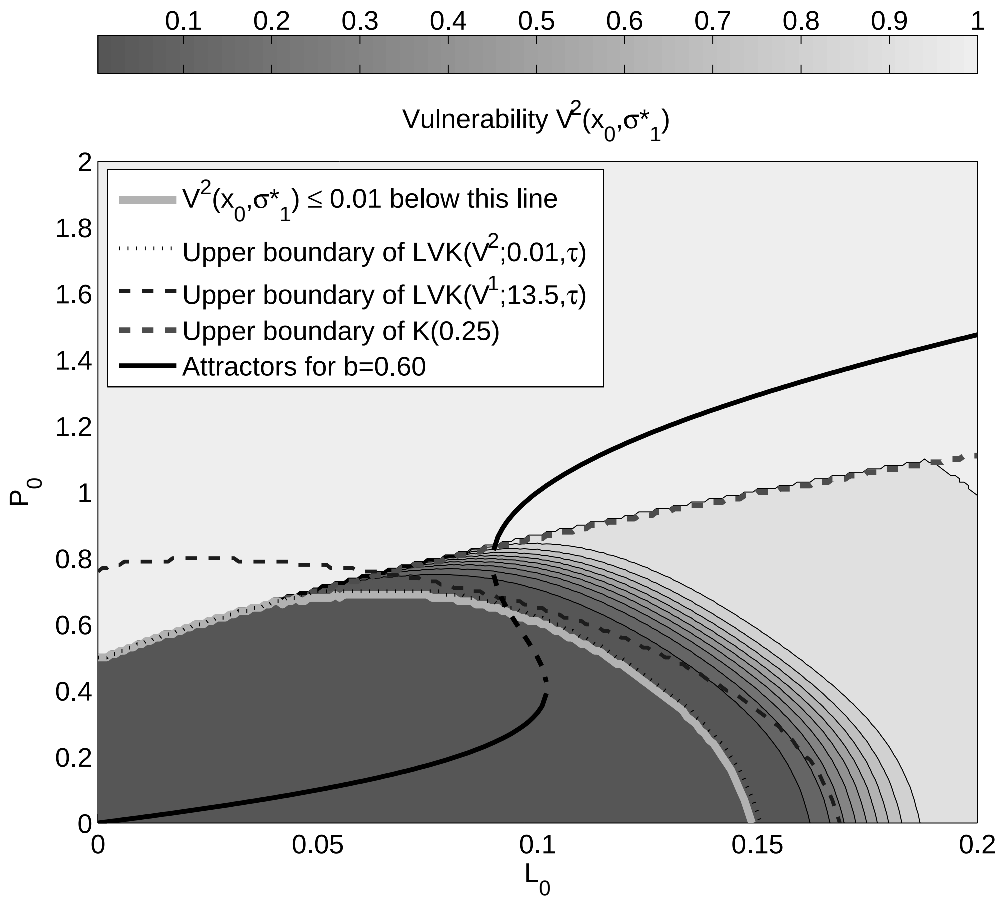

FIGURE 4.5 – Probability of leaving $K(0.25)$ if σ_1^*, the strategy which minimizes $V^1(x_0, \sigma, \tau)$, is being used ($\tau = 100$).

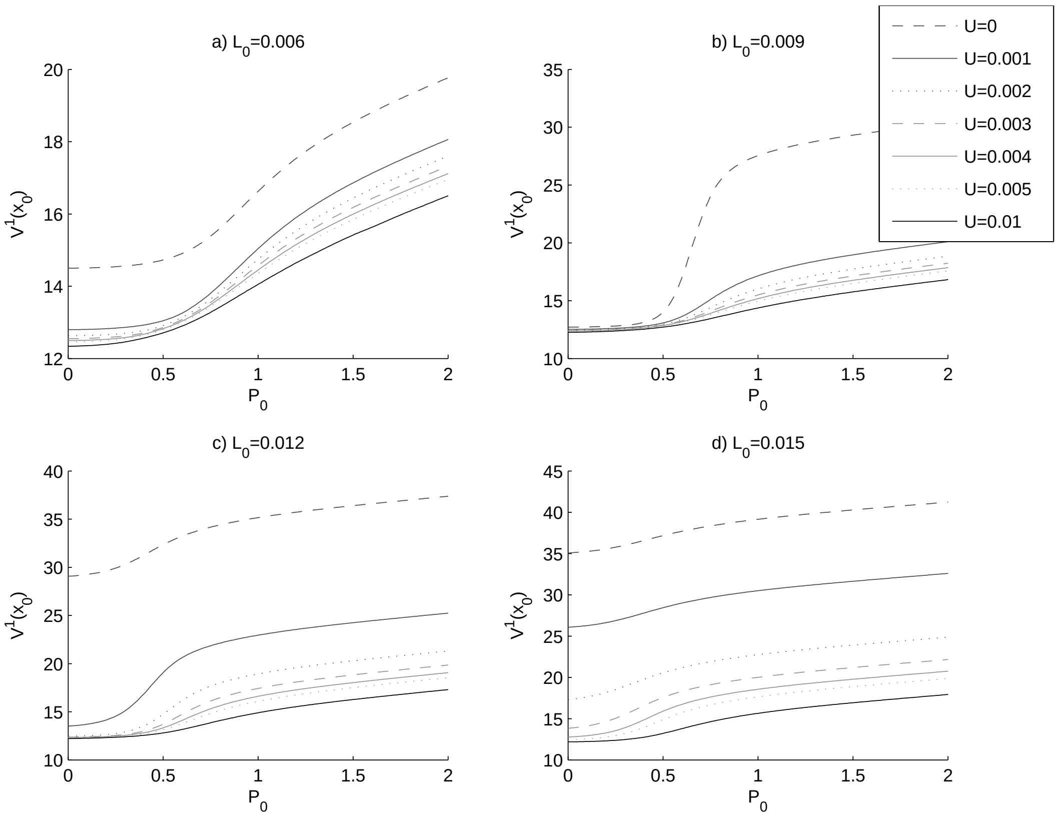

FIGURE 4.6 – Influence of U on $V^1(x_0, \tau)$ for different values of $x_0 = (P_0, L_0)$, with $\tau = 100$.

tion of vulnerability concepts on an very simple model, we focus on the impacts of possible modifications of the rainfall regime. Lower precipitation may affect the outflow from the lake, which lowers b, but also the Ph inputs if it is accompanied of rainfall events of lower intensity (Schindler *et al.*, 2008). However, Ph is disproportionately released from the soil into the lake by important runoff events (Sharpley *et al.*, 2008; Rodríguez-Blanco *et al.*, 2013), and more extreme precipitation is expected Jeppesen *et al.* (2009), so that more extreme events may balance the lower quantity of total precipitation when it comes to releasing Ph into the lake. As a result, we simply explore the consequences of a shift Z in b, namely a 15% decrease. This changes the uncertainty vector $w_t = (\mathbf{W}_t, 0.60)$ into $(\mathbf{W}_t, 0.51)$. Again, $\tau = 100$, and $U = 5 \times 10^{-3}$ unless stated otherwise.

Because of Z, the dynamic of the system is modified, and therefore the indicators V^1 and V^2 respectively become V_Z^1 and V_Z^2. The strategy that minimizes the expected value of total harm V_Z^1 is noted u_1^+. The vulnerability to the shift Z can be given by $V^1(x_0, \tau; Z) = V_Z^1(x_0, \tau) - V^1(x_0, \tau)$ (Figure 4.7). It is always positive and takes very high values if L_0 is high at the time of the shift. This is true in particular in some regions of $LVK(V^1; 13.5, \tau)$. $V_Z^1(x_0)$ has a minimum of 14.48, which is why we derive a new LVK, namely $LVK(V_Z^1; 14.9, \tau)$, which represents the zone in which vulnerability to computable uncertainty after the change is within 10% of its new minimal value. In fact, the least-harm strategy brings the state of the system towards the new attractors for which P is low. This supposes a decrease in L, and an economic vulnerability to Z – provided a policy that balances ecological and economic harm is applied.

As for u_1^* in the case $b = 0.60$, one can test the impact of u_1^+ on the probability of crossing the threshold $h(x) = 0.25$. The strategy u_1^+ is found to be near optimal for minimizing V_Z^2 so it can be used for both purposes of minimizing the expected value of harm within the long run, as well as the probability of crossing the threshold. However, vulnerability as the added probability of crossing after Z occurs is high in a zone of the state space which was safe as long as $b = 0.60$ (Figure 4.8).

Finally, one can assess vulnerability to Z for different values of U (Figure 4.9). This supposes to know, for each value of U, the strategies u_1^* and u_1^+ that respectively minimize $V^1(x_0, \sigma, \tau)$ and $V_Z^1(x_0, \sigma, \tau)$. Figure 4.9 shows the impact of U on the specific vulnerability to Z for states in which the lake may well be after having been managed for a long time under $b = 0.60$ using u_1^*. It shows that the effect of increasing U, which was almost null for $U \geq 5 \times 10^{-3}$ before the shift Z, can become important when the shift occurs. This is the case for V^1 for $P_0 = 0.45$ and 0.60, and for V^2 if $P_0 = 0.45$. However, it is interesting to note that even $U = 0.01$ cannot prevent the system from crossing the threshold if $x_0 = (0.6, 0.09)$. Thus, adapting by anticipating

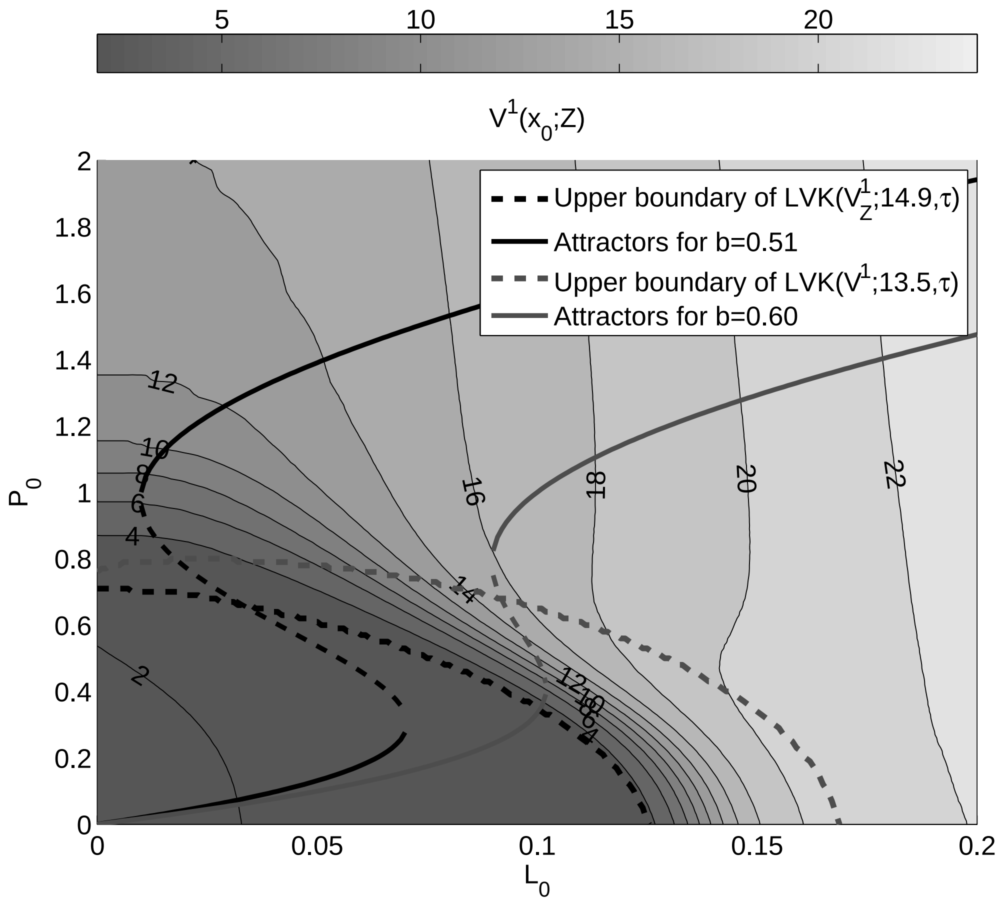

FIGURE 4.7 – Vulnerability $V^1(x_0,\tau;Z) = V_Z^1(x_0,\tau) - V^1(x_0,\tau)$ to a 15% decrease in b, and incidence on the LVK ($\tau = 100$).

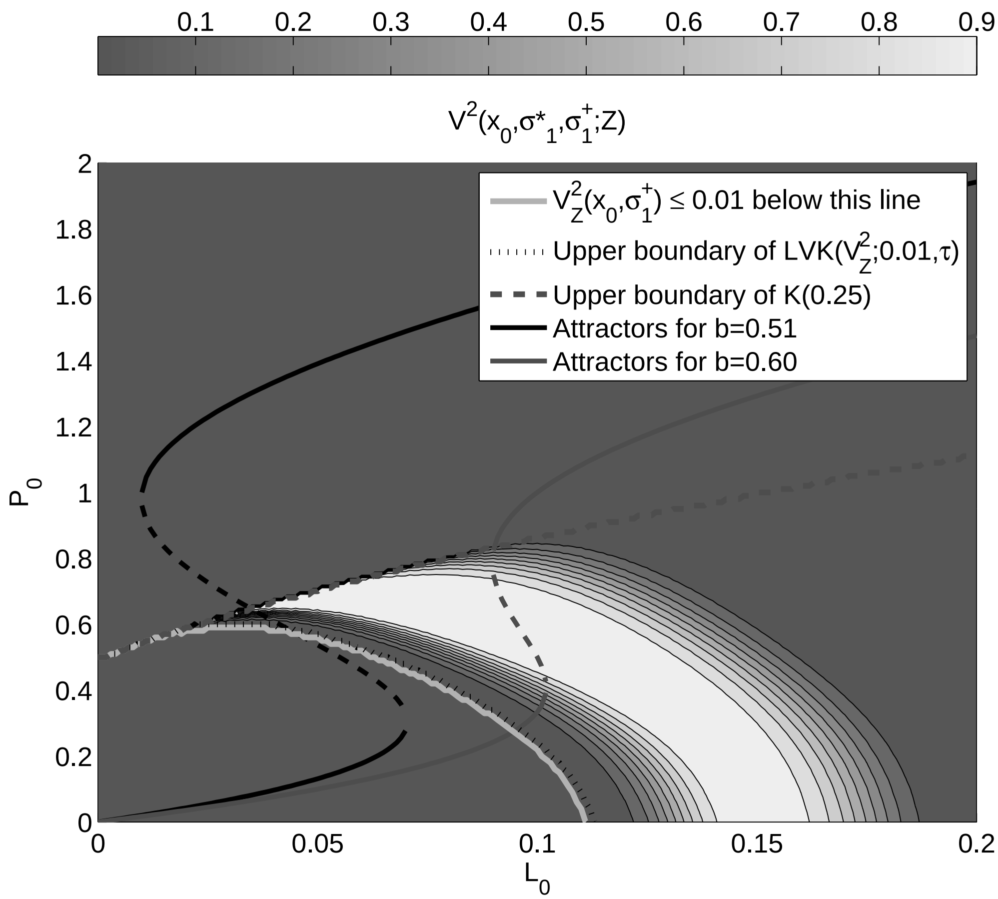

FIGURE 4.8 – Vulnerability to a 15% decrease in b as the added probability of crossing the threshold $h(x) = 0.25$, assuming the strategies used are σ_1^* and σ_1^+ which respectively minimize $V^1(x_0, \tau)$ and $V_Z^1(x_0, \tau)$, with $\tau = 100$.

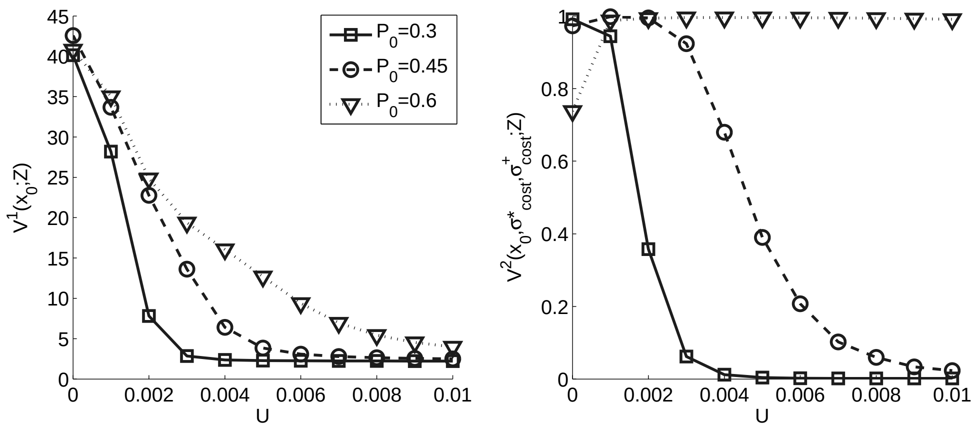

FIGURE 4.9 – Vulnerability to a 15% decrease in b for different initial states (P_0, L_0) such that $L_0 = 0.09$, and for different values of U. For each U, the strategies used are those which respectively minimize $V^1(x_0, \tau)$ and $V^1_Z(x_0, \tau)$, with $\tau = 100$.

on Z by increasing U can sometimes be very beneficial, but this also depends on the chosen indicator.

4.5 Discussion

This discussion is articulated around the concepts vulnerability is related to. The respective links of vulnerability with adaptation and with risk shall be discussed, before tackling the relationship between vulnerability and its so-called components.

4.5.1 Adaptation and adaptive capacity indicators

The ability to act is described through the ability of these actions to have beneficial consequences, and this has been designated in many different ways : coping capacity, ability to cope, ability to adjust, adaptation, transformation, adaptive capacity. . .Even though there is to date no comprehensive framework precisely defining these terms together, a few guidelines can be extracted from the literature. Thus, adaptation refers to an answer to potential future harm, be it anticipatory adaptation to circumstances that have not yet been met, or reactive adaptation undertaken so that past harmful events may not happen again (Fankhauser *et al.*, 1999). The notion of success in at least alleviating harm is implicit. Adaptation can be interpreted as a proof that the system has adaptive capacity (Smit et Wandel, 2006), the latter being measurable as the vulnerability reduction achieved through a modification of the system (Luers *et al.*, 2003; Sandoval-Solis *et al.*, 2011).

Besides, this framework can foster a discussion on the distinction between adaptation and coping, made for instance by the 2012 IPCC report (Field *et al.*, 2012, pp. 51 and 72). This distinction is implicit in concept such as that of coping range, the range of perturbations a system can cope with, and which can be dynamically influenced by climatic changes but also by adaptation (Smit *et al.*, 2000; Smit et Pilifosova, 2003; Smit et Wandel, 2006) Thus, coping is described as being the immediate reaction to a hazard, while adaptation can also refer to action undertaken before the hazard hits. The aim of adaptation is to increase the capacity to cope to a particular event or change (Smit *et al.*, 2000; Yohe et Tol, 2002), and it takes place over longer time scales than coping (Smit et Wandel, 2006). Even though this distinction can be ambiguous, since coping can be seen as a form of emergency adaptation (Turner II *et al.*, 2003), coping can be related to control strategies in the dynamical system framework we propose.

Indeed the strategy used to cope with computable uncertainty – or with a specific hazards – is chosen among a set of already available strategies, and implementing it supposes no change in the system, nor in the available strategy or in the definition of harm. By contrast, adaptation can be thought of as through such modifications. For instance, in Section 4.4 one can relate an increase in U with adaptation, and this changes the way the system can cope with both computable events, and a specific event Z.

Adaptation can either be planned or autonomous (Fankhauser *et al.*, 1999). Planned adaptation is instigated with the purpose of lowering vulnerability, and it can result in a change in any of the system's parameters, or in a change of the set $\Sigma(\tau)$ of available action strategies, in a reduction of the uncertainty (affecting ω), or in a change in the shape or size of the constraint set. Autonomous adaptation can also result in any of the above, and may also arise from the dynamic itself. It is then a phenomenon that is not captured by equation (4.1), which generally implies that the dynamical system representation should be updated. Such surprises are unexpected phenomena that can arise either from the environment, the public or the institutions (Janssen, 2002) and while the literature is most concerned with the negative ones (and rightly so), they can also be positive.

Since adaptation is a manifestation of adaptive capacity, adaptive capacity can be demonstrated when vulnerability is reduced as the result of an adaptation A, which can be any of the changes described in the previous paragraph. Similar to the case of specific vulnerability, the merits of adaptation are best judged when vulnerability statistic V is chosen to measure it. The consequences of the adaptation A modify the vulnerability function $V(x_0, \sigma, \tau)$ into a new function $V_A(x_0, \sigma, \tau)$. Following up on the analogy with specific vulnerability, for any two strategies σ_1 and σ_2, the adaptive capacity C_A demonstrated by the adaptation A can be measured at the initial state x_0 :

$$C_A(x_0, \sigma_1, \sigma_2, \tau; A) = V(x_0, \sigma_1, \tau) - V_A(x_0, \sigma_2, \tau) \tag{4.28}$$

so that adaptive capacity is positive, provided adaptation did lower vulnerability. This definition is analogous to that of Luers *et al.* (2003), yet more precise since it connects adaptive capacity with the implementation of definite coping strategies both before and after adaptation. The proposed adaptive capacity indicator judges adaptive capacity on its efficiency in lowering a given vulnerability indicator. Other frameworks define instead adaptive capacity based on its determinants (Yohe et Tol, 2002), or as the inventory of the resources that can be allocated to adaptation (Nelson *et al.*, 2007; McDowell et Hess, 2012). These evaluation criteria are complimentary, since the former focuses on potential results while the latter emphasizes the causes that make adaptations possible.

Section 4.3.3 has shown that there are cases in which SDP techniques can find the action strategy that minimizes the vulnerability indicator. If σ_1 and σ_2 in equation (4.28) are the strategies that respectively minimize V and V_A then we can omit the strategies in the expression of the capacity C_A demonstrated by the adaptation A :

$$C_A(x_0, \tau; A) = V(x_0, \tau) - V_A(x_0, \tau) \tag{4.29}$$

One could use these formulas to assess adaptive capacity associated to an increase of U in the case of the lake : it is null when vulnerability does not decrease, but it can be very high in other cases, for instance when U is doubled from 10^{-3} to 2×10^{-3}. Yet, some management policies are called maladaptative since they increase vulnerability (Burton, 1997; Smit *et al.*, 2000). This leads to a negative adaptive capacity, one can state instead that there is a vulnerability to these policies, and use equation (4.7) to quantify it.

Some frameworks (e.g. Turner II *et al.*, 2003) rather use the concept of resilience to describe the capacity of a system to cope and adapt. In reality, resilience in its own is a long-standing concept (Holling, 1973), which has given way to a large body of literature in many fields (Brand et Jax, 2007). Vulnerability and resilience, due to their distinct disciplinary origins, conceptual center of interests, and methodological approaches, seem to be strongly complementary concepts (Miller *et al.*, 2010). This complementarity is yet to be fully explored and lies outside the scope of this work, but it could be achieved through the integration of both vulnerability and resilience in the same dynamical system framework. Indeed, a dynamical system formulation that uses concepts and tools from viability theory have also been used in the resilience literature

(Martin, 2004; Deffuant et Gilbert, 2011; Rougé *et al.*, 2013).

4.5.2 Vulnerability and risk

Besides all of its so-called components, vulnerability is often associated to the notion of risk. Risk is commonly defined as the convolution of the cost and pdf. It has long been related to vulnerability (Turner II *et al.*, 2003), even though there does not seem to be a clear consensus on how exactly the two notions are situated with respect to one another. It is common to consider that risk arises from the interplay of a hazard and of a system's vulnerability (Wamsler *et al.*, 2012). Indeed a hazard, in itself, does not cause harm or damage, and it is rather its interplay with vulnerable properties of a system that causes harm.

Arguably, the difference between risk and vulnerability is that computing a risk requires the knowledge of the probability of occurrence of a hazard. This can prove challenging since major hazard events are by definition extreme and rare, so that the estimation of their return period can be very uncertain and heavily dependent on the pdf used to approximate them (Esteves, 2013) ; and it has been shown that providing such estimates is far more perilous in the context of a changing climate (Felici *et al.*, 2007a,b). Vulnerability, by contrast, only requires the knowledge of computable uncertainty, represented by $\omega(t)$ in equation 4.1, as opposed to uncomputable uncertainty, be it extreme events or mechanisms designated by the expression "uncertainty and surprise" (Folke *et al.*, 2002b, 2004; Adger *et al.*, 2005) because their impact on the system is not anticipated. Specific vulnerability assessments allow for exploring the impacts of such uncertainties as scenarii, without requiring that return periods be calculated. Scenario planning has been proved to be a fitting tool to explore large uncertainties associated to climate change (Allen *et al.*, 2011; Cobb et Thompson, 2012) and the proposed framework for vulnerability may result useful in evaluating scenario outcomes.

4.5.3 Components of vulnerability

Vulnerability has often been expressed as a function of its three perceived main components, most notably exposure, sensitivity, and adaptive capacity or resilience, (e.g. Luers *et al.*, 2003; Turner II *et al.*, 2003; Luers, 2005; Parry *et al.*, 2007). However, it has been argued that the use of vulnerability-related concepts is largely a matter of which disciplinary point of view is taken on a problem (Miller *et al.*, 2010), but also that confronting different viewpoints on vulnerability could be warranted in order to have a comprehensive vision of a given situation (Fuchs, 2009). For instance, Li *et al.*

(2013) mention a total of five ways to describe vulnerability to floods as a function of three components. Hence the choice made in this work not to write vulnerability explicitly as a function of related concepts, in order to keep the proposed framework general.

Nevertheless, links between vulnerability and related concepts can sometimes be inferred from this framework. For instance, sensitivity can be measured as the impact of a perturbation on the state of the system (e.g. Luers, 2005). Yet, even though this impact may be immediate in the case of an extreme event, one may want to instead consider sensitivity based on how the whole trajectory of the system is modified following the hazard's occurrence. The necessity of considering many possible future trajectories may be a challenge in measuring sensitivity, and our framework shows that in fact, it is not a prerequisite to measuring vulnerability.

As for exposure, it is defined based on the inventory of elements that may be adversely impacted by an event or change (Cardona *et al.*, 2012). Thus, an element is exposed when it there is both a value judgment on the adverse impacts associated to its possible states, and when hazards or uncertainties may affect its evolution. In that respect, defining harm functions is closely related to determining a system's exposure, and only the sensitivity of the system following a hazard event may determine if the exposed elements can truly be harmed. In that sense our framework illustrates the common notion (Luers, 2005; Smit et Wandel, 2006) that exposure and sensitivity only work in tandem.

4.6 Conclusions and perspectives

Hinkel (2011) contends that vulnerability indicators are mainly fit for identifying who may be vulnerable and where. A dynamic systems perspective on the matter may transcend this grim diagnosis by fostering the development of fully dynamic indicators at the interface between the representation of social-ecological system and its management (Figure 4.10). The benefits of explicitly considering the temporal dimension of vulnerability are enumerated hereafter.

1) Vulnerability becomes a descriptive concept based on a system representation, which comprises both a description of the system's dynamics and the normative choices which were made when assigning a harm value to each possible trajectory. This clarification of the normative and descriptive aspects is arguably a clarification of the science-policy interface as well.

2) The choice of the indicator itself is normative. For instance, choosing between an expected value of future harm, or a value-at-risk – related to the probability of crossing

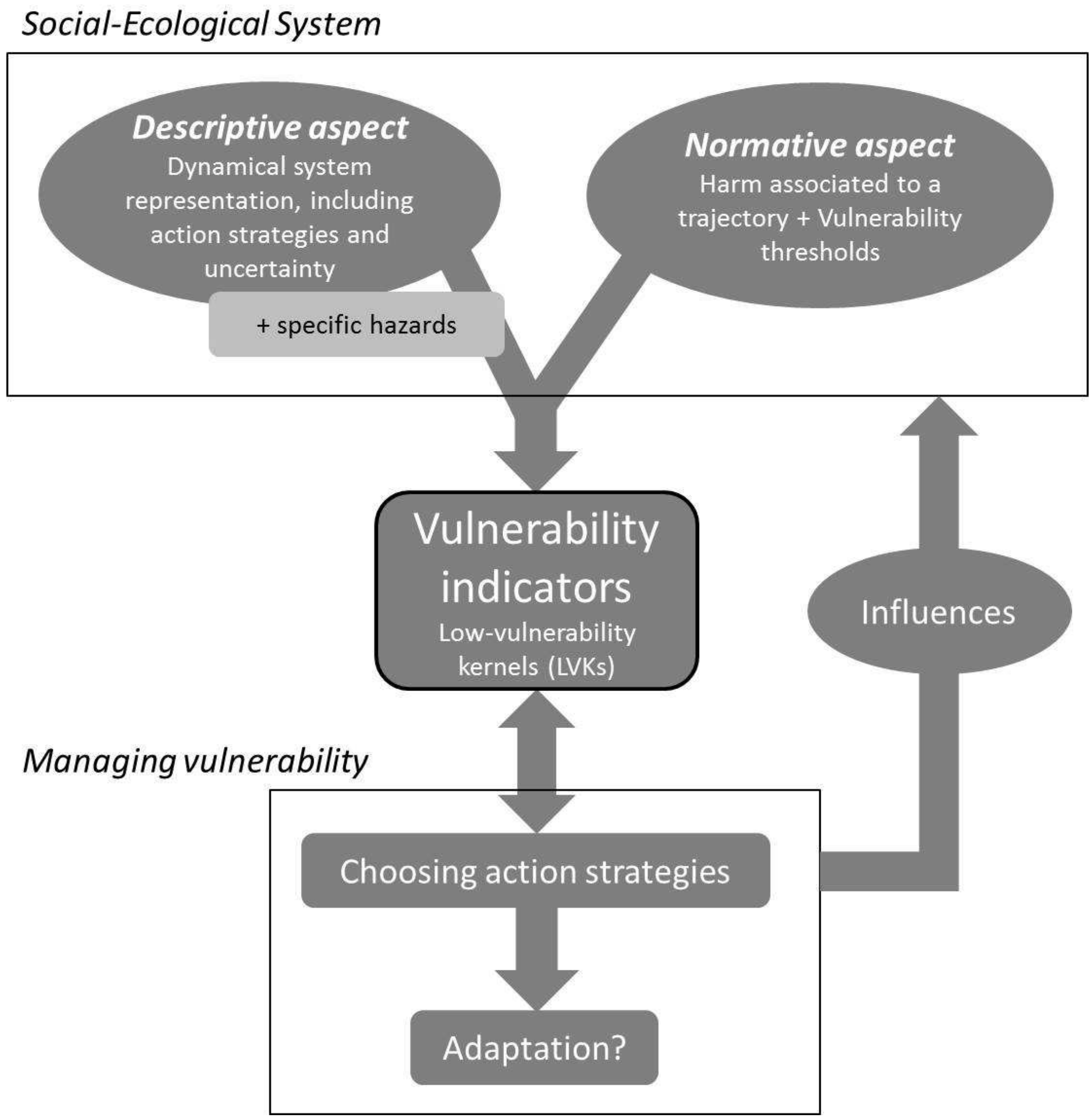

FIGURE 4.10 – Schematic representation of the proposed framework, which replaces vulnerability indicators at the interface between a social-ecological system and its management.

a threshold of harm – is motivated by different policy objectives. This difference is hard to make explicit unless trajectories are used.

3) An indicator becomes associated to a given combination of computable events and of specific extreme hazards. This clarifies which events the assessment, and therefore the vulnerability indicators, do not take into account. This link between considered hazards and indicator values is crucial when it comes to make policy decisions.

4) A fully dynamic framework is a prerequisite to integrating the impact of strategies in the assessment of vulnerability indicators, so as to understand how to best cope with a given situation. Even though a closed set of equations, let alone a way to optimize the implemented strategy, may not be available often in practice, it should be kept in mind that strategies dynamically influence vulnerability.

5) Eventually, using a stochastic controlled dynamical system perspective is a way to understand under which conditions a proposed adaptation measure may prove effective, since its success is shown to be related to the dynamic, the considered events, and the already available coping strategies.

Acknowledgements

This work would not have been possible without funding from Région Auvergne.

4.7 Appendix : Stochastic viability kernels as LVKs

4.7.1 Proof of equation (4.19)

Recall the definition of the stochastic viability kernel for a decision horizon τ and a threshold confidence level β :

$$\mathrm{Viab}(\beta,\tau)=\{x_0\in K_0(l)|\exists\sigma\in\Sigma(\tau),\mathbb{P}[\forall t\in[0,\tau],\theta_t(x_0,\sigma,\omega)\in K_t(l)]\geq\beta\} \tag{4.30}$$

From equation (4.8) we have the following equivalence between the situation within the constraint set and the value of the static harm function :

$$[\forall t\in[0,\tau],\theta_t(x_0,\sigma,\omega)\in K_t(l)]\Leftrightarrow[\forall t\in[0,\tau],h_t(x_0,\sigma,\omega)\leq l] \tag{4.31}$$

and this equation, taken along with the definition of $H_l(x_0,\sigma,\omega)$ in equation (4.10), yields :

$$[\forall t\in[0,\tau],\theta_t(x_0,\sigma,\omega)\in K_t(l)]\Leftrightarrow[H_l(x_0,\sigma,\omega)\leq l] \tag{4.32}$$

Then, injecting this latter result into equation (4.30) and using the definition of the probability of vulnerability (equation (4.15)) directly leads to equation (4.19).

4.7.2 $LVK(V_M^{\beta};v,\tau)$ as a stochastic viability kernel

By definitions of $LVK(V_M^{\beta};v,\tau)$ in equation (4.9) and V_M^{β} in equation (4.17) :

$$LVK(V_M^{\beta};v,\tau)=\{x_0\in\mathbb{X}|\exists\sigma\in\Sigma(\tau),\mathbb{P}(H_M(x_0,\sigma,\omega)\leq v)\geq\beta\} \tag{4.33}$$

Then, using the definition of $H_M(x_0,\sigma,\omega)$ from equation (4.12), equation (4.33) becomes :

$$LVK(V_M^{\beta};v,\tau)=\{x_0\in\mathbb{X}|\exists\sigma\in\Sigma(\tau),\mathbb{P}[\forall t\in[0,\tau],h_t(x_0,\sigma,\omega)\leq v]\geq\beta\} \tag{4.34}$$

so that introducing the set of states $K_t(v)=\{x\in\mathbb{X}|h_t(x)\leq v\}$ such that the static harm function is below the threshold value v leads to :

$$LVK(V_M^{\beta};v,\tau)=\{x_0\in\mathbb{X}|\exists\sigma\in\Sigma(\tau),\mathbb{P}[\forall t\in[0,\tau],\theta_t(x_0,\sigma,\omega)\in K_t(v)]\geq\beta\} \tag{4.35}$$

which corresponds to the definition of the stochastic viability kernel given by equation (4.18), solely replacing $K_t(l)$ with $K_t(v)$.

4.8 Appendix : Stochastic dynamic programming algorithms

4.8.1 Viability maximization

In what follows we use for simplicity a discretization of the state space into a discrete set which we note X. Thus, we can define the transition probability from any state x to any state y given the decision u. We note this function $p(x, y|u)$.

Let us have $K_t(l)$ as defined in equation (4.8), SDP works using a value function G which is initialized at the final date τ, then recursively updated backwards from τ to the initial date 0. Initialization reads :

$$G(T,x) = \begin{cases} 1 & \text{if} \quad x \in K_T(l) \\ 0 & \text{if} \quad x \notin K_T(l) \end{cases} \tag{4.36}$$

and the recursive transition equation is :

$$\forall t \in [0,\tau-1],\ G(t,x) = \max_{u \in U(t,x)} \left(\sum_{y \in K_t(l)} p(x,y|u)\ G(t+1,y) \right) \tag{4.37}$$

Doyen et De Lara (2010) prove that $G(0,x)$ was the maximal probability for the system to remain within $K_t(l)$ at all dates during $[0,\tau]$, and that therefore, the stochastic viability kernel is the set of states such that $G(0,x) \geq \beta$.

4.8.2 Cost minimization

We use the same discretization as in Appendix 4.8.1, as well as the notation $p(x, y|u)$. However, the value function G changes so that initialization now reads :

$$G(\tau,x) = h_\tau(x) \tag{4.38}$$

and the recursive transition equation is :

$$\forall t \in [0,\tau-1],\ G(t,x) = h_t(x) + \min_{u \in U(t,x)} \left(\sum_{y \in X} p(x,y|u)\ G(t+1,y) \right) \tag{4.39}$$

The reunion of equations (4.38) and (4.39) constitutes the well-known Bellman equation, and the associated optimality principle ensures that $G(0,x)$ is the lowest possible value for the expected vulnerability indicator V_S^E (for a proof, see for instance Section A.3 of De Lara et Doyen (2008)).

– 5 –

Performance indicators in water resources : a joint stochastic dynamic programming algorithm for reliability and vulnerability

En préparation.
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Three indicators, called reliability, resilience and vulnerability, are commonly used to describe the performance of a water supply system. This work focuses on reliability and vulnerability and proposes a novel way to optimize the expected value of an indicator which uses stochastic dynamic programming to combine both. Reliability is a measure of how often the system is in a satisfactory state. Vulnerability measures how severe or damaging a failure is, and several different indicators can be used. A composite index is then the sum of individual indicators, each of which can be optimized separately using stochastic dynamic programming, whether they express a cost or the probability of crossing a threshold. A change of variable enables the optimization of the composite index. An application to a single reservoir storing water for consumption illustrates the method, highlighting how it makes the trade-off between the two types of indicators explicit.

5.1 Introduction

It is common in the field of water resources management to describe the performance of a static property of interest – e.g. the minimal satisfactory value of water supply from a reservoir system – through a set of three indicators called reliability, resilience and vulnerability (Hashimoto *et al.*, 1982). This work aims at developing a composite indicator of reliability and vulnerability, and at proposing a stochastic dynamic programming (SDP) algorithm to optimize this composite indicator. It does not represent the third indicator, namely resilience. Indeed, resilience has been found to be redundant with vulnerability in some cases (Kundzewicz et Laski, 1995; McMahon *et al.*, 2006), so that it has been recommended to use only vulnerability and reliability (Kjeldsen et Rosbjerg, 2004), a recommendation followed by Jain (2010) when proposing a sustainability index. Hence our focus solely on reliability and vulnerability indicators.

SDP is a backward induction algorithm that is well-suited to optimize the expected value of a criterion defined for a trajectory of a system (De Lara et Doyen, 2008). It has for instance proved effective for cost minimization problems (e.g. Loucks et van Beek, 2005), and more recently, for the maximization of the probability of respecting state constraints during a given time horizon (Doyen et De Lara, 2010). The latter is also called a viability criterion, a term springing from a branch of control mathematics called viability theory (Aubin, 1991). SDP also yields the control decisions that maximize each type of criterion.

However, the decisions that minimize cost may not maximize viability, and *vice versa*. This is an important limitation, because then, using these two types of criteria together is only marginally useful : one is just warned about the existence of a trade-off between them, without having real means to quantify it. Besides, while minimizing cost is a common endeavor in many fields, avoiding the crossing of a threshold is an issue that is focused upon in various domains such as that of vulnerability assessments (e.g. Luers *et al.*, 2003; Eakin et Luers, 2006), or those studied through viability theory. The latter has been applied to diverse fields such as economics (Aubin, 2003), the management of harvested ecosystems (Béné *et al.*, 2001; Sabatier *et al.*, 2010), social sciences and material sciences (Deffuant et Gilbert, 2011). We show in Section 5.2 that SDP can be used to remove this hurdle, by minimizing a composite vulnerability indicator that is a linear combination of a cost criterion along with one or several viability criteria.

Such a composite indicator can then be related to reliability and vulnerability (Section 5.3), indirectly fostering their computation using SDP. Previous studies have only used SDP on one of these indicators, e.g. vulnerability understood as a cost (Raje et Mujumdar, 2010), or reliability (Nalbantis et Koutsoyiannis, 1997). Others have used the three performance indicators in parallel with optimization of another objective via SDP (e.g. Onta *et al.*, 1991). In general, these indicators have not been used by researchers and practitioners as an objective to optimize, but rather as a descriptive tool that helps understand trade-offs between different alternatives (e.g. Vogel et Bolognese, 1995; Maier *et al.*, 2001; Kjeldsen et Rosbjerg, 2004; Jain et Bhunya, 2008). The exception (Moy *et al.*, 1986) concerns cases were linear programming is applicable, and where uncertainty is not accounted for. The proposed composite indicator, which we shall call R-V indicator, is to be high when reliability is low and vulnerability is high.

Once such R-V indicators have been defined, Section 5.4 applies them to a simple case of reservoir operation, assuming the only objective is to deliver a target water supply each year. It shows the pertinence of the proposed algorithm and of the indicators which computation it enables. Finally, Section 5.5 discusses both the algorithm and its application.

5.2 Computing the composite vulnerability index

5.2.1 System dynamics

We consider a system and its uncertain and controlled dynamics. We are interested in its evolution during a given period spanning between an initial date 0 and a final date T. In discrete time, the transition between two consecutive dates is given by :

$$x(t+1) = f(t, x(t), u(t), w(t)) \tag{5.1}$$

In equation (5.1), the dynamic f is the transition function between two dates, and it is in general dependent on time. The state space is $\mathbb{X} \subset \mathbb{R}^n$ and the vector $x(t)$ is the state of the system at a date t. The vector $u(t)$ is the decision vector. The set of decisions that are available depends both on the date and on the state, and this decision space is noted $U(t,x) \subset \mathbb{R}^q$. A strategy σ associates to any date t and state x a decision $u(t,x)$ chosen among the set of possible decisions $U(t,x)$. The set of all the strategies σ available within a horizon T is noted $\Sigma(T)$. $w(t)$ is the vector representing the uncertainty and variability that affect the system at date t.

A possible sequence of events can be called a scenario (De Lara et Doyen, 2008), and be noted $\omega = (w(0), w(1), \dots, w(T-1))$. The space of all the scenarios is noted Ω, and for the computation of vulnerability indicators, one needs to assume the existence of a probability $\mathbb{P}$ defined over $\mathscr{P}(\Omega)$, the set of all the subsets of Ω. The triplet $(\Omega, \mathscr{P}(\Omega), \mathbb{P})$ is a probability space. In this section, the uncertainty vectors $w(t)$ from equation (5.1) are assumed to be i.i.d (independent and identically distributed) at all dates. We also note $E_{\mathbb{P}}[.]$ expected value in the scenario space.

An initial state x_0 at $t = 0$, a strategy σ and a scenario ω define only one possible trajectory, which we note $\theta(x_0, \sigma, \omega)$. Each successive state $x(t)$ belonging to a given trajectory may also be noted $\theta(t, x, \sigma, \omega)$, so that equation (5.1) leads to the following recursive sequence :

$$\begin{cases} \theta(0, x_0, \sigma, \omega) = x_0 \\ \forall t \in [1, \tau], x(t) = \theta(t, x_0, \sigma, \omega) \end{cases} \tag{5.2}$$

and to designate the states along a trajectory that goes through x at date t, one can also use $x(\tau) = \theta(\tau, x(t), \sigma, \omega)$ when $\tau \geq t$.

5.2.2 Harm criteria

We extract a composite criterion from two types of harm functions from Chapter 4. These respectively correspond to cost and viability criteria. The exit criterion is the opposite of a viability criterion.

Viability criterion

The viability criterion assesses whether a trajectory constantly respects constraints during $[0, T]$. Contrary to Doyen et De Lara (2010) who define this criterion, we assume that there is no uncertainty on the constraints. Noting $K(t)$ the constraint set at each date, the viability criterion is defined for a trajectory, hence the notation $\gamma(t, x_0, \sigma, \omega) = \gamma(\theta(t, x_0, \sigma, \omega))$. The criterion reads :

$$\gamma(t, x_0, \sigma, \omega) = \prod_{\tau=t}^{T} \mathbb{1}_{K(\tau)}[\theta(\tau, x_0, \sigma, \omega)] \tag{5.3}$$

where $\mathbb{1}_{K(t)}$ is the indicator function of the set $K(t)$, defined by :

$$\mathbb{1}_{K(t)}(x) = \begin{cases} 1 & \text{if} \quad x \in K(t) \\ 0 & \text{if} \quad x \notin K(t) \end{cases} \tag{5.4}$$

Here we are interested in aggregating several such viability criteria, designed by an integer $i \in [1, J]$. Each is represented by a set $K_i(t)$, and we assume that :

$$\forall i, j \in [1, J], i \leq j \Rightarrow K_i(t) \subset K_j(t) \tag{5.5}$$

so that when $i < j$, the viability criterion "j" is respected when the viability criterion "i" is. Thus, the sets $K_i(t)$ define degrees of undesirability. We note $\mathbb{1}_{K_i(t)}(x)$ the indicator functions of these ensembles, and $\gamma_i(t, x_0, \sigma, \omega)$ the associated viability criteria.

Since we are interested in an exit criterion β_i rather than in a viability criterion, we can define for each $K_i(t)$:

$$\beta_i(t, x_0, \sigma, \omega) = \alpha_i \left(1 - \gamma_i(t, x_0, \sigma, \omega)\right) \tag{5.6}$$

where each exit criterion is weighted by $\alpha_i > 0$.

Cost criterion

The cost criterion is based on associating a cost $C(t, x)$ to each state at each date. These costs are summed up along a trajectory, leading to the criterion $\chi(t, x_0, \sigma, \omega) = \chi(\theta(t, x_0, \sigma, \omega))$:

$$\chi(t, x_0, \sigma, \omega) = \sum_{\tau=t}^{T} C(\tau, \theta(\tau, x_0, \sigma, \omega)) \tag{5.7}$$

Contrary to the viability or exit criteria, a linear combination of cost criteria is still a cost criterion.

Composite harm criterion

A linear combination of a cost criterion and of several cost criteria leads to a composite criterion which weighs the two sources of harm along the trajectory. This new criterion $\kappa(t, x_0, \sigma, \omega) = \kappa(\theta(t, x_0, \sigma, \omega))$ reads :

$$\kappa(t, x_0, \sigma, \omega) = \chi(t, x_0, \sigma, \omega) + \sum_{i=1}^{J} \beta_i(t, x_0, \sigma, \omega) \tag{5.8}$$

The goal is to minimize the value on the criterion on the whole trajectory :

$$\kappa(0, x_0, \sigma, \omega) = \chi(0, x_0, \sigma, \omega) + \sum_{i=1}^{J} \beta_i(0, x_0, \sigma, \omega) \tag{5.9}$$

Due to the multiplicity of potential trajectories, this minimization can only be done in a probabilistic sense.

5.2.3 Change in state variables

The issue with the criterion of equation (5.9) is that its expected value cannot be minimized using SDP. Indeed, for this minimization to be possible, the criterion κ must be in a so-called Whittle form (Whittle, 1982; De Lara et Doyen, 2008), a recursive relationship between $\kappa(t, x_0, \sigma, \omega)$ and $\kappa(t+1, x_0, \sigma, \omega)$. Therefore, we are looking for a transition function ψ such that κ can be written as follows :

$$\begin{cases} \forall t \in [0, T-1], \kappa(t, x_0, \sigma, \omega) = \psi\left(t, x(t), \sigma(t, x(t)), w(t), \kappa(t+1, x_0, \sigma, \omega)\right) \\ \kappa(T, x_0, \sigma, \omega) = M\left(\theta(T, x_0, \sigma, \omega)\right) \end{cases} \tag{5.10}$$

where the function M enables the initialization of the backward induction. For SDP to be used to optimize the expected value of the criterion, ψ must be under the form $\psi(t, x, u, w, C) = g(t, x, u, w) + h(t, x, u, w)C$. This is the case for a cost criterion ($\psi(t, x, u, w, C) = g(t, x, u, w) + C$) or a viability criterion ($\psi(t, x, u, w, C) = h(t, x, u, w)C$), but the composite harm criterion κ falls in neither category.

This is why we make a change of variables by replacing the state x by $y = (x, j)$ where j has integer values comprised between 0 and J. The new state $y(t)$ is given by the system (S') :

$$\begin{cases} x(t+1) = f\left(t, x(t), u(t), w(t)\right) \\ j(t+1) = \max\left\{ j(t), \sum_{i=1}^{J} \left[1 - \mathbb{1}_{K_i(t)}\left(x(t+1)\right)\right] \right\} \end{cases} \tag{S'}$$

so that j is the count of the sets $K_i(t)$ that the trajectory left at some point during the trajectory, and a variable that can only increase along the trajectory. The set $\mathbb{Y}$ of feasible states is :

$$\mathbb{Y} = \left\{(x, j) \in \mathbb{X} \times [0, J] | x \in K_{j+1}(t)\right\} \tag{5.11}$$

where we set the convention $K_{J+1}(t) = \mathbb{X}$. The harm criterion can now be written as follows (proof in Appendix 5.6) :

$$\kappa(t, y_0, \sigma, \omega) = \chi(t, x_0, \sigma, \omega) + \sum_{i=1}^{j(T)} \alpha_i \tag{5.12}$$

and by definition of the cost criterion χ (equation (5.7)) we can write :

$$\kappa(t, y_0, \sigma, \omega) = \sum_{\tau=t}^{T} C\left(\tau, \theta(\tau, x_0, \sigma, \omega)\right) + \sum_{i=1}^{j(T)} \alpha_i \tag{5.13}$$

so we can introduce the cost $C'(t,y)$ by :

$$\begin{cases} \forall t < T, C'(t,y) = C(t,x) \\ C'(T,y) = C(T,x) + \sum_{i=1}^{j} \alpha_i \end{cases} \tag{5.14}$$

and we can then rewrite κ as a cost criterion :

$$\kappa(t, y_0, \sigma, \omega) = \sum_{\tau=t}^{T} C'(\tau, \theta(\tau, y_0, \sigma, \omega)) \tag{5.15}$$

As stated at the beginning of the section, a cost criterion can be put in Whittle form with ψ under the form $\psi(t,x,u,w) = g(t,x,u,w) + C$:

$$\begin{cases} \forall t \in [0, T-1], \kappa(t, y_0, \sigma, \omega) = C'(t,y) + \kappa(t+1, y_0, \sigma, \omega) \\ \kappa(T, y_0, \sigma, \omega) = C'(T,y) \end{cases} \tag{5.16}$$

and we now minimize its expected value.

5.2.4 Composite vulnerability index

We define the composite vulnerability index as the expected value of the κ :

$$\kappa_E(t, y_0, \sigma) = E_{\mathbb{P}}[\kappa(t, y_0, \sigma, \omega)] \tag{5.17}$$

Since κ is a cost criterion, κ_E is related to the following value function :

$$\begin{cases} V(T,y) = C'(T,y) \\ \forall t \in [0, T-1], V(t,y) = C'(t,y) + \min_{u \in U(t,x)} E_{\mathbb{P}}[V(t+1, f(t,y,u,w))] \end{cases} \tag{5.18}$$

In particular, one can prove (see e.g. De Lara et Doyen, 2008, Appendix A.4) that :

$$V(0, y_0) = \min_{\sigma \in \Sigma(T)} \kappa_E(0, y_0, \sigma) \tag{5.19}$$

and we can define this minimal value κ_E^* as a function of the initial state and date alone :

$$V(0, y) = \kappa_E^*(0, y) \tag{5.20}$$

so that any composite indicator made of a linear combination cost and exit criteria can be computed through equation (5.18), provided the exit criteria are based on sets that verify equation (5.5).

Eventually, if we set :

$$j(x) = \arg\min \{j \in \mathbb{N}, (x, j) \in \mathbb{Y}\} \tag{5.21}$$

then we write the mimimal value of κ_E for the initial state x_0 :

$$\kappa_E^*(x_0) = \kappa_E^*(0, x_0, j(x_0)) \tag{5.22}$$

5.3 Application to reliability and vulnerability indicators

The results from Section 5.2 are applied to the reliability and vulnerability indicators, as they are known in the field of water resources planning and management. In what follows, we assume that the state space is discretized in a set $\mathbb{X}_d$.

5.3.1 Description of the static criteria of satisfaction

R-R-V performance indicators are based on static notions of satisfactory and unsatisfactory performance, which they translate into dynamic indicators. Satisfactory performance is described by a subset of the state space, $K_1(t)$. When performance is unsatisfactory, the degree of dissatisfaction can be given by a sequence of other sets $K_j(t)$, growing in the sense of equation (5.5). This sequence is finite because the state space is discrete.

One can also define performance at a state x and date t based on a cost, $c(t, x)$. The cost is positive and is null when the performance is satisfactory, so that we have the following relationship between this cost and $K_1(t)$:

$$K_1(t) = \{x | c(t, x) = 0\} \tag{5.23}$$

and one can also relate all the sets $K_i(t)$ with a cost level :

$$\forall i \in [2, J], K_i(t) = \{x | c(t, x) \leq c_i\} \tag{5.24}$$

5.3.2 Trajectory-based reliability and vulnerability

Let x_0, σ and ω : they define a unique trajectory, and this section deals with reliability and vulnerability indicators defined along it, and how they relate to the viability and cost criteria defined in Section (5.2.2).

The definition of reliability as a trajectory-based criterion can be made in two ways (Hashimoto *et al.*, 1982). On one hand, it is the fraction of the time when the system is in a satisfactory state :

$$\mathrm{Rl}_1(x_0,\sigma,\omega) = \frac{1}{T+1}\sum_{t=0}^{T} \mathbb{1}_{K_1(t)}[\theta(t,x_0,\sigma,\omega)] \tag{5.25}$$

Maximizing Rl_1 amounts to minimizing a cost $(1-\mathrm{Rl}_1)$ so that this first reliability criterion is essentially a cost criterion. On the other hand, reliability is the fact of not reaching an unsatisfactory state during $[0,T]$:

$$\mathrm{Rl}_2(x_0,\sigma,\omega) = \prod_{t=0}^{T} \mathbb{1}_t^0[\theta(t,x_0,\sigma,\omega)] \tag{5.26}$$

where one can recognize the viability criterion γ of equation (5.3). This equivalence between this reliability criterion and a viability criterion was also the focus of Chapter 3.

As for vulnerability criteria, they are defined in a variety of ways on a given trajectory (Loucks, 1997; Sandoval-Solis *et al.*, 2011). One can distinguish between two broad families, one being vulnerability as a cost along a trajectory :

$$\mathrm{Vu}_1(x_0,\sigma,\omega) = \sum_{t=0}^{T} c(t,\theta(t,x_0,\sigma,\omega)) \tag{5.27}$$

while the other emphasizes the worst state along the trajectory :

$$\mathrm{Vu}_2(x_0,\sigma,\omega) = \max_{0\leq t\leq T} c(t,\theta(t,x_0,\sigma,\omega)) \tag{5.28}$$

This definition is related to the sets $K_i(t)$ as defined in equation (5.24). If a set is defined for each cost level, Vu_2 can be written as a function of all the thresholds that are crossed :

$$\begin{cases} \mathrm{Vu}_2(x_0,\sigma,\omega) = \sum_{i=1}^{J} \alpha_i \left[1-\prod_{t=0}^{T} \mathbb{1}_t^i[\theta(t,x_0,\sigma,\omega)]\right] \\ \forall i \in [2,I], \alpha_i = c_i - c_{i-1} \end{cases} \tag{5.29}$$

where the c_i are defined by equation (5.24). A particular case arises when cost is defined as a distance to the property of interest which defines satisfactory performance : then, $\alpha_i = 1$ for all i.

To summarize, both Vu_1 and Rl_1 are related to a cost, while Rl_2 is a viability criterion and Vu_2 can be interpreted as linear combination of exit criteria. Therefore, any linear combination of reliability and vulnerability criteria can be expressed by a composite harm criterion κ under the form of equation (5.9).

5.3.3 The composite R-V indicator

Since the expected value of κ can be minimized thanks to SDP (equations (5.18) to (5.22), one can get a composite reliability-vulnerability indicator along with the decision strategy σ that minimizes it. Noting it RV we have :

$$RV(x_0) = \kappa_E^*(x_0) \tag{5.30}$$

This indicator depends on the weights of the different reliability and vulnerability criteria that are considered while computing κ. As such, it enables the elicitation of trade-offs between reliability and vulnerability. In particular, the knowledge of κ_E^* at a given initial state and for different values of the weights leads to approximating the Pareto front for these indicators.

Besides, the computation of $RV(x_0)$ also yields the strategy σ^* that minimizes it. This gives in particular the decision to be taken at the initial date of the assessment, which can for instance be the current date in the physical world. $RV(x_0)$ is a vulnerability indicator in the sense of Chapter 4 : a low-vulnerability kernel can be associated to it.

5.3.4 The performance kernel

A low value of $RV(x_0)$ corresponds to a high reliability and a low vulnerability of the property of interest. In other words, failures are expected to be a rare occurrence, and when they happen, they are expected to be of mild severity. Since reliability and vulnerability indicators are usually called performance indicators in the field of water resources management, a low value of $RV(x_0)$ arguably measures a high system performance. The set of initial system configurations x_0 such that $RV(x_0)$ is below the threshold value ρ is defined as the performance kernel $\mathscr{P}(RV;\rho,T)$ defined as :

$$\mathscr{P}(RV;\rho,T) = \{x_0 \in \mathbb{X} | RV(x_0) \leq \rho\} \tag{5.31}$$

and we propose an overall indicator of the performance of the system with respect to the property of interest, namely the size of the performance kernel relative to that of the state space :

$$\Phi(RV;\rho,T) = \frac{1}{\sum_{x\in\mathbb{X}_d}\lambda(x_i)}\sum_{x\in\mathbb{X}_d}\mathbb{1}_{\mathscr{R}(RV;\rho,T)}(x_i)\lambda(x_i) \tag{5.32}$$

5.4 Application

This application is inspired from You et Cai (2008c), except that we consider a horizon T instead of a rolling two-period scheme.

5.4.1 Model formulation

We are interested in the operation of a single reservoir for water supply. The state variables are reservoir storage S and the delivery D of water for consumption. The decision is the release R from the reservoir, and the inflow I is stochastic. The state transition reads :

$$\begin{cases} S(t+1) = S(t) + I(t) - R(t) \\ D(t+1) = \min\{R(t), D_0\} \end{cases} \tag{5.33}$$

where the first equation is the water balance, and the second expresses that D matches the water demand D_0 whenever possible. We set a yearly time step, and the demand D_0 is assumed to be known and constant for every year. $D(t+1)$ is the water delivery at year t, and the date $t+1$ is artificially given so as to respect the formulation of equation (5.1). By convention we set $D(0) = D_0$. We also have the hard constraints :

$$0 \le S(t) \le C \tag{5.34}$$

$$0 \le R(t) \le S(t) + I(t) \tag{5.35}$$

where C is the maximum capacity of the reservoir. Finally, the inflow $I(t)$ is assumed to be log-normal with mean I_0 and scale parameter η, and to be i.i.d.

The static criterion expresses whether the demand is met, so the set of satisfactory states K_1, constant through time, is :

$$K_1 = \{(S,D) | D = D_0\} \tag{5.36}$$

In other words, satisfaction amounts to having release equal or greater than D_0.

All the quantities introduced above are made non-dimensional by scaling them by the reservoir capacity C. Consequently, at any date, $S(t) \in [0,1]$, and in what follows, a resolution of 0.01 is taken for all the variables.

5.4.2 Performance criteria

Three criteria are used in this case-study. First, Rl_1 from equation (5.25) translates into :

$$\mathrm{Rl}_1(x_0,\sigma,\omega)=\frac{1}{T}\sum_{t=1}^{T}\mathbb{1}_{D=D_0}[\theta(t,x_0,\sigma,\omega)] \tag{5.37}$$

since there are only T release decisions over $[0,T]$. Then, Vu_1 from equation (5.27) represents the sum of water deficits $D_0-D(t)$ over $[0,T]$, normalized on $[0,1]$:

$$\mathrm{Vu}_1(x_0,\sigma,\omega)=\frac{1}{T}\sum_{t=1}^{T}\left(1-\frac{D(t)}{D_0}\right) \tag{5.38}$$

and we also keep track of the highest deficit along the trajectory, so Vu_2 from equation (5.28) becomes Vu_2 :

$$\mathrm{Vu}_2(x_0,\sigma,\omega)=\max_{1\leq t\leq T}\left(1-\frac{D(t)}{D_0}\right)^3 \tag{5.39}$$

where Vu_2 is like the other two criteria. The exponent 3 was chosen as a good compromise between the need for skewing the criterion against the more severe failures, and that for . Noting D_i the discrete points between 0 and D_0, with a resolution of $\Delta S=0.01$, one can rewrite Vu_2 as a sum of exit criteria like in equation 5.29. The change of variables $y=(x,j)$ of Section 5.2.3 leads to rewriting Vu_2 as follows :

$$\mathrm{Vu}_2(y_0,\sigma,\omega)=\left(1-\frac{\Delta S\times j(T)}{D_0}\right)^k \tag{5.40}$$

Then, one can define two composite criteria. The first composite criterion weighs Vu_1 with Vu_2 :

$$\kappa_1(\lambda,y_0,\sigma,\omega)=\lambda\mathrm{Vu}_1(x_0,\sigma,\omega)+(1-\lambda)\mathrm{Vu}_2(y_0,\sigma,\omega) \tag{5.41}$$

The second weighs (1—Re_1) and Vu_2, which mirror the reliability and vulnerability indicators initially proposed by Hashimoto *et al.* (1982) to describe the performance of a water supply system with respect to a delivery target :

$$\kappa_2(\lambda,y_0,\sigma,\omega)=\lambda(1-\mathrm{Rl}_1(x_0,\sigma,\omega))+(1-\lambda)\mathrm{Vu}_2(y_0,\sigma,\omega) \tag{5.42}$$

and the objective is to minimize the expected value of these criteria. For $i=[1,2]$ the objective is :

$$\kappa_i^*(\lambda, y_0) = \min_{\sigma \in \Sigma(T)} E_{\mathbb{P}}[\kappa_i(\lambda, y_0, \sigma, \omega)] \tag{5.43}$$

and, using equation (5.22), we can write, still for $i = [1, 2]$:

$$\kappa_i^*(\lambda, x_0) = \kappa_i^*(\lambda, x_0, j(x_0)) \tag{5.44}$$

5.4.3 Operating policies

SDP is used to minimize the two objectives given above, with $T = 20$. Due to a yearly time step, this value of the decision horizon is deemed sufficient in practice, because the past behavior of the statistical distribution of water inflows cannot be extrapolated into the future (Koutsoyiannis, 2006), especially not in a stationary way (Milly *et al.*, 2008). Besides, the decision horizon should not exceed the forecast horizon (You et Cai, 2008a).

In particular, SDP yields the optimal release decision at the initial (current) date. This policy assumes that past releases have no influence on $j(T)$, which is only a mean to keep track of the thresholds that will be crossed until T. The release decision at year t is assumed to be done only once the total amount of water available, $S + I$, is known. Of course, there may be uncertainty within year t regarding the release rate depending on future inflow during the same year. Such decisions would be captured by intra-year dynamics which are not captured by equations (5.33) and (5.35), so we choose to ignore intra-year uncertainty.

Respective release decisions at the date $t = 0$ depending on total available water are displayed in Figure 5.1 for $\kappa_1^*(x_0)$, and Figure 5.2 for $\kappa_2^*(x_0)$. On both figures, intermediate values of λ lead to policies that compromise between those chosen for $\lambda = 0$ or $\lambda = 1$. Though they both are for given values of the parameters, the behavior they display is generic across all parameter values.

On Figure 5.1, $\lambda = 1$ corresponds to the policy that minimizes the expected value of the sum of the future shortages, known as the standard operating policy or SOP (Hashimoto *et al.*, 1982). It consists in (A) releasing all the available water when the demand is not met, (B) filling the reservoir when there is enough water to meet the demand, and (C) releasing the surplus. Alternatively, $\lambda = 0$ is the case where only Vu_2 is considered. As a result, the corresponding policy is a hedging policy, defined as a policy where the current demand is purposefully not met so as to store more water in the reservoir and make possible future shortages less severe (Draper et Lund, 2004). Through economic analysis over a current and a future period, You et Cai (2008b)

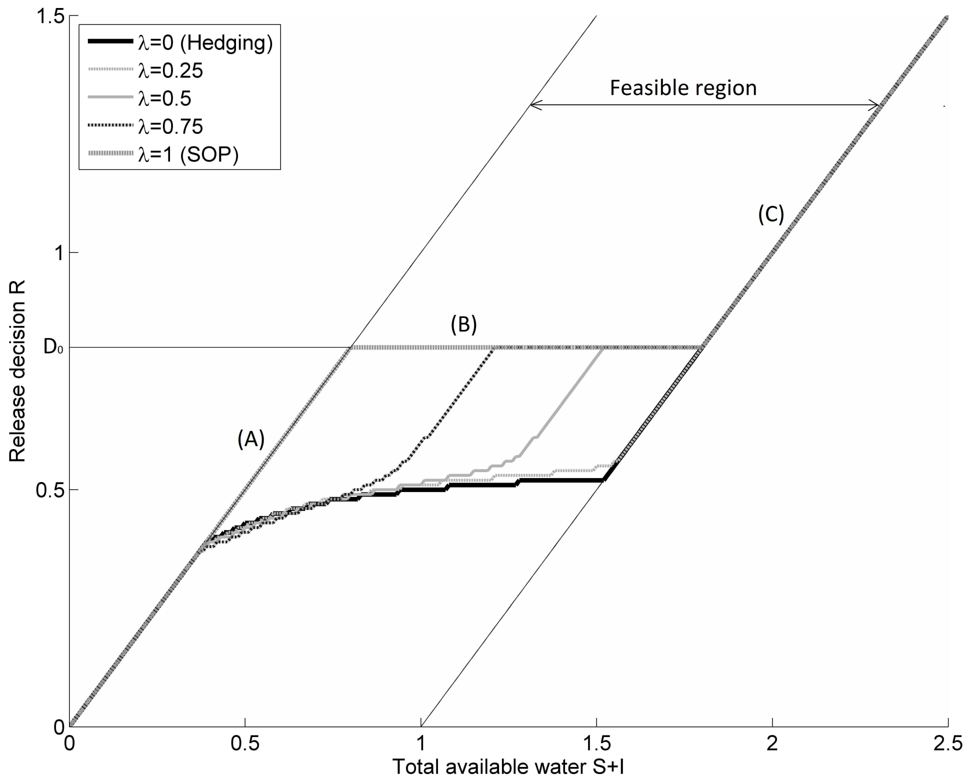

FIGURE 5.1 – Optimal policies for $\kappa_1^*(\lambda, x_0)$ and different values of λ. Known cases correspond to $\lambda = 0$ (hedging) and $\lambda = 1$ (Standard Operating Policy or SOP). $D_0 = 0.8$, $I_0 = 0.8$ and $\eta = 0.8$.

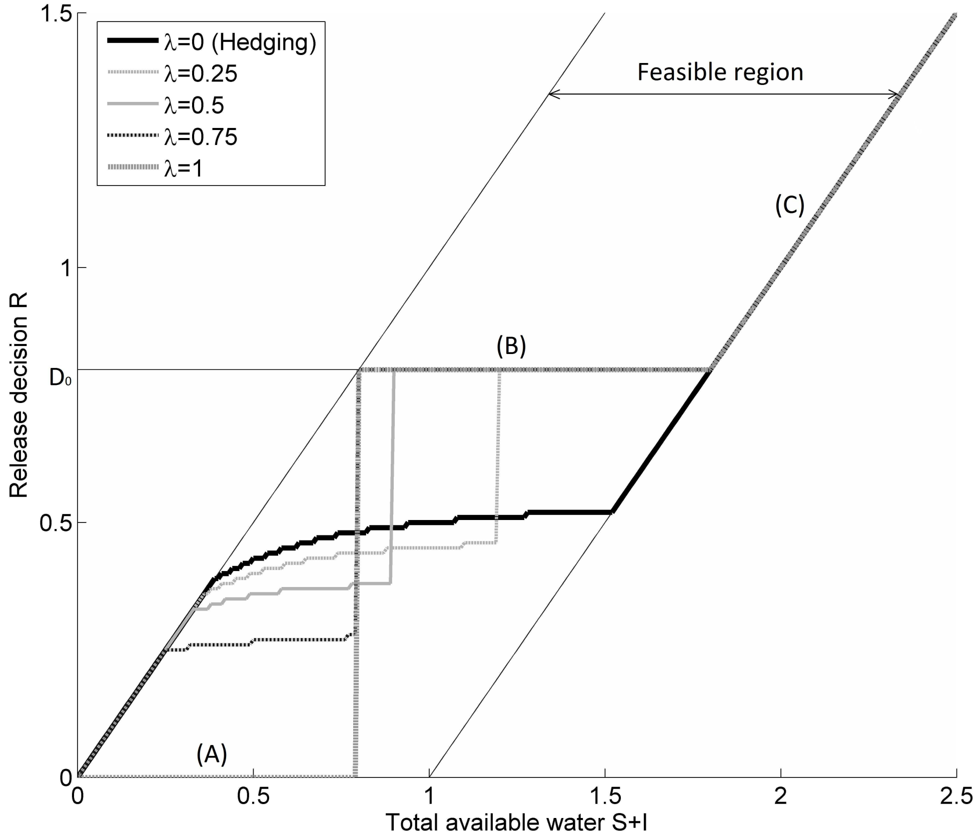

FIGURE 5.2 – Optimal policies for $\kappa_2^*(\lambda, x_0)$ and different values of λ. A known case corresponds to $\lambda = 0$ (hedging). $D_0 = 0.8$, $I_0 = 0.8$ and $\eta = 0.8$.

find a concave hedging curve between the part (A) of the curve where the reservoir is empty, and the part (C) where it is full. Through a minimization of the expected value of the maximum failure, we also find a concave hedging curve. In between those two cases, the compromise between hedging and SOP yields mixed policies that amount to hedging for low values of $S+I$, so the reservoir gets partially filled even though the demand is not met. Then when the reservoir is filled at 41% (in the case $\lambda = 0.5$) or 72% (for $\lambda = 0.75$), the additionnal water is released until the demand is met : this amounts to switching back to SOP. Such mixed rules do not appear in the literature, and it is the proposed composite indicator that enables to show their relevance.

On Figure 5.2, $\lambda = 1$ corresponds to (A) storing all the water when $S + I < D_0$, then (B) meeting the demand and filling the reservoir with the remaining water, and (C) releasing all the excess water. This difference apart, the results are the same as for Figure 5.1, and intermediate values of λ again give rise to new policies where hedging is prioritized until a threshold value of $S+I$ where the optimal release decision is that given by the case $\lambda = 1$.

Both these policies are optimal at the initial date and for $T = 20$. It should be noted that the optimal policy at a given date changes with the time horizon considered. As T grows, so does the possibility of a severe shortage, which means that the expected value of $j(T)$ increases. To hedge against this risk, the operating policy gets more conservative, and leads to storing less water. However, the optimal policy remains qualitatively unchanged with respect to the horizon, as illustrated in Figure 5.3 which represents the operating rule in the same conditions as for Figure 5.1, except for the decision horizon. The comparison between both figures also reveals that the quantitative difference between the decisions at $T = 20$ and $T = 50$ is relatively small, as it never goes over 0.06.

5.4.4 Performance indicators

In this section, we keep only one R-V indicator, so $RV(x_0, \lambda) = \kappa_2^*(x_0, \lambda)$. This choice enables us to examine the trade-offs between reliability and vulnerability (Figures 5.4 and 5.5) using for each λ the optimal strategy $\sigma^*(\lambda)$. One can thus get a Pareto front, where λ close to 0 corresponds to hedging more water in order to avoid more severe future water shortages, and λ close to 1 corresponds to maximizing reliability. It is interesting to note the sensitivity of reliability and vulnerability to a change in inflow standard deviation between the two figures, which highlights the importance of climate variability. Initial water storage is also crucial, especially when it comes to the expected maximal deficit.

Besides having an in-depth look at the indicators that compose RV, one can de-

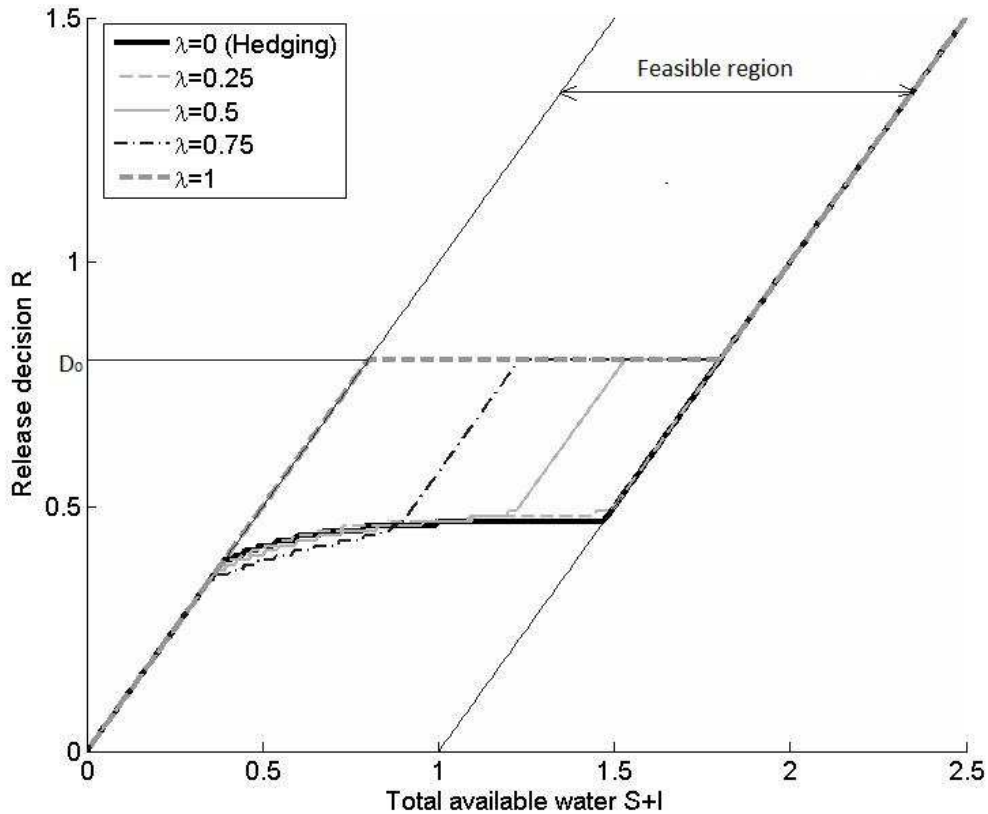

FIGURE 5.3 – Same as Figure 5.1, but with $T = 50$.

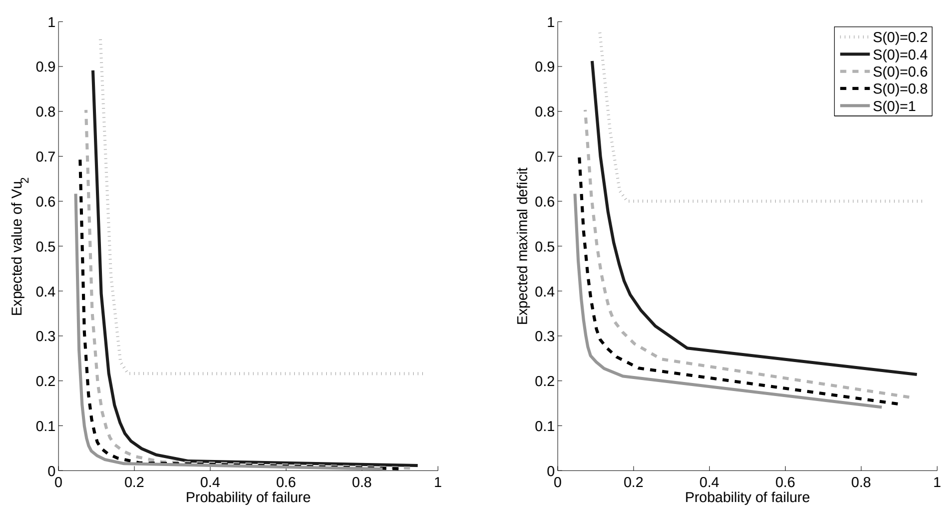

FIGURE 5.4 – Trade-off of the probability of failure $E_{\mathbb{P}}[1 - Rl_1(x_0, \sigma^*(\lambda), \omega)]$ with $E_{\mathbb{P}}[Vu_2(x_0, \sigma^*(\lambda), \omega)]$ (left) and the expected maximal deficit (right). $D_0 = 0.5$, $I_0 = 0.5$ and $\eta = 0.3$. $S(0)$ is the initial storage.

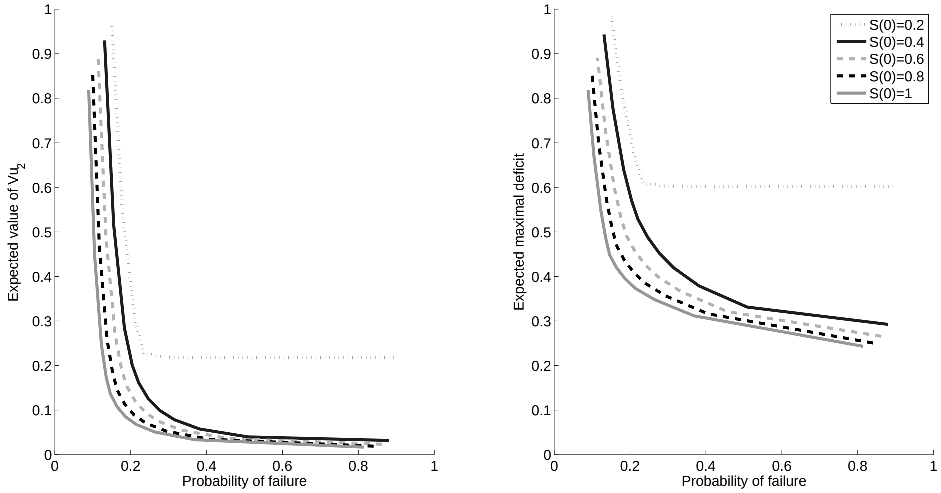

FIGURE 5.5 – Same as for Figure 5.4, but now with $\eta = 0.4$.

termine the performance kernel $\mathscr{P}(RV;\rho,T)$ so as to infer the overall performance of the system. We choose $\rho = 0.1$, and plot the performance Φ for $D_0 = 0.5$ and different values of the inflow mean I_0 and variance η (Figure 5.6). For all three values of λ (0, 0.5 and 1) there is a range of combinations of I_0 and η for which the water inflow is too scarce and too variable. The performance kernel is empty, which highlights that the reservoir cannot be in a state where RV is low. In contrast, most other combinations of these parameters yield a performance of 0.7 when $\lambda = 0$, 0.79 when $\lambda = 0.5$ and 1 when $\lambda = 1$. This means that even for a low initial storage, the reservoir can function in a satisfactory way. When $\lambda \neq 1$, large water deficits $D_0 - D(t)$ are penalized, so the system cannot be expected to have a very low $RV(x_0)$ when the initial storage is very low. Indeed, if the first inflow is low, then the water deficit of the first year can be very high. For $\lambda = 1$ however, only reliability is taken into account, so even if initial water storage is low, what matters is whether the reservoir can quickly be filled, which is the case when the annual mean inflow is greater than the annual demand.

5.5 Discussion

This paper presents an SDP algorithm which computes the expected value of a criterion which is a linear combination of cost and exit criteria. Even though it is presented to cases where the viability criteria are delimited by certain constraints, this hypothesis is no longer needed when the state space is discretized. Indeed, it is then possible to compute, at each point within the state space, the sum of α_i coefficients weighted by the probability that the state is outside $K_i(t)$. Then, all the states can be ranked

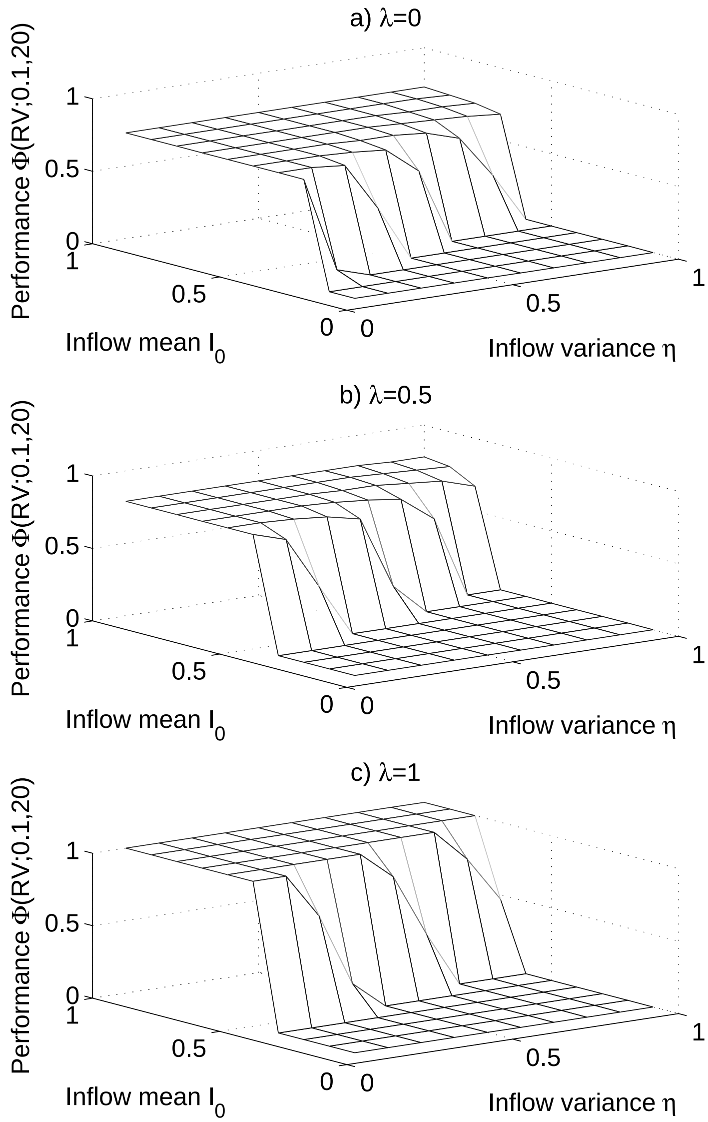

FIGURE 5.6 – Overall system performance for $RV(x_0,\lambda) = \kappa_1^*(x_0,\lambda)$, and $D_0 = 0.5$. For instance, a value of 0.74 means that the top 74% of the initial storage values are within the performance kernel $\mathscr{P}(RV;0.1,20)$.

according to the value of this sum, and be put into newly defined sets $K'_m(t)$ which reflect this ranking. Their boundaries are not uncertain so one gets back to the case in which the algorithm is proposed.

It is also possible to use the same kind of scheme to optimize a compromise between the expected value a harm function as defined in Chapter 4, and the probability of reaching a set at a certain horizon. If this set is a stochastic viability kernel, then this probability is a probability of resilience (Chapter 2), and the algorithm can be used to find a compromise between resilience and vulnerability.

The application to a simple case of reservoir operation showcases how the algorithm can be used to compute trade-offs between indicators, and to propose operating policies. However, a limitation of the proposed method appears to be its dependence on the time horizon, which can be illustrated through the application.

5.6 Appendix : Proof of equation (5.12)

Let us define the sets $\mathbb{Y}_i = \mathbb{X} \times [0, i-1]$. The viability criterion $\gamma_i(0, y_0, \sigma, \omega)$ is equal to 1 if the trajectory did not cross the boundary of i of the constraint sets and to 0 otherwise. :

$$\begin{aligned}\gamma_i(0, y_0, \sigma, \omega) &= \mathbb{1}_{\mathbb{Y}_i}(\theta(T, y_0, \sigma, \omega)) \\ &= \mathbb{1}_{[0,i-1]}(j(T))\end{aligned}$$

As a consequence we can write the weighted sum of exit criteria :

$$\begin{aligned}\sum_{i=1}^{J} \beta_i(0, y_0, \sigma, \omega) &= \sum_{i=1}^{J} \alpha_i \left(1 - \mathbb{1}_{[0,i-1]}(j(T))\right) \\ &= \sum_{i=1}^{j(T)} \alpha_i \left(1 - \mathbb{1}_{[0,i-1]}(j(T))\right) + \sum_{i=j(T)+1}^{J} \alpha_i \left(1 - \mathbb{1}_{[0,i-1]}(j(T))\right)\end{aligned}$$

on the other hand, by definition of the indicator function :

$$\mathbb{1}_{[0,i-1]}(j(T)) = \begin{cases} 0 & \text{if} \quad j(T) \geq i \\ 1 & \text{if} \quad j(T) < i \end{cases}$$

so that we finally get :

$$\sum_{i=1}^{J} \beta_i(0, y_0, \sigma, \omega) = \sum_{i=1}^{j(T)} \alpha_i \tag{5.45}$$

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