Contributions en commande optimale et au problème du temps de crise - applications en irrigation

Kenza Boumaza

THÈSE POUR OBTENIR LE GRADE DE DOCTEUR DE L' UNIVERSITÉ DE MONTPELLIER

En Mathématiques et Modélisation

École doctorale I2S - Information, Structures, Systèmes

Unité de recherche : UMR MISTEA, INRAE

Contributions en commande optimale et au problème du temps de crise - applications en irrigation

Présentée par BOUMAZA Kenza
Le 16 Novembre 2021

Sous la direction de Alain Rapaport, Térence Bayen et Sébastien Roux

Devant le jury composé de

M. Piernicola Bettiol, Professeur, Université de Bretagne Occidentale	Rapporteur
Mme. Maria Do Rosario De Pinho, Professeur, Université de Porto	Rapporteure
Mme. Emmanuelle Augeraud-Veron, Professeur, GREThA Bordeaux	Examinatrice
M. Olivier Cots, Maître de conférences, ENSEEIHT	Examinateur
M. Tewfik Sari, Directeur de recherche, INRAE Montpellier	Examinateur
M. Térence Bayen, Professeur, Université d'Avignon	Codirecteur
M. Sébastien Roux, Ingénieur de recherche, INRAE Montpellier	Codirecteur
M. Alain Rapaport, Directeur de recherche, INRAE Montpellier	Directeur

UNIVERSITÉ DE MONTPELLIER

à mes chers parents,
à mes frères et soeurs.

Remerciements

Cette thèse a été réalisée au sein de L'UMR Mathématiques, Informatique et Statistique pour l'Environnement et l'Agronomie (MISTEA) de l'Institut National de Recherche Agronomique (INRAE). J'ai bénéficié d'un financement du gouvernement algérien dans le cadre du programme de Bourses d'Excellence (2018-2021). Je remercie le gouvernement algérien, pour cette opportunité qui aura marqué ma vie étudiante et personnelle.

Je tiens tout d'abord à remercier mes directeurs de thèse, M. Térence Bayen et M. Sébastien Roux pour les précieux conseils, leur patience infinie et leur bonne humeur communicative. Je remercie tout particulièrement M. Alain Rapaport, pour sa gentillesse et sa disponibilité qui ont marqué cette expérience académique.

Mes remerciements et toute ma reconnaissance vont ensuite à M. Piernicola Bettiol et Mme. Maria Do Rosario De Pinho, qui ont accepté d'être rapporteurs de cette thèse. Leurs observations furent pertinentes et bienveillantes, et ont été précieuses pour l'amélioration du manuscrit de ma thèse. Un grand merci à M. Tewfik Sari d'avoir présider le jury de ma soutenance, et je remercie Mme. Emmanuelle Augeraud-Veron et M. Olivier Cots d'avoir examiné mon travail.

Je remercie M. Pascal Neveu, directeur du laboratoire MISTEA pour son accueil, ainsi que tout les membres du labo ayant été présents pour moi lors de ces trois années de recherches: Anne, Brigitte, Céline, Bertrand, Bénédicte, Isabelle, Ismael, Malika, Maria, Nadine, Patrice et tous mes autres collègues.

Merci à mes amies du laboratoire: Emna, Fatima, Manel et Nesrine. Leurs amitiés et leurs soutien ont été indispensables dans la production de ce travail. Mon expérience au sein du laboratoire n'aurait pas été la même sans nos moments de complicité et de bonne humeur. Merci à mes amies les Daltonnes : Radia, Thiziri et Zaineb pour les moments de joie qui m'ont aidé à surmonter les moments les plus difficiles. Merci également à tout mes camarades boursiers algériens que j'ai rencontré grâce à ce programme.

Évidemment, réaliser cette thèse n'aurait pas été possible sans la présence de ma deuxième famille ici en France, les Boutonnistes (*l'sang!*): Anis, Asma, Astrid, Emmanuel, Ismael, Laurent, Liana, Livais, Logan, Manya, Malak, Matthias, Marina, Merouane, Milan, Natacha, Rodrigue, Shailesh, Stefania, Sonia et tout les autres voisins que j'ai rencontré à la cité universitaire de Boutonnet.

Enfin, je remercie ma famille pour leur soutien et leur affection: ma mère Razika , mon père Mohamed, mon petit frère Chakib, ma petite soeur Chahla, et mon grand frère Hamza. Un grand merci à ma famille en France, dont ma tante Feriel, mes cousines Amélia et Nouna, mon cousin Mustapha et sa compagne Alex pour le soutien.

Il ne serait pas possible de remercier ici toutes les personnes que j'ai pu rencontrer et qui m'ont soutenu d'une façon ou d'une autre: j'en suis infiniment reconnaissante et je les remercie chaleureusement.

Je dédie cette thèse à mon grand-père Houcine Boumaza et à mon voisin Aldy Kizito Kalyongo, décédés lors de la réalisation de ma thèse et dont les présences étaient si précieuses.

Résumé.

Cette thèse s'inscrit dans le domaine du contrôle optimal et se divise en deux parties.
La première partie vise l'étude d'un problème de contrôle optimal issu de la modélisation agronomique, qui consiste à maximiser la production de biomasse au moment de la récolte sous une contrainte sur l'eau nécessaire à l'irrigation dans un contexte de ressources limitées. Le problème est présenté comme un problème de contrôle optimal sous contrainte d'état avec une cible, soumis à un système dynamique non-autonome dont le deuxième membre est non-lisse. Tout d'abord, nous examinons la contrainte d'état en comparant les trajectoires au-dessus du seuil à partir duquel la production de biomasse est maximale. Ensuite, nous appliquons le Principe du Maximum non-lisse et nous montrons qu'une solution optimale peut avoir un ou plusieurs arcs singuliers uniquement aux points de non-différentiabilité de la dynamique. Enfin, nous proposons trois stratégies différentes d'irrigation et nous faisons une comparaison numérique entre les trois.
La seconde partie porte sur l'étude du problème de temps de crise, qui consiste à minimiser le temps passé par une trajectoire solution d'un système contrôlé général en dehors d'un ensemble donné K. La caractéristique principale de ce type de problème est la discontinuité de l'intégrande par rapport à l'état au bord de l'ensemble K. Nous proposons d'abord une nouvelle méthode de régularisation du problème en utilisant un contrôle auxiliaire et une fonction de pénalité. Nous montrons la convergence de la suite de solutions optimales du nouveau problème régularisé vers une solution optimale du problème original. Nous proposons ensuite une méthode de régularisation plus générale où nous appliquons le Principe du Maximum de Pontryagin à la suite des problèmes de contrôle optimal régularisé, et nous étudions la bornitude ainsi que la convergence des extrémales. Sous une hypothèse plus générale que celle habituellement requise dans la littérature, qui exige toute solution optimale à traverser l'ensemble des contraintes de manière transversale, nous dérivons des conditions nécessaires d'optimalité pour le problème du temps minimal de crise.

***Mots clés*:** Contrôle optimal, contrainte d'état, Principe du maximum de Pontryagin, temps de crise minimum, arc singulier, dynamique non-lisse, condition de franchissement transverse, régularisation, modélisation de cultures, irrigation.

Abstract.

This thesis is about optimal control theory and modeling, and it is divided into two parts. The first part investigates an optimal control problem derived from agronomic modeling, which consists in maximizing the biomass production at harvesting time under a constraint on the water required for irrigation in a context of limited resources. The problem is presented as a state-constrained optimal control problem with a target, subject to a non-autonomous non-smooth (w.r.t. the state) dynamical system. First, we address the state constraint by comparing the trajectories above the highest biomass production threshold. Next, we apply the non-smooth Pontryagin Maximum Principle and show that an optimal solution can have one or several singular arcs located at the points of non-differentiability of the dynamics. Finally, we propose three different strategies of irrigation and make a numerical comparison between all of them.

The second part deals with the study of the minimal time of crisis problem which amounts to minimize the time spent by a trajectory solution of a general controlled system outside a given set K. The main characteristic of this kind of problem is the discontinuity of the integrand with respect to the state at the boundary of the set K. We first propose a new regularization method of the minimal time crisis problem using an additional control and a penalty function. We show the convergence of the sequence of optimal solutions of the new regularized problem to an optimal solution of the original problem. We then propose a more general regularization scheme where we apply Pontryagin Maximum Principle to the sequence of the smooth optimal control problems, and we study the boundedness and the convergence of the resulting extremals. Finally, we derive necessary optimality conditions for the minimal time of crisis under a more general hypothesis than the usual one considered in the literature, which requires any optimal solution to hit the boundary of the constraint set transversely.

***Keywords*:** Optimal control, state constraint, Pontryagin maximum principle, minimal time of crisis, singular arc, non-smooth dynamics, transverse crossing, regularization, crop modeling, irrigation.

SOMMAIRE

Première partie

Introduction Générale

I.1 L'optimisation des modèles d'irrigation.

Ressource naturelle indispensable à la vie, l'eau douce se raréfie. Avec environ 70 pour cent de la consommation mondiale d'eau, le secteur agricole est le plus grand consommateur d'eau douce. La FAO[1] prévoit que d'ici 2050 les besoins en eau pour l'agriculture augmenteront de 50 pour cent pour répondre aux demandes d'une population en augmentation constante. Les répercussions futures du changement climatique sur le stress hydrique seront principalement visibles dans la région méditerranéenne. Cette crise de l'eau représente une grave menace pour la durabilité de l'environnement à l'avenir. Il convient alors d'accorder la priorité à une gestion rationnelle des ressources en eau disponibles lors des processus d'irrigation tout en maintenant une production agricole élevée. Pour relever ce défi, il est indispensable d'élaborer des approches visant à optimiser l'efficacité de l'irrigation en introduisant de nouveaux scénarios d'irrigation plus efficaces. Ainsi, l'optimisation des stratégies d'irrigation constitue un problème majeur en gestion d'eau, dont les approches basées sur la modélisation pourraient répondre à ce problème.

I.1.1 Différentes méthodes d'optimisation pour l'irrigation.

La recherche de stratégies d'irrigation optimales repose principalement sur une combinaison de techniques de modélisation et de méthodes mathématiques en vue de l'optimisation. Plus précisément, il s'agit d'une part d'être en mesure de modéliser la croissance végétale en tenant compte de disponibilité de l'eau, et d'autre part d'en déterminer une stratégie optimale pour gérer cette ressource et obtenir un rendement optimal.

I.1.1.1 Types de modèles pour l'optimisation de l'irrigation.

La partie de modélisation consiste à décrire un système dynamique qui suit l'évolution de la plante en fonction de l'humidité du sol au cours temps. Plusieurs modèles visant cette démarche ont été établis. Les modèles considérés ici se classent dans deux catégories: les modèles de simulation complexes et les modèles mathématiques simplifiés.

1. Organisation des Nations unies pour l'alimentation et l'agriculture

a. **Les modèles complexes.** Depuis de nombreuses années, les modèles de la FAO [68] sont largement utilisés pour l'étude de la croissance des plantes via l'évapotranspiration et leurs besoin en eau. Nous citons principalement deux approches le plus souvent utilisées: FAO-33 [35, 1], qui établie un lien entre l'évapotranspiration de la plante et son rendement, ainsi que l'approche Aquacrop [69, 71] qui représente de façon explicite le lien, à un instant t, entre le stress hydrique et la production de biomasse. Cette dernière est utilisée dans plusieurs études d'optimisation de l'irrigation. Une autre approche à citer est l'approche développée dans le logiciel `Optirrig` (cf. [25, 56]), ce logiciel est développé pour la conception, l'analyse et l'optimisation des stratégies d'irrigation.

b. **Les modèles simplifiés.** Les modèles FAO cités sont complexes et présentent plusieurs variables, ces modèles conviennent mieux pour les études numériques en vue de l'optimisation. Pour être en mesure de réaliser une étude mathématique et analytique, il est nécessaire d'envisager des modèles simplifiés ou réduits comme dans les travaux de [52, 53, 64, 82]. Les études analytiques sur des modèles simples permettent une meilleure compréhension et une information qualitative sur une politique d'irrigation optimale.

Parmi les modèles simplifiés, un modèle qui présente un intérêt particulier l'étude des modèles de croissance de plantes est le modèle *ToyCrop* qui tient en compte un nombre faible de variables et qui prend plusieurs processus clé telle que l'évapotranspiration de la plante, le stress hydrique ou encore la croissance de biomasse.

Le modèle *ToyCrop* [47]. Le modèle *ToyCrop* est un modèle simplifié qui a été évalué sur un autre modèle plus complexe développé et codé dans le logiciel `Optirrig`. Cette évaluation consiste à vérifier son aptitude à reproduire différents jeux de données en ajustant ses paramètres. Le modèle simplifié *ToyCrop* est donné comme suit:

$$\dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)), \tag{I.1.1}$$

$$\dot{B} = k_3\varphi(t)K_S(S). \tag{I.1.2}$$

La première variable d'état dans le modèle, S dont la valeur est comprise entre 0 et 1, représente la teneur en eau du sol. L'évolution de S (voir équation (I.1.1)) dépend principalement des apports effectués par l'irrigation et les pertes dues à la transpiration de la plante et l'évaporation du sol. Le présent modèle ne tient pas

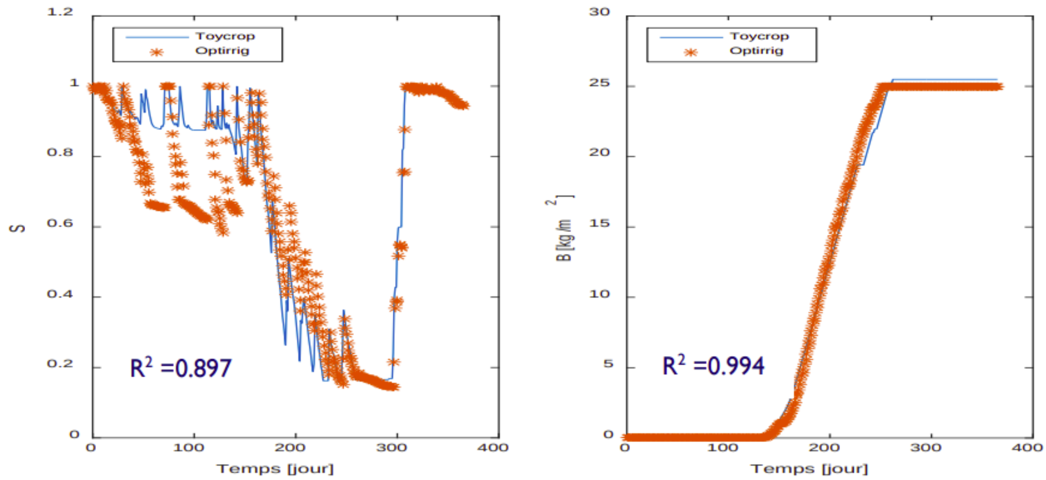


FIGURE I.1 – Simulation du taux d'humidité dans le sol et la production en Biomasse, stratégie tour d'eau, année 2011 [47].

compte les apports en précipitations ni les pertes par drainage: c'est un modèle plus adapté aux cultures sous serre. La fonction $K_S(\cdot)$ dans (I.1.1) est une fonction de limitation de la transpiration en fonction du taux d'humidité disponible à la plante. $K_S(\cdot)$ qui est définie sur $[0, 1]$ à valeurs dans $[0, 1]$ et est régie par deux seuils de teneur en eau : $S^\star$ le point du début de fermeture des stomates au-dessous duquel la plante ferme progressivement ses stomates afin de limiter les pertes par transpiration et S_w le point de flétrissement d'une plante à partir duquel la plante flétrit. Quant à la fonction $K_R(\cdot)$ c'est une fonction de limitation de l'évaporation en fonction de l'humidité du sol : plus le sol est humide, plus la quantité d'eau perdue par évaporation augmente. La fonction $K_R(\cdot)$ est également définie sur $[0, 1]$ et prend ses valeurs dans $[0, 1]$, elle est régie par un seuil d'humidité S_h au-dessous duquel l'évaporation ne se produit pas. La fonction $\varphi(\cdot)$ représente l'efficience d'interception du rayonnement supposée croissante au cours du temps, elle est utilisée comme proportion entre les deux fonctions K_S et K_R qui évolue en fonction du couvert végétal. La biomasse est la deuxième variable d'état du système notée B. Sa croissance est modélisée par l'équation (I.1.2), elle atteint sa valeur maximale lorsque l'humidité du sol est au-dessus du seuil $S^\star$. Dans (I.1.1), u représente l'apport d'eau en irrigation, c'est la variable de contrôle qui permet de piloter le système afin d'obtenir une croissance de plante. Le bon choix de cette variable de contrôle va favoriser une meilleure croissance de biomasse. Dans cette thèse, nous examinons ce modèle avec des hypothèses plus générales.

I.1.1.2 Méthodes d'optimisation de l'irrigation.

Diverses méthodes d'optimisation ont été employées pour identifier les stratégies d'irrigation pertinentes.

a. **Approche numérique.** L'étude numérique de modèles d'irrigation complexes a fait l'objet de plusieurs études: Dans [33], Domineguez et al. résolvent un problème à quatre variables en utilisant un solveur appelé LINDO pour l'optimisation. Des algorithmes évolutionnaires ont été proposés dans [13, 49, 62, 77]. Des approches de programmation dynamiques ont aussi été réalisées. comme par exemple dans les travaux de Schutze et al [62] et Zhang [80].

b. **Approche analytique.** Lorsque le modèle est simplifié, une étude analytique en utilisant les outils mathématique peut être abordée. Essentiellement, la théorie du contrôle optimal est largement utilisée dans ce type de problème ([52, 53, 64, 82]). Nous décrirons les travaux réalisés dans le paragraphe suivant. Des approches probabilistes ont également été utilisées comme on peut le voir dans les travaux de Vico et Porporato dans [74] ou ils établissent un lien entre l'occurrence des précipitations et l'irrigation, d'une part, et la croissance des plantes et les rendements à la récolte, d'autre part. À partir de ce lien, ils définissent la fonction de densité de probabilité qui permet de prendre des décisions en situation d'incertitude pour les diverses stratégies d'irrigation.

I.1.1.3 Le contrôle optimal et l'optimisation de l'irrigation.

Dans la modélisation mathématique, un système dynamique décrit un état qui évolue au cours du temps suivant une loi d'évolution appelée la dynamique. Par exemple: l'évolution de la croissance des cultures au cours du temps. Lorsqu'une variable dépendante du temps est ajoutée au système dynamique, l'évolution de l'état au cours du temps est par conséquent influencée. On parle dans ce cas de systèmes contrôlés. De manière générale, un système contrôlé s'écrit:

$$\begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \\ x(0) &= x_0, \end{cases} \tag{I.1.3}$$

où $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ est la dynamique, $x(t)$ désigne l'état du système à l'instant $t \in [0, T]$. La variable du contrôle $u(\cdot)$ est une fonction mesurable bornée qui prend des valeurs dans l'ensemble des contrôles admissibles $\mathcal{U}$, ainsi défini:

$$\mathcal{U} := \{u : [0, T] \to U;\ u(\cdot) \text{ mesurable } \},$$

où U est un sous-ensemble non-vide de $\mathbb{R}^m$. L'évolution de l'état dans ce cas dépend du temps et du contrôle $u(\cdot)$ qui à son tour peut varier en temps ou en espace. Le choix de cette variable va alors contrôler, commander ou diriger l'évolution du système. Le choix de l'ensemble $\mathcal{U}$ assez riche permet de garantir l'existence d'une solution optimale sous certaines conditions.

Nous distinguons trois types de contrôles: les contrôle en boucle ouverte (aussi appelé *time-varying* ou *openloop* en anglais) $t \mapsto u(t)$, les contrôles en boucle fermée (aussi appelé *feedback* en anglais) $x \mapsto u(x)$ et les contrôles en boucle fermée non-autonomes (aussi appelé *time-varying feedback* en anglais) $(t, x) \mapsto u(t, x)$. Dans un système de contrôle en boucle ouverte, l'action de commande du contrôleur au cours du temps ne dépend que de la condition initiale. Un exemple est un arrosage automatique qui ne prendrait pas en considération la mesure au cours du temps de l'humidité du sol réelle. Dans un système de contrôle en boucle fermée, l'action de commande du contrôleur prend en compte l'évolution de l'état du système. Dans l'exemple de l'arrosage automatique, le programmateur fonctionnerait en fonction de l'humidité du sol.

Un domaine particulier de la théorie du contrôle est la théorie du contrôle optimal ou de la commande optimale, une théorie apparue après la seconde guerre mondiale qui consiste à déterminer un contrôle de telle sorte que l'état doit satisfaire une certaine condition initiale tout en optimisant un certain coût et en vérifiant des contraintes données. Cette théorie a été introduite pour faire face à des problématiques de guidage et d'optimisation dans plusieurs domaines notamment les problèmes liés à l'aéronautique. La théorie du contrôle a eu également des applications sur la planification de l'irrigation des cultures où généralement l'irrigation est la variable de contrôle employée. Dans [52, 53], par exemple, S. O. Lopes et al. ont considéré un problème de contrôle optimal avec un modèle simplifié qui prend en compte l'apport naturel d'eau provenant des pluies. Le problème de commande optimale consiste à minimiser l'eau utilisée pour l'irrigation sous une contrainte sur le stress hydrique. Les auteurs ont ensuite caractérisé la solution optimale analytiquement à fin de valider les résultats numériques en utilisant le célèbre Principe du Maximum de Pontryagin avec contrainte d'état. Dans les travaux de U. Shani et al. [64], les auteurs utilisent les outils de la théorie du contrôle optimal sur un modèle dynamique de croissance des plantes dans le but de trouver des stratégies optimales d'irrigation en tenant compte à la fois du coût de l'eau apporté par l'irrigation, l'évolution de la production de biomasse et l'humidité du sol. Dans leur étude, ils démontrent que la politique optimale d'irrigation consiste en 3 phases:

- Tout d'abord, l'humidité du sol doit atteindre une certaine valeur optimale à partir du point initial dans les meilleurs délais.
- Ensuite, elle doit être maintenue sur cette valeur.
- Puis cesser l'irrigation à un certain moment avant la récolte.

Cette politique ne considère pas la production maximale de biomasse étant donné que le critère d'optimisation correspond au profit du rendement.

I.1.2 Critère et contraintes.

Le choix du critère à optimiser et des contraintes à imposer sont primordiales à la définition des problématiques liées à la gestion de l'irrigation. Choisir les critères et les contraintes adéquats est alors important pour mettre en place le problème de contrôle optimal et l'examiner in fine.

Travaux existants. Plusieurs critères dans la littérature ont été utilisés: Dans [64] par exemple, les auteurs ont pris comme critère la maximisation du profit résultant de la différence du prix de la production et le coût de l'eau. La solution optimale obtenue dans ce cas ne fournit pas une production maximale. En effet, une production de culture maximale demanderait plus d'eau pour l'irriguer. L'eau étant coûteuse, une politique qui maximise la production pourrait entraîner une perte du bénéfice. Par ailleurs, dans [52, 53] par exemple, les auteurs ont considéré la minimisation de la quantité d'eau utilisée pour l'irrigation tout en imposant une contrainte sur l'humidité du sol. La stratégie optimale est alors définie en fonction du seuil du stress hydrique imposé (l'humidité du sol ne doit pas passer au-dessous du seuil de contraintes).

Les travaux de cette thèse s'inscrivent dans la recherche d'une politique optimale d'irrigation dans le cas d'une contrainte sur la quantité d'eau disponible pour l'irrigation (pénurie d'eau).

Critère et contrainte en situation de pénurie d'eau. Chaque plante ayant un seuil d'humidité nécessaire pour garantir un épanouissement et une croissance optimale, il est important de considérer les contraintes qui pèsent sur la disponibilité de l'eau qui est souvent limitée. Une pénurie en eau où l'eau disponible n'est pas suffisante pour permettre une bonne croissance de la plante, entraîne une perte en production de biomasse. Par conséquent, la prise en compte d'une production optimale de cultures doit être envisagée comme un critère à optimiser afin d'obtenir une bonne stratégie d'irrigation. L'objectif consiste alors à atteindre le meilleur compromis possible entre l'exploitation d'eau disponible pour l'irrigation et la maximisation de production végétale sous ces conditions délicates. Ainsi, une stratégie optimale d'irrigation peut consister à maximiser le critère de production de biomasse au moment de la récolte sous la contrainte du quota d'eau disponible. Ce problème est étudié en détail dans le chapitre 2.
Un autre critère qui peut être envisagé dans la recherche du bon compromis entre la production végétale et la disponibilité de l'eau est de considérer la minimisation du temps toléré durant lequel l'humidité du sol passe au dessous du seuil optimal pour la croissance de plante. Les contraintes seront dans ce cas le quota d'eau disponible ainsi que la production; c'est le problème de minimisation du temps de crise. Ce problème de commande optimale est étudié d'un point de vue numérique dans le chapitre 3 et dans un cadre général dans la partie III du manuscrit.

I.2 Étude et optimisation d'un modèle d'irrigation

I.2.1 Formulation du problème.

Dans cette thèse, nous nous intéressons à un modèle de culture simple, donné par un système d'équations différentielles ordinaires non-autonomes décrivant les évolutions de l'humidité du sol et de la biomasse produite au cours du temps. Afin d'obtenir une information qualitative sur une stratégie efficace en matière d'irrigation, Dans le cas où la quantité d'eau disponible pour irriguer est limitée par un quota, les stratégies à employer deviennent moins intuitives. La question naturelle qui se pose: Sous contrainte de quota en eau, quelle serait la bonne stratégie à adopter afin d'avoir une production de biomasse optimale au moment de la récolte ? C'est un problème de commande optimale: Le but est de chercher une loi de commande qui va permettre d'optimiser le critère de biomasse au moment de récolte. Formellement, le problème de commande optimale s'écrit comme suit:

$$\begin{cases} \max\limits_{u(\cdot)} B(T) \quad \text{tel que} \\ \dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)), \quad S(0) = S_0, \\ \dot{B} = \varphi(t)K_S(S)f(B), \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad B(0) = B_0, \\ S(t) \in [0,1], \ \forall t \in [0,T] \\ \int_0^T u(t)\mathrm{d}t \leqslant \bar{Q}, \end{cases} \tag{I.2.1}$$

avec $S_0 \in [S^\star, 1]$ et $B_0 > 0$. T représente le moment de la récolte et $\bar{Q}$ le quota en eau disponible, avec $u \in [0, u_{max}]$. Sans perte de généralité on prend $u_{max} = 1$. Ce modèle est une généralisation du modèle présenté dans [47] avec la fonction f qui est une fonction de croissance de biomasse.

Ce problème de commande optimale est un problème de Mayer avec contrainte d'état et de cible. Pour faire face à ce type de problèmes, de nombreux chercheurs en mathématiques se sont intéressés à la caractérisation des conditions d'optimalité pour lesquelles un contrôle optimal peut être dérivé. Des progrès considérables dans l'étude des conditions d'optimalité ont été réalisé en URSS en 1958 par le mathématicien L. Pontryagin et ses étudiants (cf. [58]), qui ont formulé le célèbre Principe du Maximum de Pontryagin (PMP), qui fournit (sous certaines hypothèses de régularité) un ensemble de conditions nécessaires pour des problèmes de contrôle optimal auquel un optimum (s'il existe) doit satisfaire. Des généralisations de ce théorème au cas de contrainte de cible et contrainte d'état ont été également établies (voir [75]). Une extension du PMP lorsque les hypothèses de régularité sur la dynamique ne sont pas respectées (comme dans notre modèle avec les deux fonctions K_S et K_R) a également été mise en place (voir [26]) sous le nom du Principe du maximum de Pontryagin généralisé de Clarke (PMP généralisé ou non-lisse).

Avant de commencer à étudier la solution optimale, nous mettons en place les hypothèses suivantes.

Assumption I.2.0a. *Les fonctions K_S et K_R sont linéaires par morceaux non décroissantes de $[0,1]$ à $[0,1]$ avec des nombres $0 < S_h < S_w < S^\star < 1$ tels que*

1. *K_S, resp. K_R est nulle sur $[0, S_w]$, resp. $[0, S_h]$, et positive en dehors de cet intervalle.*
2. *K_S est égale à 1 sur $[S^\star, 1]$ et concave croissante sur $[S_w, S^\star]$.*
3. *$K_R(1) = 1$ et K_R est convexe croissante sur $[S_w, S^\star]$.*

Cette hypothèse généralise les expressions trouvées dans la littérature (cf. [47, 57]) qui est de cette forme:

Assumption I.2.0b. *Les fonctions K_S et K_R sont linéaires par morceaux non décroissantes de $[0,1]$ à $[0,1]$. En particulier, il est souvent considéré les fonctions suivantes:*

$$K_S(S) = \begin{cases} 0 & S \in [0, S_w], \\ \dfrac{S - S_w}{S^\star - S_w} & S \in [S_w, S^\star], \\ 1 & S \in [S^\star, 1], \end{cases} \qquad K_R(S) = \begin{cases} 0 & S \in [0, S_h], \\ \dfrac{S - S_h}{1 - S_h} & S \in [S_h, 1], \end{cases} \tag{I.2.2}$$

où $0 < S_h < S_w < S^\star < 1$.

Assumption I.2.1. *La fonction φ est C^1 croissante avec $\varphi(0) \geq 0$ et $\varphi(T) \leq 1$.*

Assumption I.2.2. *k_1, k_2 sont des paramètres positifs avec $k_2 \geq 1$.*

Notons que dans le cas où $\bar{Q}$ est suffisamment grand, le problème de commande optimale (I.2.1) est trivial. En effet, quelque soit le contrôle dont la trajectoire associée S reste dans le domaine au-dessus du seuil hydrique $S^\star$ en tout temps $t \in [0, T]$, la biomasse produite est maximale et elle est donnée par la valeur

$$B_T^* := k_3 \int_0^T \varphi(t) \mathrm{d}t.$$

Le problème de commande optimale se pose alors et devient pertinent lorsqu'on se place en cas de pénurie d'eau où le quota en eau $\bar{Q}$ n'est pas suffisamment grand et oblige les trajectoires à passer au dessous du seuil $S^\star$. C'est ce cas que nous allons traiter dans cette thèse.

I.2.2 Résumé du Chapitre 2

Dans ce premier chapitre, nous nous intéressons au problème de commande optimale (I.2.1) et à la caractérisation de la solution optimale pour ce problème, nous suivons ces étapes.

1. Étude de la contrainte d'état en comparant les trajectoires au dessus du seuil $S^\star$.

2. Application du PMP généralisé et étude de l'optimalité des arcs singuliers.

3. Étude des différentes possibilités de structures optimales.

Étude de la contrainte d'état.

Pour étudier le problème (I.2.1), nous traitons dans un premier temps, la contrainte d'état en comparant les trajectoires lorsqu'elles sont au-dessus du seuil $S^\star$. Pour cela, nous introduisons un contrôle dit "MRAP" (*Most Rapid Approach Path*), ce contrôle permet à sa trajectoire associée de rejoindre au plus vite le seuil $S*$ et y rester. De manière simple, nous définissons le contrôle MRAP entre $(0, S_0)$ et $(T, S^\star)$ comme suit:

$$\tilde{u}(t) := \begin{cases} 0 & t \in [0, \underline{t}) \\ u_{S^\star}(t) & t \in (\underline{t}, T] \end{cases}$$

avec $\underline{t} := \sup\{t \in [0,T] \text{ s.t. } \underline{S}(t) > S^\star\} < T$ le premier instant où la trajectoire associée au contrôle nulle touche $S^\star$ et $u_{S^\star}(t) := \frac{\varphi(t)+(1-\varphi(t))K_R(S^\star)}{k_2}$ c'est le contrôle qui permet de rester sur l'arc $S^\star$. Le contrôle MRAP est utilisé comme un outil de

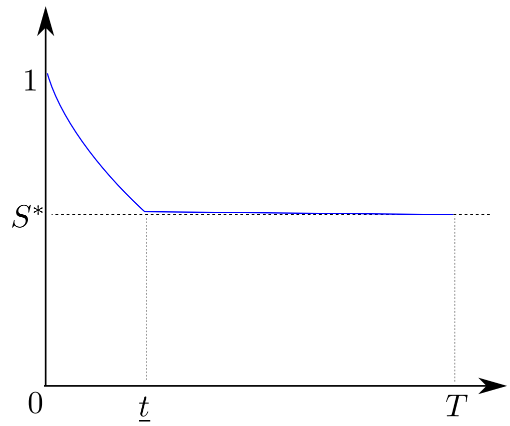


FIGURE I.2 – Trajectoire associée au contrôle MRAP avec $S_0 = 0$

comparaison entre trajectoire comme le montre la proposition suivant:

Proposition I.2.1. *Soit $u(\cdot)$ un contrôle admissible et $S(\cdot)$ sa trajectoire associée telle que $S(t) \geq S^\star$ pour tout $t \in [0,T]$. Alors, le contrôle MRAP $\tilde{u}(\cdot)$ avec sa trajectoire associée $\tilde{S}(\cdot)$ vérifient:*

$$\tilde{S}(t) \leq S(t) \quad \textit{et} \quad \int_0^T \tilde{u}(t)\, dt \leq \int_0^T u(t)\, dt.$$

Ce résultat (voir la preuve détaillée dans la Proposition 2.4.1) fait valoir que pour toutes les trajectoires au-dessus du seuil $S^\star$, la trajectoire associée au contrôle MRAP correspond à celle consomme le moins d'eau. De plus, pour tout contrôle admissible $u(\cdot)$ tel que la trajectoire associée $S(t) \geq S^\star$ pour tout $t \in [0,T]$, la biomasse produite au temps final $B(T)$ est maximale. Le contrôle MRAP serait donc un bon candidat pour la solution optimale. Cependant nous avons l'hypothèse suivante:

(H1) $\underline{t} < T$ et $\bar{Q} < \int_0^T \tilde{u}(t)\mathrm{d}t.$

Cette hypothèse correspond à des situations de pénurie d'eau où l'eau disponible n'est pas suffisante pour maintenir l'humidité dans le domaine au-dessus $S^\star$ et

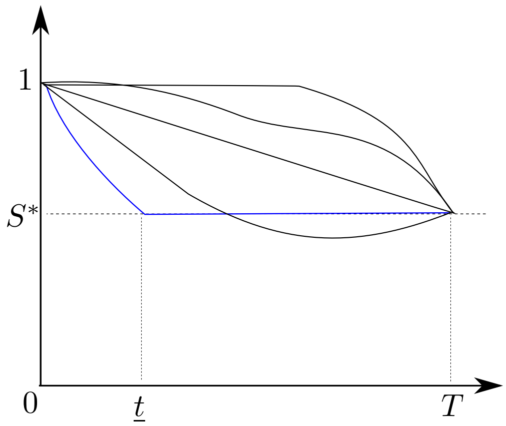


FIGURE I.3 – Comparaison entre la trajectoire associée au MRAP (en bleu) avec d'autres trajectoires

ainsi obtenir une production maximale au moment de la récolte. Il en résulte la proposition ci-dessous.

Proposition 1.2.2. *Supposons que l'hypothèse* (H1) *soit satisfaite. Alors, toute solution optimale satisfait les propriétés suivantes.*

(i) $u(t) = 0$ *pour presque tout* $t \in [0, \underline{t}]$,

(ii) $S(t) \leq S^\star$ *pour tout* $t \in [\underline{t}, T]$,

(iii) $\int_0^T u(t)\mathrm{d}t = \bar{Q}$.

Ce premier résultat sur la structure de la solution optimale indique qu'une solution doit atteindre au plus vite le domaine d'au-dessous de $S^\star$ et y demeurer tant que l'eau disponible pour l'irrigation n'est pas épuisée. Ce résultat montre en outre que la contrainte d'état n'est jamais saturée pour une solution optimale i.e., la solution optimale ne peut pas rester, rejoindre ou passer au-dessus de 1. (Voir Proposition 2.4.2 pour la preuve détaillée)
Nous allons maintenant étudier le comportement de la solution optimale à partir de l'instant $\underline{t}$, le domaine d'au-dessous de $S^\star$.

Étude du domaine au-dessous de $S^\star$

Dans cette partie, nous sommes intéressés par la structure de la solution optimale sur l'intervalle de temps $[\underline{t}, T]$. Nous commençons en reformulant le problème de contrôle optimal de la manière suivante:

$$\max_{u(\cdot)} \int_0^T \varphi(t) K_S(S(t))\, dt \tag{1.2.3}$$

tel que

$$\dot{S} = k_1\Big(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)\Big), \quad S(0)=1 \tag{1.2.4}$$

$$\dot{V} = u(t), \quad V(0) = 0 \tag{1.2.5}$$

avec la cible

$$V(T) \leq \bar{V} := \frac{\bar{Q}}{F_{max}}. \tag{1.2.6}$$

C'est un problème de Lagrange avec cible. Compte tenu du fait que la dynamique n'est pas suffisamment lisse, le PMP classique ne peut pas être appliqué. Nous appliquons alors le PMP généralisé [28]: Soit $H : [0,T] \times [0,1] \times [0,\bar{V}] \times [0,1] \to \mathbb{R}$ le *Hamiltonien* associé à ce problème de contrôle tel que:

$$\begin{aligned} &H(t, S, \lambda_S, \lambda_V, u) := \\ &\quad \lambda_S k_1\Big(k_2 u - (\varphi(t)K_S(S) + (1-\varphi(t))K_R(S))\Big) + \lambda_V u + \lambda_0 \varphi(t) K_S(S), \end{aligned} \tag{1.2.7}$$

ainsi que le système adjoint:

$$\dot{\lambda}_S \in \varphi(t)\Big(\lambda_S k_1 - \lambda_0\Big)\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S k_1 \partial_C K_R(S(t)), \tag{1.2.8}$$

$$\dot{\lambda}_V = 0, \tag{1.2.9}$$

et $\partial_C K_S$, $\partial_C K_R$ désigne le sous-différentiel généralisé de Clarke des fonctions Lipschitz K_S, K_R. Le Principe de Maximum de Pontryagin affirme que pour une solution optimale $S(\cdot)$, $V(\cdot)$, $u(\cdot)$, il existe un état adjoint $\lambda(\cdot) = (\lambda_S(\cdot), \lambda_V(\cdot))$ absolument continue solution du système adjoint (1.2.8)-(1.2.9) et un scalaire positif λ_0 tels que

$$\lambda_0 + |\lambda_S(t)| + |\lambda_V(t)| \neq 0, \quad t \in [0,T], \tag{1.2.10}$$

qui satisfont à la condition de transversalité

$$\lambda_S(T) = 0, \quad \lambda_V \leq 0, \tag{1.2.11}$$

ainsi que la condition de maximisation

$$H(t, S(t), \lambda_S(t), \lambda_V(t), u(t)) = \max_{v\in[0,1]} H(t, S(t), \lambda_S(t), \lambda_V(t), v), \quad \text{p.p. } t \in [0, T]. \tag{1.2.12}$$

Dans notre cas d'étude, le Hamiltonien est linéaire en contrôle, la fonction de commutation $\phi(\cdot)$ est alors:

$$\phi(t) := \lambda_S(t)k_1k_2 + \lambda_V, \tag{1.2.13}$$

La condition de maximisation (1.2.12) donne pour presque tout $t \in [0, T]$:

$$\begin{cases} u(t) = 1 & \text{si } \phi(t) > 0, \\ u(t) \in [0, 1] & \text{si } \phi(t) = 0, \\ u(t) = 0 & \text{si } \phi(t) < 0. \end{cases} \tag{1.2.14}$$

La condition de maximisation avec la fonction de commutations donne les propriétés suivantes des vecteurs adjoint:

1. Pour toute solution optimale, on a $\lambda_0 = 1$.
2. Pour toute solution optimale, on a $\lambda_S(t) \geq 0$ pour tout $t \in [0, T]$. De plus, on a $\lambda_V < 0$.

Par suite de ces propriétés, nous obtenons le résultat suivant.

Corollaire 1.2.1. *Pour toute solution optimale, il existe $\bar{t} < T$ tel que $u(t) = 0$ pour presque partout $t \in [\bar{t}, T]$. De plus, on a $S(T) < S^\star$.*

En d'autres termes, dans une stratégie optimale, il faut cesser l'irrigation vers la fin avant le moment de la récolte et laisser la plante poursuivre sa croissance avec le reste de l'humidité sur le sol.

Optimalité des arcs singuliers. Le problème de commande optimal est réduit à la sélection du meilleur contrôle $u(\cdot)$ parmi les extrémales. Si la condition de maximisation est atteinte sur des points où le Hamiltonien H ne dépend pas du contrôle u, alors il est nécessaire de faire une étude supplémentaire pour obtenir des informations sur le contrôle optimal. Un contrôle $u(\cdot)$ est dit contrôle singulier si la fonction de commutation reste égale à zéro sur un intervalle de temps non vide. Un arc singulier est alors la solution associée au contrôle singulier. Ci-dessous sont le résultat principal sur l'existence et l'optimalité des arcs singuliers pour notre problème.

Proposition 1.2.1. *Un arc singulier sur un intervalle fermé I non réduit à un singleton satisfait $S(t) = \tilde{S}$, $t \in I$ où $\tilde{S}$ est un point non-différentiable de $K_S(\cdot)$ ou $K_R(\cdot)$ qui appartient à $(S_w, S^\star]$.*

Voir Proposition 2.5.2 pour la preuve. Cette propriété remarquable est due à la conjonction de la non-différentiabilité des fonctions $K_S(\cdot)$ ou $K_R(\cdot)$ et au fait que $\varphi(\cdot)$ soit une fonction du temps, sinon un arc singulier ne serait pas possible.
Nous montrons ensuite qu'un arc singulier ne peut se produire qu'aux points au-dessus de S_w où les fonctions de stress hydrique $K_S(\cdot)$ et $K_R(\cdot)$ ne sont pas différentiables. À partir de S_w et en deçà la plante continue de consommer de l'eau mais ne produit plus de biomasse. En effet, pour tout $S \leq S_w$ on a $K_S(S) = 0$ et par conséquent $\dot{B} = 0$. Par contradiction on peut considérer qu'il est optimal de rester sur ou au-dessous de S_w et on construit un autre contrôle qui donne la même production de biomasse mais avec une consommation d'eau plus faible. Ensuite, à partir de ce contrôle, on en construit un autre qui épuisera toute l'eau disponible de façon à atteindre un meilleur objectif, ce qui est contradictoire avec l'hypothèse.

Structure de la solution optimale. Nous nous intéressons aux diverses structures optimales possibles, en tenant compte l'optimalité des arcs singuliers au points où les fonctions K_S et K_R ne sont pas lisses. Nous définissons en premier lieu, les sous-ensembles de points de non-différentiabilité des fonctions $K_S(\cdot)$ ou $K_R(\cdot)$ comme suit.

Définition 1.2.3. *Pour tout $S \in (S_w, S^\star]$, soit $\mathcal{C}(S)$ l'ensemble des points de non- différentiabilité $\tilde{S} \geq S$ dans $(S_w, S^\star]$ et $n(S) = card\,\mathcal{C}(S)$. On définit alors la suite croissante de points de non-différentiabilité des fonctions $K_S(\cdot)$ et $K_R(\cdot)$, $\tilde{S}_i(S)_{i=1\ldots n(S)}$ telle que $\mathcal{C}(S) = \{\tilde{S}_1(S), \ldots, \tilde{S}_{n(S)}(S)\}$.*

Pour tout $S \in (S_w, S^\star]$, l'ensemble $\mathcal{C}(S)$ est non vide. Il contient au moins $S^\star$ comme plus grand élément, i.e., $\tilde{S}_{n(S)}(S) = S^\star$. Nous définissons ensuite les stratégies d'irrigation suivantes.

Définition 1.2.4. *Pour $S_m \in (S_h, S^\star]$ et une séquence de nombres non décroissants $V_i \in (0, \bar{V}]$, $i \in \{1, \ldots, n(S_m)\}$ dont au moins un est égal à $\bar{V}$, on définit le contrôle par retour* *d'état non stationnaire (time-varying feedback) comme suit :*

$$\psi^{SMS}_{S_m,\{V_i\}}(t,S,V) := \begin{cases} 0 & \text{si } V = \bar{V} \text{ or } S > S_m \text{ avec } V = 0, \\ \tilde{u}_S(t) & \text{si } S = \tilde{S}_i(S_m) \text{ pour } i \in \{1,\dots,n(S_m)\} \text{ avec } V < V_i, \\ 1 & \text{sinon.} \end{cases} \tag{1.2.15}$$

Cette stratégie appelée "SMS" pour *Satured Multiple Shot* consiste à démarrer l'irrigation, en une ou plusieurs étapes, lorsque le niveau d'humidité $S(t)$ atteint un seuil déclencheur pour l'irrigation S_m. Si le niveau d'humidité $S(t)$ atteint une valeur $\tilde{S}_i(S_m)$ pour un certain $i \in \{1,\dots,n(S_m)\}$ le débit est saturé pour maintenir S sur cette étape aussi longtemps que le volume d'eau utilisé $V(t)$ reste en dessous de la valeur V_i. Dans le cas où V_i n'est pas assez grand ou si $\tilde{S}_i(S_m)$ ne peut être atteint, alors il n'y aura pas de saturation du débit pour cette valeur. La trajectoire générée S présente alors au plus $n(S_m)$ apports saturés croissants par étape. Remarquons également qu'une fois que $S(\cdot)$ a atteint S_m alors $S(t) \leq S^\star$ pour tout temps futur.

Définition 1.2.5. *Soit $S_m \in (S_h, S^\star]$. on définit le contrôle par retour d'état non stationnaire (time-varying feedback)comme suit :*

$$\psi^{SOS}_{S_m}(t,S,V) := \begin{cases} 0 & \text{si } V = \bar{V} \text{ ou } S > S_m \text{ avec } V = 0, \\ \tilde{u}_{S^\star}(t) & \text{si } S = S^\star \text{avec } V < \bar{V}, \\ 1 & \text{sinon.} \end{cases} \tag{1.2.16}$$

La stratégie "SOS" pour *Satured One Shot* consiste à irriguer les cultures par une seule étape lorsque le taux d'humidité $S(t)$ atteint S_m. L'eau est délivrée au débit maximal ($u = 1$) tant que le taux d'humidité $S(\cdot)$ est inférieur à $S^\star$, ou le maintenir $S = S^\star$ (avec la commande singulière $\tilde{u}_{S^\star}(\cdot)$), jusqu'à ce que le volume total d'eau $\bar{V}$ soit épuisé. Cette stratégie est paramétrée par la valeur unique S_m ou de manière équivalente le moment où l'irrigation est déclenchée. Cette stratégie est un cas particulier de la stratégie SMS (1.2.15) où un seul seuil est saturé.

Définition 1.2.6. *Soit $t_S \in (0,T)$ le temps de déclenchement*[2] *associé à un seuil d'humidité $S_m \in (S_h, S^\star]$. On définie le contrôle open-loop comme suit :*

$$u^{OS}_{S_m}(t) := \begin{cases} 0 & \text{si } t < t_S \text{ or } t > \min(t_S + \bar{V}, T)), \\ 1 & \text{si } t \in [t_S, \min(t_S + \bar{V}, T)). \end{cases} \tag{1.2.17}$$

2. Le temps de déclenchement est l'instant de la première commutation du contrôle u=0 à u=1 ou u singulier.

La stratégie OS pour *One Shot* est un contrôle "bang-bang" classique en boucle ouverte, elle consiste à trouver le seuil de déclenchement S_m qui fournit la meilleure production de biomasse. Une fois que l'irrigation commence, elle dure $\bar{Q}/F_{max}$. Cette stratégie est un cas particulier de la stratégie SOS, lorsque la quantité d'eau n'est pas suffisante pour atteindre et saturer le seuil $S^\star$. Nous avons alors les résultats suivants sur l'optimalité des stratégies SMS et SOS.

Théorème 1.2.7. *Sous les hypothèses I.2.0a, I.2.1 et I.2.2 il existe une valeur $S_m \in (S_h, S^\star]$ et une suite de nombres non décroissants V_i, $i = 1, \dots, n(S_m)$ avec au moins un égal à $\bar{V}$ tels que le contrôle SMS* (1.2.15) *avec*

$$\int_0^{t_M} \psi^{SMS}_{S_m,\{V_i\}}(t, S(t), V(t))\, dt = \bar{V} \quad \textit{pour un } t_M < T \tag{1.2.18}$$

est optimale.

Théorème 1.2.8. *Sous les hypothèses I.2.0b, I.2.1 et I.2.2, il existe une valeur $S_m \in (S_h, S^\star]$ telle que la stratégie SOS* (1.2.16) *avec*

$$\int_0^{t_M} \psi^{SOS}_{S_m}(t, S(t), V(t))\, dt = \bar{V} \quad \textit{pour un } t_M < T, \tag{1.2.19}$$

est optimal.

La preuve de ces résultats repose principalement sur la connexité de l'ensemble

$$C := \{t \in [\underline{t}, T] \text{ s.t. } \phi(t) \geq 0\}.$$

En effet, si C est connexe alors cela veut dire que la fonction de commutation ne peut pas passer de strictement négatif à positif plus de deux fois. Par conséquent les structures du contrôle optimal possibles sont :

1. Sous l'hypothèse I.2.0a et tant que S est au dessous de $S^\star$: $u = 0 \to u$ singulier $\to u = 1 \to u$ singulier $\cdots u = 1$ ou singulier $\to u = 0$. Ce qui correspond au contrôle SMS.

2. Sous l'hypothèse I.2.0b: $u = 0 \to u = 1 \to u$ singulier $\to u = 0$. Ou bien I.2.0b: $u = 0 \to u$ singulier $\to u = 0$ ce qui correspond au contrôle SOS. Ou bien encore $u = 0 \to u = 1 \to u = 0$ si S ne dépasse pas $S^\star$.

La structure des stratégies SMS et SOS se distingue des solutions obtenues dans [64] et [53]. En effet, dans les travaux cités la politique du contrôle optimal a la structure d'une stratégie SOS mais avec un niveau de déclenchement qui correspond à l'arc singulier. Toutefois, les modèle ainsi que les critère pris en considération sont différents (avec des apports pluviométriques et sans quota d'eau) ce qui en résulte une étude analytique différente. Ces différences entre ces modèles, leur critère et leurs solutions optimales sont intéressantes et enrichissent l'analyse mathématique de notre problème.

1.2.3 Résumé du chapitre 3

Dans ce chapitre nous nous intéressons à une application du problème de temps de crise minimal sur le modèle de culture introduit au Chapitre 2 à fin de caractériser une stratégie d'irrigation optimale dans le cas de contrainte en eau. Le problème s'écrit comme suit: Soit K le sous-ensemble défini par $K = \{S \in [0,1];\ S \geq S_{crisis}\}$ où $S_{crisis} \in [S_w, S^\star]$ un paramètre donné qui représente un seuil de stress hydrique, nous considérons le problème de contrôle optimal appliqué au modèle d'irrigation:

$$\inf_{u \in \mathcal{U}} \int_0^T \mathbb{1}_{K^c}(S(t))\,\mathrm{d}t, \tag{1.2.20}$$

avec la cible qui représente la contrainte sur la disponibilité d'eau pour l'irrigation

$$V(T) \leq \bar{V}$$

où $\bar{V} > 0$ est un paramètre représentant le quota en eau disponible. Le problème de contrôle optimal est sujet au système contrôlé suivant

$$\begin{cases} \dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)), \quad S(0) = 1 \\ \dot{B} = \varphi(t)K_S(S), \quad B(0) = 0 \\ \dot{V} = u(t), \quad V(0) = 0. \end{cases} \tag{1.2.21}$$

Le caractère particulier du problème est d'une part la discontinuité de l'intégrande, et la non-différentiabilité par rapport à l'état d'autre part, ce qui présente une grande difficulté pour traiter le problème analytiquement. Dans ce chapitre, nous examinons une approche numérique faisant appel au solver tels BocopHJB [17] qui permet la résolution des problèmes de contrôle optimal en temps continu et repose sur le principe de programmation dynamique.

Les résultats numériques montrent qu'une solution optimale du problème (1.2.20)

consiste à rejoindre le seuil de crise S_{crisis} au plus vite avec le contrôle nul et ensuite y rester jusqu'à épuisement de l'eau disponible pour l'irrigation. Ce type de stratégie fait partie de la classe des stratégies SOS rencontrées dans le chapitre 2 où le seuil de déclenchement coïncide avec le seuil de crise. Cette stratégie ne produit pas forcément la meilleure production de biomasse cependant elle reste une stratégie adéquate qui consomme le moins d'eau et minimise le temps de stress de la plante.
Nous relatons dans ce chapitre des difficultés numériques rencontrées et des solutions optimales qui traversent de façon non transverse l'ensemble K. Ce sont ces deux points qui ont motivé les études théoriques (dans un cadre plus général) des chapitres 4 et 5.

1.3 Minimisation du temps de crise.

Une crise est un événement qui se produit dans de nombreux domaines tels que la santé, l'agriculture ou l'environnement. Typiquement, le bon fonctionnement d'un système est caractérisé par certaines contraintes, qui doivent être satisfaites en permanence. Lorsque ces contraintes sont violées, on entre dans une *période de crise*. Par exemple, un problème en agriculture est formé par l'étude des variations de l'humidité du sol et des rendements. Les contraintes sont alors définies par des seuils. La période de crise représente le temps passé sous les seuils imposés, pénalisant la croissance des plantes si elle est inférieure à ces seuils. Une façon intéressante de traiter ce problème serait donc de minimiser la période de crise.

1.3.1 Formulation du problème et contexte scientifique.

Formellement, Le problème de minimisation du temps de crise est défini comme suit: Soit $T > 0$ fixé, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ une fonction, K un sous-ensemble non vide de $\mathbb{R}^n$, U un sous-ensemble non vide de $\mathbb{R}^m$, nous considérons un système dynamique contrôlé général comme suit:

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)), \\ x(0) &= x_0, \end{cases} \tag{S}$$

avec $x_0 \in \mathbb{R}^n$. On définit l'ensemble des contrôles admissibles $\mathcal{U} \subset \mathbb{R}^m$ comme suit:

$$\mathcal{U} := \{u : [0,T] \to U \ ; \ u \text{ mesurable}\}.$$

Le problème du temps de crise sur l'horizon fini $[0,T]$ est un problème de Lagrange défini comme suit:

$$\inf_{u \in \mathcal{U}} \int_0^T \mathbb{1}_{K^c}(x_u(t))\, \mathrm{d}t, \tag{TC}$$

où $x_u(\cdot)$ est la solution du problème de Cauchy (S) associée au contrôle u et $\mathbb{1}_{K^c}$ est la fonction caractéristique du complémentaire de l'ensemble de contraintes K. *i.e.*,

$$\mathbb{1}_{K^c}(x) = \begin{cases} 1 & \text{si } x \in K^c, \\ 0 & \text{si } x \in K. \end{cases}$$

Nous supposerons que cet ensemble K est défini comme un ensemble de niveau d'une fonction de classe C^1 que nous allons noter φ, définie sur $\mathbb{R}^n$ à valeur dans $\mathbb{R}$. i.e.,

$$K := \{x \in \mathbb{R}^n, \quad \varphi(x) \leq 0\}.$$

Le complémentaire de l'ensemble K, noté K^c, est appelé *ensemble de crise*. L'objectif du problème de temps de crise est alors de trouver un contrôle $u \in \mathcal{U}$ qui permet de minimiser le temps pendant lequel une trajectoire $x_u(\cdot)$ est à l'extérieure de K.

Le problème du temps de crise minimal a été introduit et proposé initialement par L. Doyen et P. Saint-Pierre [34] comme une extension de la théorie de viabilité, un domaine des mathématiques qui consiste à étudier l'évolution des systèmes dynamiques sous des contraintes sur l'état (voir [4, 3]). Les notions de viabilité dépendent de ce qui se passe à l'intérieur d'un domaine de contrainte donné. En effet, une trajectoire $x(\cdot)$ solution de (S) est dite viable dans l'ensemble de contrainte K si

$$\forall t \in [0, +\infty[, \quad x(t) \in K.$$

Ce type de problèmes de contrôle optimal avec contrainte d'état ont été largement étudiés dans la littérature (voir [75]). Une condition essentielle dite *inward pointing condition* sur le bord de l'ensemble, oblige les trajectoires à rester dans l'ensemble de contrainte. Lorsque cette condition n'est pas satisfaite, on s'intéresse à un outil qui joue un rôle fondamental dans la théorie de viabilité qui est *le noyau de viabilité* associé au système. Cela correspond au plus grand ensemble de conditions initiales dans K à partir desquelles il existe une solution du système (S) qui ne quitte pas l'ensemble K pour tout temps:

$$\text{Viab}_f(K) := \{x_0 \in K \; ; \; \exists u \in \mathcal{U} \text{ s.t. } x_{u,x_0}(t) \in K, \; \forall t \in [0, +\infty[\},$$

où $x_{u,x_0}(\cdot)$ est solution du système (S).

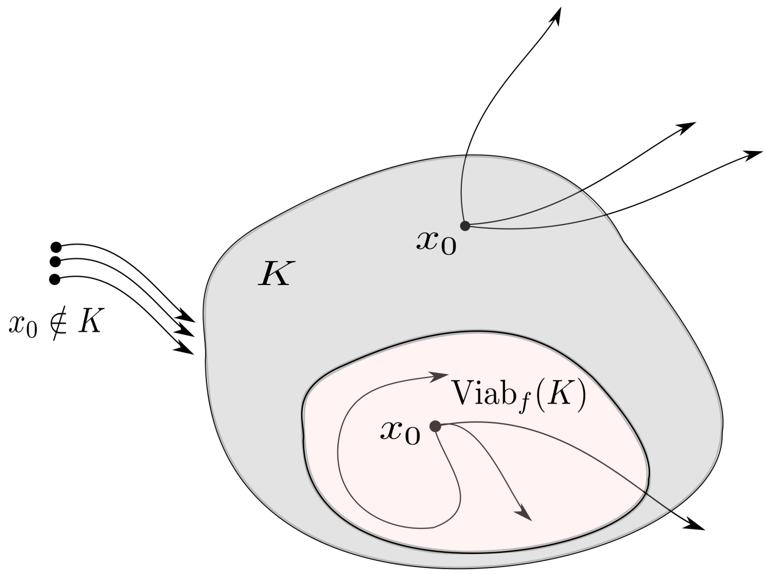


FIGURE 1.4 – Trajectoires issues de K. (Le temps de crise vaut 0 dans l'ensemble de couleur rose)

Si la condition initiale x_0 n'est pas dans le noyau de viabilité de l'ensemble K ($x_0 \notin$Viab$_f(K)$), le problème de minimisation du temps de crise devient pertinent. On chercherait alors à trouver le temps minimal qu'une trajectoire peut rester en dehors de l'ensemble de contrainte K. Si la condition initiale x_0 se trouve à l'intérieur de l'ensemble K et s'il existe un contrôle u dont la trajectoire associée reste dans K en tout temps i.e., $x_0 \in$Viab$_f(K)$, alors le problème de commande optimale est trivial. En effet, le temps de crise dans cas vaut zéro. Cependant, si le noyau de viabilité est non vide et n'est pas l'ensemble K, alors le problème de temps de crise n'est pas trivial dans ce cas: pour les conditions initiales dans K et non pas dans le noyau de viabilité (voir fig. 1.4).

Remarque. Si la dynamique n'est pas autonome (comme pour le problème d'irrigation), on peut se ramener au cadre autonome en ajoutant le temps dans la dynamique i.e. $\dot{t} = 1$ ce qui permet également d'ajouter une contrainte sur le temps dans l'ensemble K, par exemple $t \leq T$ pour étudier la propriété de viabilité sur un horizon fini.

Exemple sur le modèle proie-prédateur. Une application en écologie du problème du temps de crise a été étudié par T. Bayen et A. Rapaport [11], plus spécifiquement, elle a été réalisée sur un modèle de *proie-prédateur*. Les modèles proie-prédateur ont fait l'objet de nombreuses études dans la littérature, avec divers objectifs : contrôle optimal, stabilisation et autres (voir [19, 39, 40, 79]). On considère le système proie-prédateur contrôlé suivant :

$$\begin{cases} \dot{x}_1 & = x_1(r - x_2), \\ \dot{x}_2 & = -x_2(m - x_1) - ux_2, \end{cases} \tag{1.3.1}$$

où $r > 0$ et $m > 0$. x_1 représente le nombre de proies, x_2 le nombre de prédateurs, et $u(\cdot) \in [0, u_{max}]$ la variable de contrôle. Le problème de minimisation du temps de crise consiste à préserver les proies des prédateurs, en maintenant autant que possible leur densité au-dessus d'un seuil donné $\bar{x_1} > 0$. L'ensemble de contrainte est alors définie comme suit :

$$K = \{(x_1, x_2) \in \mathbb{R}_+^\star \times \mathbb{R}_+^\star;\ x_1 \geq \bar{x}\}.$$

Le caractère très intéressant dans cet exemple est que les trajectoires ont un comportement oscillatoire. Par conséquent, de part le caractère oscillatoire, les solutions vont visiter un nombre fini de fois l'ensemble de crise. Le problème de minimisation du temps de crise consiste alors à calculer le meilleur contrôle $u(\cdot)$ pour rester le moins de temps dans la crise. Le contrôle est du type bang-bang, les trajectoires optimales atteignent le noyau après avoir franchi plusieurs fois la frontière de l'ensemble K (voir fig. 1.5).

Le problème de minimisation du temps de crise entre dans la classe des problèmes de Lagrange dont la particularité est la discontinuité de l'intégrande. La discontinuité étant par rapport à l'état x empêche l'application du *Le Principe du Maximum de Pontryagin* (PMP). Ainsi, recourir à des méthodes de régularisation ou à des théorèmes qui généralisent le PMP comme le *Le Principe du Maximum Hybride* (HMP) sont des approches plus adaptées pour traiter ce type de problème.

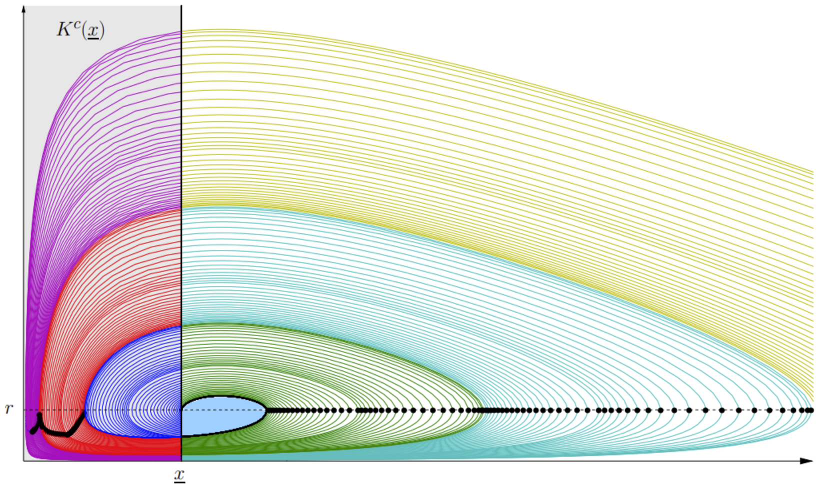


FIGURE 1.5 – Les trajectoires optimales pour le temps de crise minimal. Les couleurs des trajectoires sont changées à chaque temps de croisement qui sont représentés par les points noirs [11].

Les systèmes hybrides. Les systèmes hybrides sont des systèmes dynamiques définies par échantillonnage. Plus précisément, les systèmes hybrides sont décrits par un un ensemble fini de systèmes de contrôle et une séquence finie de temps appelés *temps de commutation* ou *temps de traversée* qui sont fixés et qui partitionnent l'intervalle de temps en sous-intervalles. Sur chaque sous-intervalle, l'état du système s'écoule en fonction de l'un des systèmes. À un moment de commutation, l'état subit un saut instantané et un autre système s'installe et ainsi de suite. Un problème de contrôle optimal hybride consiste alors à chercher des stratégies pour minimiser ou maximiser un critère défini. Étant donné le caractère naturellement discontinue du système, ce type de problème ne satisfait pas les hypothèses classiques pour l'application du PMP. Divers auteurs se sont intéressés au développement d'un principe de maximum qui s'applique aux systèmes *hybrides*. Nous citons par exemple [27], [23], [70], [38], [61], où les auteurs établissent des conditions nécessaires d'optimalité avec de différentes preuves et approches. Dans [41], les auteurs proposent une régularisation C^1 du système Hybrid affine en le contrôle pour lesquels ils ont pu appliquer les principes de maximum de Pontryagin habituels. Ils ont ensuite étudié la question de la convergence des extrémales résultants lorsque le paramètre de régularisation tend vers zéro sous certaines hypothèses générales.

Le problème du temps de crise est un cas particulier des problèmes hybrides. En effet, si on considère la dynamique augmentée avec $\dot{L} = \mathbb{1}_{K_C}(x)$ et le critère terminal $L(T)$, on obtient un système hybride. Cependant, la grand différence est que les instants de commutation ne sont pas connus à l'avance et dépendent de l'état $x(\cdot)$ contrôlé par $u(\cdot)$. Dans notre travail, dans la même lignée que [41], nous allons traiter le problème du temps de crise en passant par une régularisation générale en relâchant les conditions sur les trajectoires optimales.

1.3.2 Historique du problème et travaux existants.

Le problème du minimisation du temps de crise a été initialement introduit par L. Doyen et P. Saint-Pierre [34] comme une extension de la théorie de viabilité. Le problème de contrôle optimal a été traité uniquement par une approche de programmation dynamique et la solution optimale était caractérisée par la fonction valeur comme solution de l'équation Hamilton-Jacobi associé au problème. L'existence d'une solution optimale a été démontrée dans [9, 34]. La question des conditions nécessaires d'optimalité a été étudiée plus dans [9] en utilisant le *Principe du Maximum Hybride*. Les auteurs dans [8] étudient les conditions nécessaires d'optimalité de second ordre en utilisant le PMP sur une reformulation du problème du temps de crise. Dans les travaux de T. Bayen et R. Rapaport [9, 10], sous l'hypothèse que l'ensemble des contraintes K est convexe, les auteurs se sont intéressés principalement aux conditions nécessaires d'optimalité en utilisant les outils de l'analyse convexe et en passant par une régularisation de Moreau Yosida pour K convexe. Dans le même esprit que dans les travaux de T. Haberkon et E. Trélat [41], les auteurs établissent des conditions nécessaires par passage à la limite des solutions optimales du problème régularisé. Cette technique requiert une condition capitale : elle oblige les trajectoires optimales à traverser la frontière de l'ensemble de contraintes K de manière transverse. La condition est dite *condition de franchissement transverse*. Comme dans [41], cette hypothèse est cruciale pour la dérivation des conditions nécessaires d'optimalité et plus particulièrement pour définir le saut du vecteur adjoint au moment où la trajectoire optimale touche le bord de l'ensemble K (au temps de croisement). Dans [2, 66], une technique de régulation utilisant le produit de convolution est introduite dans le but d'obtenir les conditions nécessaires d'optimalité. La condition de franchissement transverse a été utilisée pour démontrer la *non-dégénérescence*[3] du vecteur adjoint au temps de croisement. La discontinuité du vecteur adjoint est un phénomène ordinaire dans les problèmes de contrôle optimal hybride ou les problèmes de contrainte d'état

3. On parle de la dégénérescence lorsque le principe du maximum donne des conditions qui ne sont pas informatives.

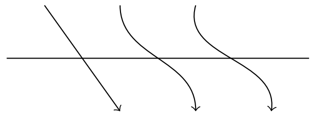


FIGURE 1.6 – Condition de franchissement transverse

(contrairement au cas des problèmes de commande optimale classique où le vecteur adjoint est absolument continu). C'est pourquoi, des difficultés théoriques et numériques peuvent survenir en raison du comportement "non-classique" possible du vecteur adjoint caractérisé par des sauts au temps de commutation pour les problèmes hybrides et au moment où la trajectoire touche le bord de l'ensemble de contrainte pour les problèmes de contrainte d'état. Par conséquent, une grande partie de la littérature est consacrée à l'analyse du comportement du vecteur adjoint et à certaines conditions de qualification des contraintes (voir [14, 15, 43]).

L'objectif de la deuxième partie de cette thèse est d'examiner le comportement du vecteur adjoint dans le cadre du problème du temps de crise et voir de près sa relation avec le comportement de la trajectoire au bord de l'ensemble de contrainte K. Nous analyserons dans quelle mesure cette condition est nécessaire pour mettre en place les conditions nécessaires d'optimalité. Cette étude nous permettra d'introduire une condition auxiliaire généralisant ainsi [2, 9, 41, 66].

1.3.3 Étude du problème de minimisation du temps de crise.

Dans cette sous-section, nous résumons nos principaux résultats, correspondants aux travaux [20, 7] dont le contenu est repris dans les chapitres 4 et 5 respectivement. Nous considérons deux fonctions $\varphi : \mathbb{R}^n \to \mathbb{R}$ et $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Dans ce qui suit, nous supposons que les hypothèses suivantes sont satisfaites :

(H1) L'ensemble U est compact et f est une fonction de classe C^1 qui satisfait la propriété de la croissance linéaire, *i.e.*, il existe $c \geq 0$ tel que pour tout $(x, u) \in \mathbb{R}^n \times U$, on a $|f(x,u)| \leq c(|x|+1)$.

(H2) Pour tout $(x, p) \in \mathbb{R}^{2n}$, l'ensemble

$$\bigcup_{u \in U} \begin{bmatrix} f(x,u) \\ -D_x f(x,u)^\top p \end{bmatrix}$$

est un sous-ensemble compact convexe de $\mathbb{R}^{2n}$.

(H3) On suppose que φ est de classe C^2, que l'ensemble K est à intérieur non vide et est le sous-ensemble 0 de φ :

$$K = \{x \in \mathbb{R}^n \ ; \ \varphi(x) \leq 0\}.$$

(H4) Pour chaque $x \in \partial K$ (le bord de K), on a $\varphi(x) = 0$ et $\nabla\varphi(x) \neq 0$.

Sous ces hypothèses, l'existence d'une solution optimale, que nous notons (x^*, u^*), est garantie par [9, 34]. Pour que le problème (TC) soit pertinent, nous supposons que les trajectoires solution de (S) quittent l'ensemble K ou entrent dans K au moins une fois. C'est-à-dire que nous excluons le cas où les trajectoires restent dans K ou dans K^c en tout temps $t \in [0, T]$: le temps de crise vaut alors soit 0 soit T. Nous définissons ensuite le *temps de croisement* comme l'instant où une trajectoire solution de (S) touche le bord de l'ensemble K en entrant ou en sortant de K.

Définition 1.3.1. *Soit $x_u(\cdot)$ une solution de (S) associée à un contrôle $u(\cdot) \in \mathcal{U}$. Soit $\tau \in (0, T)$, alors*

1. *On dit que τ un* temps de croisement sortant (de K vers K^c) *si il existe $\eta > 0$ tel que $[\tau - \eta, \tau + \eta] \subset [0, T]$ et $x_u(t) \in K$ avec $t \in [\tau - \eta, \tau]$ et $x_u(t) \notin K$ tel que $t \in (\tau, \tau + \eta]$,*
2. *On dit que τ est un* temps de croisement entrant (de K^c vers K) *si il existe $\eta > 0$ tel que $[\tau - \eta, \tau + \eta] \subset [0, T]$ et $x_u(t) \notin K$ avec $t \in [\tau - \eta, \tau]$ et $x_u(t) \in K$ tel que $t \in (\tau, \tau + \eta]$,*
3. *Un temps de croisement τ est dit* transverse *si le contrôle $u(\cdot)$ est continu à droite et à gauche, de plus*

$$f(x(\tau), u(\tau^+)) \cdot \nabla\varphi(x_u(\tau)) \neq 0 \quad \text{et} \quad f(x(\tau), u(\tau^-)) \cdot \nabla\varphi(x_u(\tau)) \neq 0. \tag{1.3.2}$$

La façon dont la trajectoire touche la frontière est cruciale pour fournir les conditions nécessaires d'optimalité pour le problème du temps de minimisation du temps de crise. Nous proposons de nouvelles définitions qui sont plus générales que les temps de croisement transverses.

Définition 1.3.2. 1. *Un temps de croisement sortant τ est dit* semi-transverse à gauche *si $\dot{x}$ est continue à gauche en $t = \tau$ et nous avons:*

$$f(x(\tau), u(\tau^-)) \cdot \nabla\varphi(x(\tau)) > 0 \quad \text{et} \quad f(x(\tau), u(\tau^+)) \cdot \nabla\varphi(x(\tau)) = 0. \tag{1.3.3}$$

2. *Un temps de croisement sortant τ est dit* semi-transverse à droite *si $\dot{x}$ est continue à gauche en $t = \tau$ et nous avons:*

$$f(x(\tau), u(\tau^-)) \cdot \nabla\varphi(x(\tau)) = 0 \quad \text{et} \quad f(x(\tau), u(\tau^+)) \cdot \nabla\varphi(x(\tau)) > 0 \qquad (1.3.4)$$

3. *Un temps de croisement entrant τ est dit* semi-transverse à gauche *si $\dot{x}$ est continue à gauche en $t = \tau$ et nous avons:*

$$f(x(\tau), u(\tau^-)) \cdot \nabla\varphi(x(\tau)) < 0 \quad \text{et} \quad f(x(\tau), u(\tau^+)) \cdot \nabla\varphi(x(\tau)) = 0 \qquad (1.3.5)$$

4. *Un temps de croisement entrant τ est dit* semi-transverse à droite *si $\dot{x}$ est continue à gauche en $t = \tau$ et nous avons:*

$$f(x(\tau), u(\tau^-)) \cdot \nabla\varphi(x(\tau)) = 0 \quad \text{et} \quad f(x(\tau), u(\tau^+)) \cdot \nabla\varphi(x(\tau)) < 0 \qquad (1.3.6)$$

Un temps de croisement τ est dit semi-transverse *si il satisfait une des conditions:* (1.3.5), (1.3.3), (1.3.6), (1.3.4). *Si le temps de croisement est semi-transverse à gauche et à droite, on dit alors qu'il est* tangent. *(voir fig. 1.7 pour illustration).*

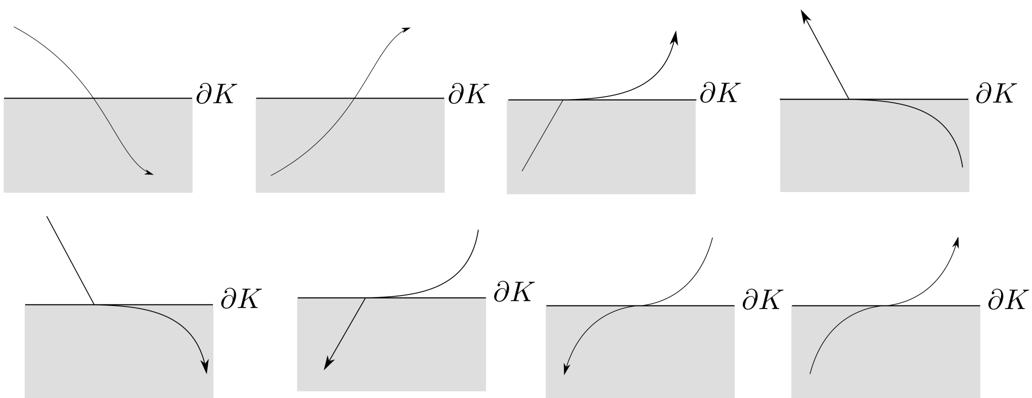


FIGURE 1.7 – Illustration des différents type de points de croisement.

Nous citerons ci-après les principaux résultats obtenus dans cette thèse.

Résumé du chapitre 4.

Le premier chapitre de la deuxième partie de cette thèse a été motivé principalement par une méthode de régularisation de la fonction caractéristique de l'ensemble de crise en utilisant une approche de pénalisation. Dans un premier lieu, nous proposons dans ce travail une nouvelle formulation du problème de crise en temps minimal introduisant un contrôle auxiliaire v qui prend ses valeurs dans $\{\pm 1\}$ i.e., $v \in \mathcal{V} := \{v : [0,T] \longrightarrow \{-1,1\};\ v \text{ mes}\}$. La nouvelle formulation du problème est définie comme suit :

$$\inf_{\substack{(u,v)\\ \in \mathcal{U}\times\mathcal{V}}} \int_0^T \frac{1+v(t)}{2} \mathrm{d}t, \tag{1.3.7}$$

sous la contrainte mixte état-contrôle :

$$v\varphi(x_u) - |\varphi(x_u)| = 0. \tag{1.3.8}$$

où $x_u(\cdot)$ est solution du système (S). La discontinuité du contrôle étant un phénomène commun en théorie du contrôle optimal, grâce à cette nouvelle formulation, le problème de discontinuité de l'intégrande par rapport à l'état du problème initial a été remplacé par un problème de contrôle optimal régulier impliquant une contrainte état-contrôle mixte avec un contrôle supplémentaire. Pour faire face à la contrainte mixte, nous proposons une nouvelle procédure de régularisation du problème du temps de crise minimal avec un contrôle supplémentaire et la méthode de pénalité.

$$\inf_{\substack{(u,v)\\ \in \mathcal{U}\times\mathcal{V}}} J_n(u,v) = \inf_{\substack{(u,v)\\ \in \mathcal{U}\times\mathcal{V}}} \int_0^T \left[\frac{1+v(t)}{2} + nP(u,v)\right] \mathrm{d}t \tag{TCR$_n$}$$

Le principe de la méthode est le suivant : Plus n est grand plus $P(u,v)$ doit être proche de 0 pour que le terme $nP(u,v)$ ne tende pas vers l'infini. Ainsi, à la limite $P(u,v) = 0$, le problème régularisé est équivalent au problème limite intuitivement. Cette nouvelle approche débouche sur l'étude d'une suite de problèmes de contrôle optimal régulier et sans contraintes mixtes. Cependant, le manque de convexité de l'ensemble des vitesses augmentées du problème (TCR_n) entraîne une incertitude concernant l'existence d'une solution optimale pour ce problème. En effet, l'hypothèse sur la convexité de l'ensemble des vitesses augmentées est essentielle pour disposer d'une solution optimale par le théorème de Filippov [37]. Par conséquent, nous considérons ci-dessous un problème auxiliaire où la condition de convexité est alors satisfaite, ce qui va nous aider à traiter le problème (TCR$_n^{\#}$).

$$\inf_{(u,v)\in\mathcal{U}\times\mathcal{V}^{\#}} \int_{\tau}^T \left(\frac{1+v(t)}{2} + nP(x(t),v(t))\right) \mathrm{d}t, \tag{TCR$_n^{\#}$}$$

où $\mathcal{V}^{\#}$ est l'ensemble de tout les contrôles admissibles $v : [0,T] \to \mathrm{co}(\Omega)$, $\mathrm{co}(\Omega) := [-1,1]$, et $x(\cdot)$ une solution de (S). L'ensemble des vitesses augmentées

$$\left\{ \begin{bmatrix} f(x,u) \\ \frac{1+v}{2} + nP(x,v) + w \end{bmatrix},\ (u,v,w) \in U \times [-1,1] \times \mathbb{R}_+ \right\}$$

est alors convexe pour tout $x \in \mathbb{R}^n$. L'existence d'une solution optimale pour ce problème découle ainsi directement du théorème de Filippov [37]. Nous considérons $V_n,\ \overline{V}_n,\ V$ comme fonctions valeur des problèmes (TCR$_n^{\#}$), (TCR$_n$), (TC) respectivement, nous obtenons dans la Proposition 4.2.1 du Chapitre 4 le résultat d'encadrement suivant:

Proposition 1.3.3. *Pour tout $n \in \mathbb{N}$ et pour tout $(\tau, y) \in [0,T) \times \mathbb{R}^n$, on a*

$$V_n(\tau,y) \leq \overline{V}_n(\tau,y) \leq V(\tau,y).$$

Grâce aux arguments classiques du théorème de compacité des trajectoires, nous obtenons aussi les résultats suivants en matière de convergence : Soit x_n solution optimal du problème (TCR$_n$). Alors, à une sous-suite près:

- $(x_n)_n$ converge uniformément vers x^* sur $[0,T]$.
- $(\dot{x}_n)_n$ converge faiblement dans $L^2([0,T],\mathbb{R}^n)$ vers $\dot{x}^*$.
- $(V_n(\cdot,\cdot))_n$ converge ponctuellement vers $V(\cdot,\cdot)$.

Avec ces résultats de convergence et la proposition 1.3.3, nous déduisons que $(\overline{V}_n(\cdot,\cdot))_n$ converge ponctuellement vers $V(\cdot,\cdot)$ ainsi que l'existence d'une trajectoire optimale $\bar{x}_n$ du problème (TCR$_n^{\#}$) pour chaque $n \in \mathbb{N}$ associée au deux contrôles $(\overline{u}_n, \overline{v}_n) \in \mathcal{U} \times \mathcal{V} \subset \mathcal{U} \times \mathcal{V}^{\#}$. En reprenant les propriétés de la compacité des trajectoires, on obtient la convergence uniforme de $(\bar{x}_n)_n$ vers x^* et la convergence faible de $(\dot{\bar{x}}_n)_n$ vers $\dot{x}^*$. Notons que considérer le contrôle supplémentaire avec seulement deux valeurs possibles $\{\pm 1\}$ malgré le manque de convexité de l'ensemble des vitesses augmentées est assez efficace d'un point de vue numérique. En effet, avec un logiciel comme BocopHJB [17] dont le principe est de discrétiser les variables de temps, d'état et de contrôle, prendre un contrôle qui n'a que deux valeurs est plus rapide pour le temps de calcul. Les exemples numériques illustrent ces résultats et valident l'efficacité de cette méthode. Cette technique de régularisation permet également aux trajectoires de quitter et d'entrer dans un ensemble K sans la nécessité d'une condition de franchissent transverse sur les trajectoires optimales ou sur la convexité de l'ensemble K.

Résumé du chapitre 5

Dans ce chapitre, nous étudions les conditions nécessaires d'optimalité pour le problème du temps de crise sans exiger d'hypothèses sur le comportement d'une trajectoire optimale au bord de l'ensemble de contrainte K notamment le caractère transverse que nous allons essayer de relâcher. Néanmoins, nous partons du principe qu'il y a au moins un nombre fini de points de croisement. Nous considérons une approche basée sur une suite de problèmes de contrôle optimal approchés qui peuvent être étudiés avec les résultats classiques de la théorie du contrôle optimal. Nous allons suivre les étapes suivante :

1. Introduire la régularisation générale du temps de crise.
2. Appliquer les résultats classiques comme le PMP sur le problème régularisé et étudier les propriétés de ses extrémales.
3. Étudier les passage à la limite et enfin récupérer les conditions nécessaires d'optimalité.

Procédure de Régularisation et application du PMP. L'approche pour régulariser le problème du temps de crise est introduit comme suit : Soit $n \in \mathbb{N}$, on considère $G_n : \mathbb{R} \to [0,1]$ comme une suite de fonctions croissantes de classe C^1 qui régularise la fonction indicatrice. i.e.,

$$\lim_{n \to +\infty} G_n(\varphi(x(\cdot))) = \mathbb{1}_{K^c}(x(\cdot)),$$

tel que il existe deux suites $(a_n)_n,\ (b_n)_n$ qui tendent vers 0 avec

$$\forall \sigma \in \mathbb{R},\ \forall n \in \mathbb{N}, \begin{cases} G_n(\sigma) = 0 & \text{si} \quad \sigma \leq a_n, \\ G_n(\sigma) = 1 & \text{si} \quad \sigma \geq b_n. \end{cases}$$

La régularisation par la méthode de pénalisation introduite au chapitre 4 est un cas particulier de la régularisation considérée. Nous définissons ensuite la suite des fonctions $h_n : \mathbb{R} \to \mathbb{R}$ comme suit:

$$h_n := G_n',$$

pour tout $n \in \mathbb{N}$, son support est inclus dans $[a_n, b_n]$ et elle vérifie $\int_{\mathbb{R}} h_n(\sigma)\mathrm{d}\sigma = 1$. Le problème de commande optimale régularisé devient alors:

$$\inf_{u \in \mathcal{U}} \int_0^T G_n(\varphi(x_u(t)))\,\mathrm{d}t. \tag{P$_n$}$$

Ce nouveau problème est un problème de commande optimal qui satisfait les hypothèses classique de l'existence d'une solution optimale par le théorème de Filippov [37] pour tout $n \in \mathbb{N}$ que nous notons par (x_n, u_n). Par le théorèmes de compacité nous avons les résultats de convergence suivant :

- $(x_n)_n$ converge uniformément vers x^* sur $[0, T]$.
- $(\dot{x}_n)_n$ converge faiblement vers $\dot{x}^*$ dans $L^2([0, T]; \mathbb{R}^n)$.

Il convient de souligner que la suite des contrôles $(u_n)_n$ ne converge pas forcément vers u^*. La suite des intégrandes $(G_n(\varphi(\cdot)))_n$ étant régulière, nous sommes en mesure d'appliquer le PMP au problème (P$_n$) pour tout $n \in \mathbb{N}$. Soit $H_n : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ l'Hamiltonien du problème régularisé:

$$H_n(x, p, u) = p \cdot f(x, u) - G_n(\varphi(x)).$$

Par le PMP, il existe une fonction absolument continue $p_n : [0, T] \to \mathbb{R}^n$ qui satisfait l'équation adjointe:

$$\begin{cases} \dot{p_n}(t) &= -D_x f(x_n(t), u_n(t))^\top p_n(t) + h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t)) \quad \text{p.p. } t \in [0, T], \\ p_n(T) &= 0. \end{cases} \tag{1.3.9}$$

Une solution optimale vérifie la condition de maximisation:

$$u_n(t) \in \operatorname*{argmax}_{u \in U} p_n(t) \cdot f(x_n(t), u) \quad \text{p.p. } t \in [0, T].$$

De plus, le problème étant autonome, le Hamiltonien est conservé le long de la trajectoire optimale. i.e., Pour tout $n \in \mathbb{N}$, il existe $\tilde{H}_n \in \mathbb{R}$ tel que

$$\tilde{H}_n = H_n(x_n(t), p_n(t), u_n(t)) = p_n(t) \cdot f(x_n(t), u_n(t)) - G_n(\varphi(x_n(t))) \quad \text{p.p. } t \in [0, T].$$

Nous nous intéressons au passage à la limite de la suite des vecteurs adjoints $(p_n)_n$ à fin de récupérer les conditions nécessaire d'optimalité pour le problème initial (TC). Dans un premier temps, nous vérifions si cette suite est bien bornée et sous quelle condition. Pour cela, nous définissons la suite $(I_n)_n$ comme suit

$$I_n := \int_0^T h_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

La bornitude de la suite $(p_n)_n$ est fortement liée à la bornitude de la suite $(I_n)_n$. En effet, la fonction $h_n(\varphi(x_n(t)))$ par définition n'est pas une fonction bornée et par conséquent le second membre de l'équation différentielle (1.3.9) est non uniformément borné par rapport à n. Cela rend a priori très délicate l'étude de la limite de la suite des vecteurs adjoints $(p_n)_n$. Ce point est crucial car c'est ici que réside toute la difficulté à obtenir les conditions nécessaires d'optimalité. Nous commençons alors par élucider le lien entre la suite (I_n) et la suite $(p_n)_n$ dans la proposition 5.3.1 du Chapitre 5:

Proposition 1.3.4. *La suite $(p_n)_n$ est bornée dans $L^\infty([0,T];\mathbb{R}^n)$ si seulement si $(I_n)_n$ est bornée dans $\mathbb{R}_+$.*

Une question naturelle qui se pose ici : Quand est-ce que la suite $(I_n)_n$ est bornée ? S'intéresser à la bornitude de la suite $(I_n)_n$ est suffisant pour conclure celle de la suite $(p_n)_n$. Si la suite $(I_n)_n$ n'est pas bornée alors la suite $(p_n)_n$ n'est pas bornée également et par conséquent c'est une suite qui ne converge pas et ainsi le passage à la limite pour tirer les conditions nécessaires d'optimalité serait impossible. Nous nous intéressons à cette question et nous établissons des conditions nécessaires et suffisantes pour que cette suite soit bornée. Mais tout d'abord, nous allons supposer que l'une des suite est bornée on a alors le résultat suivant:

Théorème 1.3.5. *Soient $\tau_i, i = 1 \cdots r$, les instants de commutation de x^*. Si la suite $(I_n)_n$ ou la suite $(p_n)_n$ est bornée sur $L^\infty([0,T]\,;\,\mathbb{R}^n)$, alors il existe une fonction absolument continue par morceaux, non identiquement nulle $p : [0,T] \to \mathbb{R}^n$ associée à x^* tel que*

- *p est solution de l'équation adjointe:*

$$\left| \begin{array}{ll} \dot{p}(t) & = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{p.p. } t \in [0,T] \setminus \{\tau_1, \cdots, \tau_r\}, \\ p(T) & = 0. \end{array} \right. \tag{1.3.10}$$

- *La condition Hamiltonienne est vérifiée:*

$$u^*(t) \in \operatorname*{argmax}_{u\in U} p(t)\cdot f(x^*(t),u) \quad \text{p.p. } t\in[0,T]. \tag{1.3.11}$$

- *p admet une limite à droite et à gauche en chaque τ_i, $p(\tau_i^{\pm})$ et il existe r nombres positifs $\ell_1, ..., \ell_r$ tel que:*

$$\forall i \in \{1,...,r\}, \quad p(\tau_i^+) - p(\tau_i^-) = \ell_i \nabla\varphi(x^*(\tau_i)). \tag{1.3.12}$$

- *L'Hamiltonien est constant p.p. sur $[0,T]$, i.e.,*

$$H(x(t),p(t),u(t)) = \max_{u\in U} H(x(t),p(t),u) = -\mathbb{1}_{K^c}(x^*(T)) \tag{1.3.13}$$

Ce résultat fournit des informations sur les conditions nécessaires d'optimalité et ne requière pas l'hypothèse de franchissement transverse pour x^*. En effet, on suppose que $(I_n)_n$ est bornée sans disposer des informations a priori sur le comportement de x^* au bord de K. L'équation (1.3.12) signifie que le vecteur adjoint admet des discontinuités au temps de croisement dans la même direction que le gradient de la fonction $\varphi(x^*(\cdot))$. L'amplitude du saut est paramétrée par les valeurs de $(\ell_i)_{i\in\{1,...,r\}}$. Supposons que la solution optimale vérifie $\nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^+) \neq 0$ ou bien $\nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^-) \neq 0$, alors en utilisant la constance du Hamiltonien on trouve la formule du saut qui correspond à :

$$p(\tau_i^+) = p(\tau_i^-) + \frac{\delta_i + p(\tau_i^-)\cdot(\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^+)}\nabla\varphi(x^*(\tau_i)), \quad \text{si } \nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^+)\neq 0, \tag{1.3.14}$$

ou bien

$$p(\tau_i^-) = p(\tau_i^+) - \frac{\delta_i + p(\tau_i^+)\cdot(\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^-)}\nabla\varphi(x^*(\tau_i)), \quad \text{si } \nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^-)\neq 0, \tag{1.3.15}$$

avec $\delta_i := \operatorname{sign}(\varphi(x^*(\tau_i^+)))$. Lorsque chaque temps de croisement τ_i de la solution optimale x^* est transverse, on peut utiliser ces expressions pour déterminer explicitement les sauts du vecteur adjoint p en rétrograde à partir du temps terminal T. Ainsi, pour tout choix des suites régularisantes $(G_n)_n$ telles que $(I_n)_n$ est borné et $(x_n)_n$ converge vers x^* à une sous-suite près, on obtient les mêmes valeurs ℓ_i.

Nous nous intéressons à la bornitude de la suite $(I_n)_n$, nous introduisons d'abord une hypothèse essentielle pour cette étude:

(H$'$) Tout les temps de croisement de x^* sont transverses.

Condition suffisante pour la bornitude de $(I_n)_n$. L'hypothèse (H') est une hypothèse générique utilisée dans la littérature, elle exclut le cas où la trajectoire optimale touche le bord de l'ensemble K de manière tangente. Dans notre étude cette condition est suffisante pour faire le passage à la limite et récupérer les conditions nécessaires d'optimalité pour le problème de minimisation du temps de crise.

Proposition 1.3.6. *Sous l'hypothèse* (H'), *la suite* $(I_n)_n$ *est bornée.*

La preuve de cette proposition est présentée en détail dans la Proposition 5.5.1 du Chapitre 5. La condition (H') est une condition sur la solution optimale $x^\star$, qui n'est généralement pas connue à l'avance. Au lieu de cela, nous mettons en place des conditions sur la sous-suite $(x_n)_n$, et non sur la solution x^*, qui garantit la bornitude de la suite des intégrales $(I_n)_n$. C'est une condition suffisante et plus faible que l'hypothèse (H'). Pour $n \in \mathbb{N}$, nous définissons la fonction absolument continue ρ_n comme :

$$\rho_n(t) := \varphi(x_n(t)), \quad t \in [0, T].$$

Pour tout $n \in \mathbb{N}$, x_n est différentiable presque partout sur $[0, T]$ et donc ρ_n l'est aussi tel que:

$$\dot{\rho}_n(t) = \nabla\varphi(x_n(t))\dot{x}_n(t) \quad \text{a.e. } t \in [0, T].$$

De plus, $(\rho_n)_n$ est uniformément bornée dans $L^\infty([0, T]\,;\mathbb{R})$ sous les hypothèses (H1) et (H3). Par conséquent, pour $i \in \{1, ..., r\}$ et $n \in \mathbb{N}$, nous définissons:

$$l_{i,n}^+ := \limsup_{h\to 0} \operatorname*{ess\,sup}_{t\in[\tau_i-h,\tau_i+h]} \dot{\rho}_n(t) \quad ; \quad l_{i,n}^- := \liminf_{h\to 0} \operatorname*{ess\,inf}_{t\in[\tau_i-h,\tau_i+h]} \dot{\rho}_n(t).$$

Nous avons alors le résultat suivant sur la bornitude de la suite $(I_n)_n$. (Voir Proposition 5.5.2 pour la preuve)

Proposition 1.3.7. *Si pour tout* $1 \le i \le r$, *on a*

$$\begin{aligned} &\liminf_{n\to+\infty} l_{i,n}^- > 0 \textit{ si } \tau_i \textit{ est un temps de croisement sortant, ou bien} \\ &\limsup_{n\to+\infty} l_{i,n}^+ < 0 \textit{ if } \tau_i \textit{ est un temps de croisement entrant}, \end{aligned} \tag{1.3.16}$$

alors la suite $(I_n)_n$ *bornée.*

Condition nécessaire pour la bornitude de $(I_n)_n$. Nous supposons que la suite $(I_n)_n$ est bornée et nous examinons comportement de la solution optimale au bord de l'ensemble K.

Proposition 1.3.8. *Soit τ_i un point de croisement de x^* tel que $\dot{x}^*(\tau_i^{\pm})$ existe. Si $(I_n)_n$ est bornée, alors on a: $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x(t_i)) \neq 0$ (si la trajectoire est sortante) ou bien $\dot{x}^*(t_i^-) \cdot \nabla\varphi(x(t_i)) \neq 0$ (si la trajectoire est entrante).*

Ce résultat généralise la condition de franchissent transverse et garantit que x^* n'est pas nécessairement transverse au bord de K (on trouvera en détail la preuve de cette proposition dans la Proposition 5.5.3 du Chapitre 5). De façon plus précise, cela signifie que les trajectoires peuvent être tangentes à l'intérieur de K. Les types de points de croisement permis sont alors: semi-transverse à droite sortant et semi-transverse à gauche entrant. Autrement dit, les trajectoires optimales doivent nécessairement toucher le bord extérieur de K transversalement. En conséquence, si la trajectoire est tangente au deux côtés de K alors, forcément, $(I_n)_n$ n'est pas bornée (contradiction avec la proposition). Cependant, ces conditions de franchissement semi-transverse restent nécessaires mais pas suffisantes : un mauvais choix de la régularisation ne garantirait pas que la trajectoire puisse être tangente à l'intérieure de l'ensemble K et inversement, si une trajectoire optimale est semi-transverse au côté extérieur de K et s'il existe une régularisation adéquate telle que $(I_n)_n$ soit bornée alors le passage à la limite est possible et finalement on obtient les valeurs du saut du vecteur adjoint. En résumé, nous avons les résultats suivant:

1. Si $(I_n)_n$ est bornée, nous avons les conditions nécessaires d'optimalité.
2. Si tout les points de croisement sont transverses, alors $(I_n)_n$ bornée et par conséquent nous avons les conditions nécessaires d'optimalité.
3. Si $(I_n)_n$ est bornée, tout les temps de croisement sont au moins soit semi-transverses à droite sortants soit semi-transverses à gauche entrants (pas forcément transverses).
4. Si chaque temps de croisement est semi-transverse, alors avec une régularisation adéquate, $(I_n)_n$ est bornée et on obtient les conditions nécessaires d'optimalité.
5. S'il existe un point de croisement purement tangent, alors $(I_n)_n$ n'est pas bornée.

Nous concluons le chapitre en mentionnant un exemple développé et résolu complètement pour lequel il existe une solution optimale non transverse, et qui montre

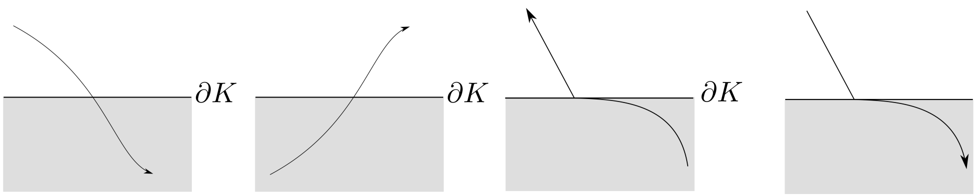


FIGURE 1.8 – Illustration des différents points de croisement pour lesquelles la condition nécessaire de bornitude de $(I_n)_n$ peut être satisfaite.

comment cette approche s'applique i.e. comment vérifier les conditions d'optimalité du Théorème 1.3.5. L'exemple est présenté en détails dans une section dédiée de la thèses (Section 5.6). Ceci illustre bien la contribution de cette thèse qui est de traiter des conditions nécessaire d'optimalité à priori sans hypothèses sur la trajectoire optimale. Toutefois, nous supposons qu'il y a un nombre fini de points de croisement et nous donnons une extension et des conditions nécessaires d'optimalité lorsqu'une solution optimale n'est pas nécessairement transverse. Ceci *relaxe* l'hypothèse couramment utilisée dans la littérature au sujet du caractère purement transverse.

Deuxième partie

Optimisation d'un modèle d'irrigation

CHAPITRE 2.0

OPTIMAL CONTROL OF A CROP IRRIGATION MODEL UNDER WATER SCARCITY

This chapter corresponds to the published paper

K. BOUMAZA, N. KALBOUSSI, A. RAPAPORT, S. ROUX, C. SINFORT, *Optimal control of a crop irrigation model under water scarcity*, Optim Control Appl Meth. 1–20, 2021. [21]

Contents

2.1 Introduction

Today, in the context of the climate crisis, tensions around water resources are growing, and agriculture in many countries, particularly in the South, must now be considered as having to be water-saving. One way to cope with this changing context is to impose irrigation quotas. But such quotas could imply yield losses as the best crop water requirements would not necessarily be fulfilled. In these particular but expected situations, optimizing the schedule of irrigation to minimize yield losses becomes crucial. Crop modeling and numerical simulations are handy tools for understanding and adjusting these compromises. More particularly the control theory as irrigation is typically a control variable of a crop model.

Numerous texts and articles have developed systematic approaches to tackle the scheduling optimization problem. We refer to [49, 50, 62, 63, 74] and references therein. However, most of the existing approaches do not allow to have analytical descriptions of an optimal solution's theoretical properties as they are based on the numerical optimization of complex models. Control theoretical tools, such as the Pontryagin's Maximum Principle, allow, as a rule, to discover structures of optimal strategies in terms of state feedback available for large classes of operating conditions and parameter values. Unlike optimization among open-loop controls that apply for precise requirements, closed-loop procedures offer adaptability and robustness for concrete applications, when state variables can be measured in real-time. However, when models have too numerous equations, variables, and parameters or do not present a robust structure, the study of optimal strategies with these techniques is most of the time out of reach of analytical characterizations. This is a motivation to consider models with a reasonable simplicity to benefit from these theoretical approaches (see e.g. [53, 64]). Even when these models are imperfect, optimal policies' derivation is of possible relevance. It generally provides simple but non-intuitive control rules that can be tested in simulation on more realistic detailed models. Sometimes, combinations of analytical description and reduced optimization can be obtained, which still offer advantages over purely numerical solutions in open-loop. This is the spirit in which this work has been conducted.

The aim of this work is investigate optimal irrigation strategies in the context of water quotas, with the help of a simplified crop model. The objective is to characterize the maximal biomass production that can be obtained for a given value of the quota (as an intrinsic performance of the system without consideration of water pricing), and for which feedback strategy it can be achieved. To the best of our knowledge, the analytical study of this problem has not been yet addressed in the literature, even for simple stress functions. The present work goes two steps further than the preliminary one presented in [48], first in relaxing assumptions on the water stress functions of the model, and secondly in providing the complete solution of the optimal control problem as well as associated numerical simulations.

The organization of the paper is as follows. Section 2.2 presents the crop model with its assumptions. Section 2.3 is dedicated to the formulation of the optimal control problem under constraint, along with some preliminary results. In section 2.4, crucial properties of the optimal solutions are proved. Then, Section 2.5 is devoted to the application of the Pontryagin Maximum Principle and the synthesis of the optimal irrigation strategy. Finally, in Section 2.6, we illustrate the theoretical results on numerical simulations and draw comparisons of several control strategies.

2.2 Model description and assumptions

We consider the dynamical model of crop irrigation introduced in [48] and inspired from [57], where $S(t)$ and $B(t)$ stand respectively for the relative soil humidity in the root zone (a quantity between 0 and 1) and the crop biomass at time t in an interval $[0,T]$ representing the crop growth season, where 0 and T are the sowing and harvesting dates. The control variable $u(t) = F(t)/F_{max} \in [0,1]$ is the ratio of the input water flow rate $F(t)$ at time t over the maximal flow F_{max} that the irrigation allows. In this model as in [12], crop evapo-transpiration is split into crop transpiration $\varphi(t)K_S(S)$ and soil evaporation $(1-\varphi(t))K_R(S)$ using the crop radiation interception efficiency $\varphi(t)$. The two functions K_S and K_R (see Assumption 2.2.1a and Fig. 3.1 below) are used to model the regulation of transpiration and evaporation by soil moisture as in [57].

Assumption 2.2.1a. *The functions K_S and K_R are piecewise linear non decreasing from $[0,1]$ to $[0,1]$ with numbers $0 < S_h < S_w < S^\star < 1$ such that*

1. *K_S, resp. K_R is null on $[0,S_w]$, resp. $[0,S_h]$, and positive outside this interval.*
2. *K_S is equal to 1 on $[S^\star,1]$ and concave increasing on $[S_w,S^\star]$.*
3. *$K_R(1)=1$ and K_R is convex increasing on $[S_w,S^\star]$.*

The value S_w represents the plant wilting point, which is usually higher than the hydroscopic point denoted by S_h. $S^\star$ is the minimal threshold on the soil humidity that gives the best biomass production. This assumption generalizes the expressions found in the literature (see for instance [57]), given by the following assumption (see Fig. 3.1).

Assumption 2.2.1b. *The functions K_S and K_R are piecewise linear non decreasing from $[0,1]$ to $[0,1]$ given by the following expressions*

$$K_S(S) = \begin{cases} 0 & S \in [0, S_w], \\ \dfrac{S - S_w}{S^\star - S_w} & S \in [S_w, S^\star], \\ 1 & S \in [S^\star, 1], \end{cases} \qquad K_R(S) = \begin{cases} 0 & S \in [0, S_h], \\ \dfrac{S - S_h}{1 - S_h} & S \in [S_h, 1], \end{cases} \tag{2.2.1}$$

where $0 < S_h < S_w < S^\star < 1$.

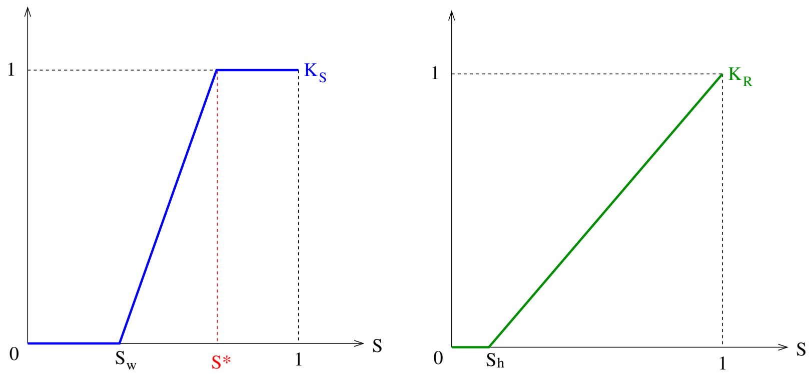


FIGURE 2.1 – Graphs of the functions K_S and K_R given by expressions (2.2.1)

The piecewise linear assumption is related to the practitioners knowledge characterizing thresholds of various stages, and also the choice of a simple representation with few parameters. We shall say that S is a *corner point* of K_S, resp. K_R when the function is non differentiable at S (therefore $S^\star$ is a corner point of K_S).

Assumption 2.2.2. *The function φ is C^1 increasing with $\varphi(0) \geq 0$ and $\varphi(T) \leq 1$.*

In many situations, one can take $\varphi(0) = 0$ at sowing date and $\varphi(T) = 1$ at harvesting date.

We shall also consider dilution rate coefficients k_1, k_2 related respectively to soil characteristics (porosity) and irrigation system.

Assumption 2.2.3. *k_1, k_2 are positive parameters with $k_2 \geq 1$.*

Finally, we denote the function f for the biomass growth rate of the crop in absence of water stress (as a function of B).

Assumption 2.2.4. *The function f is a non-negative Lipschitz continuous function with linear growth such that $f(B_0) > 0$.*

The equations of the model are then

$$\dot{S} = k_1\Big(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2u(t)\Big), \tag{2.2.2}$$
$$\dot{B} = \varphi(t)K_S(S)f(B), \tag{2.2.3}$$

with initial condition

$$S(0) = S_0 > S^\star, \tag{2.2.4}$$
$$B(0) = B_0 > 0. \tag{2.2.5}$$

Eq. (2.2.2) represents the variation of a vertically averaged soil moisture as influenced by crop evapotranspiration $\varphi(t)K_S(S) + (1-\varphi(t))K_R(S)$ and irrigation $k_2u(t)$. Eq. (2.2.3) determines the amount of biomass produced per time unit from the transpiration flux as in [69, 57] and modulated by a normalized growth kinetics function $f(\cdot)$. It is usually considered that crop is viable only if initial soil mixture S_0 is above the threshold $S^\star$. Indeed, S_0 is often equal to 1 at the chosen sowing date (typically at the end of winter season). Notice that the dynamics (2.2.2)-(2.2.3) is non-autonomous and that we consider a mild hypothesis on the function φ. The function φ, which makes the system non-autonomous, can be seen as a surrogate to the crop coefficient used in the FAO approach [1]. The condition $k_2 \geq 1$ is a *controllability* assumption, in the sense that it allows the variable S to stay equal to 1 with the constant control $u = 1/k_2$. Typical instances of growth function f are constants or the logistic law, as considered in [64]

$$f(B) = rB\left(1 - \frac{B}{B_{max}}\right), \tag{2.2.6}$$

with $B_{max} > B_0$, but other choices are possible. Note that the present model does not consider a temporal variation of the reference evapotranspiration present in many crop models and does not include either rainfall inputs: it would be more suitable for greenhouse-grown crops.

The dynamics is naturally subject to the state constraint

$$S(t) \leq 1, \quad t \in [0, T]. \tag{2.2.7}$$

and we shall consider the set $\mathcal{U}$ of *admissible* controls as measurable functions $u(\cdot)$ taking value in $[0, 1]$ such that the solution of (2.2.2)-(2.2.4) verifies the constraint

(2.2.7). Under the former assumptions, one obtains straightforwardly the following property.

Lemma 2.2.1. *For any admissible control $u(\cdot)$, the solution $(S(\cdot), B(\cdot))$ of system* (2.2.2)-(2.2.3) *with initial condition* (2.2.4)-(2.2.5) *verifies*

$$S(t) > S_h, \quad t \geq 0, \tag{2.2.8}$$

and $B(\cdot)$ is uniformly bounded on $[0, T]$.

Remark 2.2.1. *Another way to impose the state constraint* (2.2.7) *to be fulfilled is to consider that the extra water that could be brought when the soil is already saturated (i.e. $S = 1$) is indeed lost. This amounts to consider the following dynamics of S instead of equation* (2.2.2)*:*

$$\dot{S} = k_1\Big(- \varphi(t) K_S(S) - (1 - \varphi(t)) K_R(S) + k_2 \chi(S, u(t))\Big), \tag{2.2.9}$$

where the function $\chi(\cdot)$ is given by

$$\chi(S, u) := \begin{cases} \min(1/k_2, u) & \text{if } S = 1, \\ u & \text{if } S < 1. \end{cases} \tag{2.2.10}$$

Note that the right member of the ordinary differential equation (2.2.9) *remains bounded, continuous w.r.t. (t, S, u) and Lipschitz in S, which then gives existence and uniqueness of solution of* (2.2.9) *for any measurable control function $u(\cdot)$ that takes values in $[0, 1]$. However, we shall show in Section 2.4 (Proposition 2.4.2) that under water scarcity, an optimal solution never saturates this constraint, so that we no longer need to consider it.*

2.3 The optimization problem

For each control $u(\cdot)$ in $\mathcal{U}$, we associate the total water delivered to the crop during the time interval $[0, T]$ as

$$Q[u(\cdot)] := F_{max} \int_0^T u(t)\, dt \tag{2.3.1}$$

and given a quantity of water $\bar{Q} > 0$ available on the time interval $[0, T]$, we define the constraint

$$Q[u(\cdot)] \leq \bar{Q} \tag{2.3.2}$$

that defines the quota mentioned in the introduction. Then, the determination of the maximal biomass production is stated as the optimal control problem

$$\sup \Big\{ B(T), \; u(\cdot) \in \mathcal{U} \text{ that satisfies (2.3.2)} \Big\}. \tag{2.3.3}$$

By the usual argument of compactness of the set of admissible solutions, the dynamics being linear w.r.t. u (see for instance [75]), one concludes about the existence of an optimal solution of Problem (2.3.3), that we aim now to characterize. Preliminary results are available in the conference paper [48], which are much proved and generalized in the present work.

Let $\bar{B} = B(T)$ where $B(\cdot)$ is solution of $\dot{B} = \varphi(t) f(B)$ with $B(0) = B_0$, which is a uniform bound of the solutions of (2.2.3)-(2.2.5) on $[0, T]$. Note that f is necessarily positive on $[B_0, \bar{B})$ and one can consider the change of variable of the biomass

$$B \mapsto \tilde{B} = g(B) := \int_{B_0}^{B} \frac{db}{f(b)}, \quad B \in [B_0, \bar{B}),$$

which gives the simplified dynamics

$$\dot{\tilde{B}} = \varphi(t) K_S(S), \quad \tilde{B} = 0$$

instead of (2.2.3) and the true value of the biomass $B(t)$ can be recovered by $B(t) = g^{-1}(\tilde{B}(t))$, the function g being increasing and thus invertible. For instance, for the logistic law (2.2.6), one has

$$B(t) = \frac{B_{max}}{1 + \left(\frac{B_{max}}{B_0} - 1\right) e^{-r\tilde{B}(t)}}, \quad t \geq 0.$$

Therefore, maximizing $\tilde{B}(T)$ is equivalent to maximizing $B(T)$ (i.e. an optimal control maximizing $\tilde{B}(T)$ is also optimal for the problem (2.3.3)). For sake of simplicity, we shall drop the $f(B)$ term in equation (2.2.3) and consider $B_0 = 0$ without any loss of generality.

For convenience, we shall denote, for any $t_s \in [0, T]$ and $S_s \in [0, 1]$, $S_{t_s, S_s, 0}(\cdot)$, resp. $S_{t_s, S_s, 1}(\cdot)$, for the solution of the differential equation (2.2.2) with $S(t_s) = S_s$

and the constant control $u = 0$, resp. $u = 1$. We shall see in the following that the corner points, and especially the threshold $S^\star$, are playing a crucial role in the optimal synthesis. The following definitions will be useful in the following.

Definition 2.3.1. *Denote* $\underline{S}(\cdot) := S_{0,S_0,0}(\cdot)$ *and define*

$$\underline{t} := \sup\{t \in [0,T] \text{ s.t. } \underline{S}(t) > S^\star\}.$$

Define also the number

$$B_T^\star := \int_0^T \varphi(t)\,dt.$$

Straightforwardly, on has the first result.

Lemma 2.3.1.

(i) The inequality $B(T) \leq B_T^\star$ *is fulfilled for any admissible control* $u(\cdot)$.

(ii) If $\underline{t} = T$, *then any admissible control* $u(\cdot)$ *gives* $B(T) = B_T^\star$ *(in particular the control identically null).*

Let us now define *singular controls* as the ones that maintain $S(\cdot)$ constant.

Definition 2.3.2. *For any* $\tilde{S} \in (0,1)$, *define the control*

$$\tilde{u}_{\tilde{S}}(t) := \frac{\varphi(t)K_S(\tilde{S}) + (1-\varphi(t))K_R(\tilde{S})}{k_2}, \quad t \in [0,T] \tag{2.3.4}$$

and posit for $\tilde{S} = S^\star$

$$Q^\star := F_{max} \int_{\underline{t}}^T \tilde{u}_{S^\star}(t)\,dt.$$

Note that under Assumption 2.2.3, the control (2.3.4) is admissible i.e. one has $\tilde{u}_{\tilde{S}}(t) \in [0,1]$ at any $t \in [0,T]$ whatever is $\tilde{S} \in (0,1)$. Moreover one has

$$\tilde{u}_{\tilde{S}}(t) \in (0,1),\ t \in (0,T),\ \tilde{S} \in (S_h, S^\star]. \tag{2.3.5}$$

One can easily check that the following Lemma holds.

Lemma 2.3.2. *Assume* $\underline{t} < T$.

(i) For any $\bar{Q} \geq Q^\star$, the control

$$u(t) = \begin{cases} 0 & t \in [0, \underline{t}), \\ \tilde{u}_{S^\star}(t) & t \in [\underline{t}, T], \end{cases} \tag{2.3.6}$$

satisfies the constraint (2.3.2) *and gives $B(T) = B_T^\star$.*

(ii) For any $\bar{Q} < Q^\star$ and admissible control $u(\cdot)$ satisfying the constraint (2.3.2)*, one has $B(T) < B_T^\star$.*

Consequently, when $\underline{t} = T$ or $\bar{Q} \geq Q^\star$, we know that the maximal biomass production $B_T^\star$ can be reached with the control strategy (2.3.6) (other choices could be possible). We shall focus now on the complementary cases that fulfill the following conditions.

Hypothesis 1. *$\underline{t} < T$ and $\bar{Q} < Q^\star$.*

This hypothesis corresponds to situations of water scarcity, because there is not any enough water available to maintain the soil humidity constantly above or equal to the level $S^\star$ which provides the maximal production $B_T^\star$ at the harvesting time. Those situations are quite challenging from the control viewpoint because the crop has to suffer from dryness at a certain point and the question amounts to choose, to some extent, how and when, to impact as little as possible the biomass production at final time T. We start by investigating the behavior of the optimal solutions above the $S^\star$ level.

2.4 Properties of the optimal solutions with respect to threshold $S^\star$

We introduce below the MRAP (for *Most Rapid Approach Path*) to $S = S^\star$ controls. Such kind of controls have already been considered in several optimal control problems in the plane, characterizing their optimality (e.g. [54, 44, 42] or related to the so-called "turnpike" property (see e.g. [59, 73, 36]). Here, we use it in a different way. We do not pretend that these controls are necessarily optimal (and indeed they are not), but they respect the state constraint (2.2.7) and can locally improve the cost, providing then a comparison tool given in Proposition 2.4.1 below and used later on. We begin with some definitions.

Definition 2.4.1. *For $(t_s, S_s) \in [0,T) \times (S^\star, 1]$, we define the number*

$$t^+(t_s, S_s) = \begin{cases} T & \textit{if } S_{t_s,S_s,0}(t) > S^\star, \ t \in [t_s, T], \\ \inf\{t > t_s \, ; \, S_{t_s,S_s,0}(t) = S^\star\} & \textit{otherwise.} \end{cases}$$

And for any $(t_s, S_s) \in (0,T] \times (S^\star, 1]$, we define

$$t^-(t_s, S_s) = \begin{cases} 0 & \textit{if } S_{t_s,S_s,1}(t) > S^\star, \ t \in [0, t_s], \\ \sup\{t < t_s \, ; \, S_{t_s,S_s,1}(t) = S^\star\} & \textit{otherwise.} \end{cases}$$

Definition 2.4.2. *For any $(t_1, S_1) \in [0,T) \times [S^\star, 1]$ and $(t_2, S_2) \in (t_1, T] \times [S^\star, 1]$ such that S_2 is attainable from (t_1, S_1) at time t_2 with an admissible control, we associate the MRAP control $\tilde{u}(\cdot)$ on the time interval $[t_1, t_2]$ as follows:*

i) If $t^-(t_2, S_2) \geq t^+(t_1, S_1)$:

$$\tilde{u}(t) := \begin{cases} 0 & \textit{if } t \in [t_1, t^+(t_1, S_1)), \\ \tilde{u}_{S^\star}(t) & t \in [t^+(t_1, S_1), t^-(t_2, S_2)], \\ 1 & \textit{if } t \in (t^-(t_2, S_2), t_2]. \end{cases} \tag{2.4.1}$$

ii) If $t^-(t_2, S_2) < t^+(t_1, S_1)$:

$$\tilde{u}(t) := \begin{cases} 0 & \textit{if } t \in [t_1, \bar{t}(t_1, S_1, t_2, S_2)), \\ 1 & \textit{if } t \in (\bar{t}(t_1, S_1, t_2, S_2), t_2]. \end{cases}$$

where $\bar{t}(t_1, S_1, t_2, S_2)$ is the unique $\bar{t} \in [t_1, t_2]$ such that $S_{t_1,S_1,0}(\bar{t}) = S_{t_2,S_2,1}(\bar{t}) > S^\star$ (one can easily verify that the function $I(t) := S_{t_1,S_1,0}(t) - S_{t_2,S_2,1}(t)$ is decreasing on $[t_1, t_2]$ and such that $I(t_1) \geq 0$, $I(t_2) \leq 0$, which gives the existence and uniqueness of $\bar{t}(t_1, S_1, t_2, S_2)$).

These particular trajectories are depicted on Fig. 2.2 and 2.3.

Then, one has the following comparison result.

Proposition 2.4.1. *Let $S(\cdot)$ be a solution of (2.2.2) on $[t_1, t_2]$ (with $0 \leq t_1 < t_2 \leq T$) for an admissible control $u(\cdot)$ such that $S(t) \geq S^\star$ for any $t \in [t_1, t_2]$. Denote $S_1 = S(t_1)$ and $S_2 = S(t_2)$. Then, the solution $\tilde{S}(\cdot)$ of (2.2.2) on $[t_1, t_2]$ with $\tilde{S}(t_1) = S_1$ and the MRAP control $\tilde{u}(\cdot)$ (given in Definition 2.4.2) satisfies the following properties:*

$$\tilde{S}(t_2) = S_2. \tag{2.4.2}$$

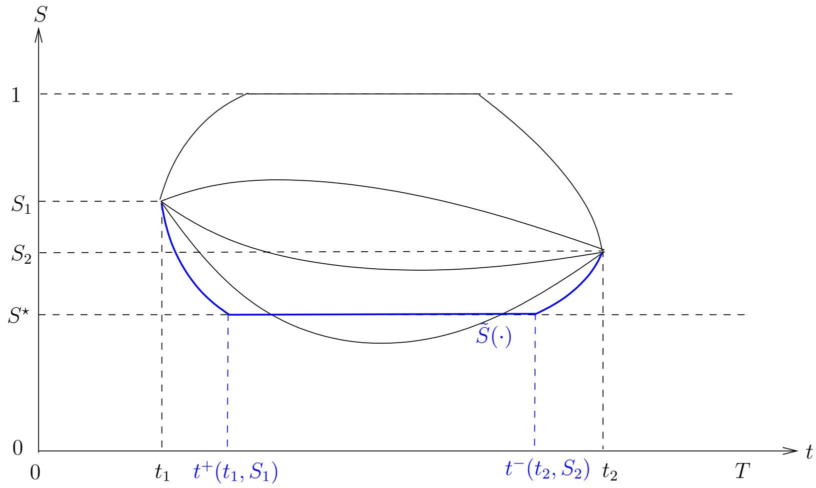


FIGURE 2.2 – The MRAP trajectory $\tilde{S}(\cdot)$ (in blue) compared to other admissible trajectories $S(\cdot)$ when $t_-(t_2, S_2) > t_+(t_1, S_1)$

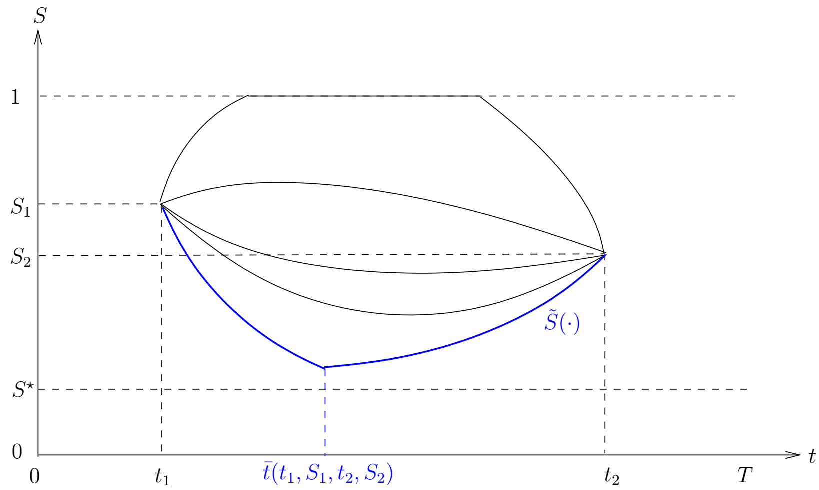


FIGURE 2.3 – The MRAP trajectory $\tilde{S}(\cdot)$ (in blue) compared to other admissible trajectories $S(\cdot)$ when $t_-(t_2, S_2) < t_+(t_1, S_1)$

$$\tilde{S}(t) \leq S(t), \quad t \in [t_1, t_2]. \tag{2.4.3}$$

$$\int_{t_1}^{t_2} \tilde{u}(t)\, dt \leq \int_{t_1}^{t_2} u(t)\, dt. \tag{2.4.4}$$

Moreover, the last inequality is strict when $S(\cdot)$ and $\tilde{S}(\cdot)$ are not identical.

Proof.

By construction, the solution $\tilde{S}(\cdot)$ verifies $\tilde{S}(t_1) = S(t_1)$ and $S(t_2) = S(t_2)$. Thus, property (2.4.2) is verified.

From standard comparison results of scalar differential equation with right hand sides that are Lipschitz continuous w.r.t. the state variable (see e.g. [76]), one has for any solution $S(\cdot)$ of (2.2.2) with $S(t_s) = S_s$ and any admissible control function $u(\cdot)$, the following frame

$$S_{t_s,S_s,0}(t) \leq S(t) \leq S_{t_s,S_s},1(t), \quad t \in [t_s, T]. \tag{2.4.5}$$

Therefore, property (2.4.3) is verified.

Consider then the function $\delta(t) := S(t) - \tilde{S}(t)$. From expression (2.2.2), one can write

$$d\delta = -k_1 \left(F(t, S(t)) - F(t, \tilde{S}(t)) \right) dt + k_1 k_2 \left(u(t) - \tilde{u}(t) \right) dt, \tag{2.4.6}$$

where we posit

$$F(t, S) = \varphi(t) K_S(S) + (1 - \varphi(t)) K_R(S).$$

Integrating (2.4.6) between $t = t_1$ and $t = t_2$, one obtains

$$\begin{aligned} \delta(t_2) - \delta(t_1) = -k_1 & \int_{t_1}^{t_2} \left(F(t, S(t)) - F(t, \tilde{S}(t)) \right) dt \\ & + k_1 k_2 \left(\int_{t_1}^{t_2} u(t)\, dt - \int_{t_1}^{t_2} \tilde{u}(t)\, dt \right). \end{aligned}$$

As F is non-decreasing w.r.t. S and $S(t) \geq \tilde{S}(t)$ for $t \in [t_1, t_2]$, one obtains

$$\int_{t_1}^{t_2} u(t)\, dt - \int_{t_1}^{t_2} \tilde{u}(t)\, dt \geq \frac{\delta(t_2) - \delta(t_1)}{k_1 k_2} = 0$$

which proves property (2.4.4).

This result leads to the following properties of the optimal solutions.

Proposition 2.4.2. *Assume that Hypothesis 2 is satisfied (water scarcity). Then, any optimal solution satisfies the following properties.*

(i) $u(t) = 0$ *for a.e.* $t \in [0, \underline{t}]$,

(ii) $S(t) \leq S^\star$ *for any* $t \in [\underline{t}, T]$,

(iii) $Q[u(\cdot)] = \bar{Q}$.

Proof.

Let $\tilde{u}(\cdot)$ be the MRAP control for $(t_1, S_1) = (0, S_0)$ and $(t_2, S_2) = (T, S^\star)$ (see Definition 2.4.2).

Consider any $S(\cdot)$ solution of (2.2.2),(2.2.4) for an admissible control $u(\cdot)$ satisfying the constraint (2.3.2). Notice first that the set

$$E := \{t \in [0, T] \text{ s.t. } S(t) < S^\star\}$$

is non-empty, otherwise one would have $B(T) = B^\star_T$, which is excluded by Lemma 2.3.2.ii. Let $t^\star := \inf E < T$. By continuity of $S(\cdot)$, one has necessarily $S(t^\star) = S^\star$ and by Proposition 2.4.1 (applied for $t_1 < t_2 \in [0, t^\star]$), one has

$$\int_0^{t^\star} \tilde{u}(t)\, dt \leq \int_0^{t^\star} u(t)\, dt. \tag{2.4.7}$$

Notice that one has $\tilde{u}(t) = \tilde{u}_{S^\star}(t)$ for $t \in [t^\star, T]$. From Hypothesis 2, the inequality

$$Q[u(\cdot)] = F_{max} \int_0^T u(t)\, dt < Q^\star = F_{max} \int_0^T \tilde{u}(t)\, dt \tag{2.4.8}$$

is fulfilled. Consequently, (2.4.7) and (2.4.8) give the inequality

$$\int_{t^\star}^T u(t)\, dt < \int_{t^\star}^T \tilde{u}_{S^\star}(t)\, dt,$$

where $\tilde{u}_{S^\star}(t) < 1$ for $t \in [t^\star, T)$ (cf property (2.3.5)). Therefore, the set

$$E_1 := \{t \in [t^\star, T] \text{ s.t. } u(t) < 1\}$$

is necessarily of non-null measure. Moreover, the set $E \cap E_1$ is also of non-null measure (otherwise one would have $u(t) = 1$ for a.e. $t \in E$ that would imply that $S(\cdot)$ is increasing on E, which contradicts $S(t^\star) = S^\star$).

If $t^\star > \underline{t}$, inequality (2.4.7) is strict (by Proposition 2.4.1 applied on $[0, t^\star]$), and one can consider a control $v(\cdot)$ such that

$$\begin{cases} v(t) = \tilde{u}(t), & t \in [0, t^\star], \\ v(t) = u(t), & t \in [t^\star, T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1, \end{cases}$$

with

$$0 < \int_{E\cap E_1} \big(v(t) - u(t)\big)\, dt \leq \int_0^{t^\star} \Big(u(t) - \tilde{u}(t)\Big)\, dt.$$

Then, one has

$$Q[v(\cdot)] \leq Q[u(\cdot)] \leq \bar{Q}$$

which guarantees that $v(\cdot)$ satisfies the constraint (2.3.2). Its associated solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies then $S_v(t) \geq S(t)$ for any $t \in [0, T]$ with

$$\int_{E\cap E_1} S_v(t)\, dt > \int_{E\cap E_1} S(t)\, dt.$$

As $S(t) < S^\star$ for $t \in E \cap E_1$, one obtains under Assumption 2.2.1a the inequality

$$\int_{E\cap E_1} \varphi(t) K_S(S_v(t))\, dt > \int_{E\cap E_1} \varphi(t) K_S(S(t))\, dt, \tag{2.4.9}$$

which yields

$$B_v(T) = \int_0^T \varphi(t) K_S(S_v(t))\, dt > \int_0^T \varphi(t) K_S(S(t))\, dt = B(T). \tag{2.4.10}$$

We conclude that an optimal solution has to verify $t^\star = \underline{t}$, that is such that

$$S(t) = \underline{S}(t), \quad t \in [0, \underline{t}]$$

or equivalently that $u(t) = 0$ for $t \in [0, \underline{t}]$ is optimal.

Consider now a solution $S(\cdot)$, $B(\cdot)$ with an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ and satisfies the constraint (2.3.2), and the set

$$F := \{t \in [\underline{t}, T] \text{ s.t. } S(t) > S^\star\}$$

is non empty. From Proposition 2.4.1, one has

$$\int_F \tilde{u}(t)\, dt < \int_F u(t)\, dt.$$

Let us consider an admissible control $v(\cdot)$ such that

$$\begin{cases} v(t) = \tilde{u}(t), & t \in F, \\ v(t) = u(t), & t \in [0, T] \setminus (F \cup (E \cap E_1)), \\ v(t) \in [u(t), 1], & t \in E \cap E_1, \end{cases}$$

with
$$0 < \int_{E\cap E_1} \Big(v(t) - u(t)\Big)\, dt \le \int_F \Big(u(t) - \tilde{u}(t)\Big)\, dt.$$
Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) = S^\star$ for $t \in F$ and $S_v(t) \ge S^\star$ for $t \in [0,T] \setminus F$ with
$$\int_{E\cap E_1} S_v(t)\, dt > \int_{E\cap E_1} S(t)\, dt.$$
As before, we obtain inequalities (2.4.9), (2.4.10), and conclude that an optimal solution has to verify $F = \emptyset$, that is such that $S(t) \le S^\star$ for $t \in [\underline{t}, T]$.

Finally, consider an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ with $S(t) \le S^\star$ for $t \in [\underline{t}, T]$ and $Q[u(\cdot)] < \bar{Q}$. As previously, one can consider another admissible control $v(\cdot)$ such that:
$$\begin{cases} v(t) = u(t), & t \in [0,T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1, \end{cases}$$
with
$$0 < F_{max} \int_{E\cap E_1} \Big(v(t) - u(t)\Big)\, dt \le \bar{Q} - Q[u(\cdot)].$$
Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) \ge S(t)$ for $t \in [0,T]$ with
$$\int_{E\cap E_1} S_v(t)\, dt > \int_{E\cap E_1} S(t)\, dt.$$
One obtains again inequality (2.4.10), which shows that the control $u(\cdot)$ cannot be optimal. Therefore, an optimal control $u(\cdot)$ has to satisfy $Q[u(\cdot)] = \bar{Q}$.

2.5 Optimal synthesis

Note first that one can write equivalently the optimization problem (2.3.3) as a (non-autonomous) scalar optimal control problem
$$\max_{u(\cdot)} \int_0^T \varphi(t) K_S(S(t))\, dt, \tag{2.5.1}$$

where $S(\cdot)$ is solution of (2.2.2), under constraints (2.2.7) and (2.3.2), or equivalently as an optimal control in the plane for the dynamics

$$\dot{S} = k_1\Big(-\varphi(t)K_S(S)-(1-\varphi(t))K_R(S)+k_2u(t)\Big), \quad S(0)=S_0, \tag{2.5.2}$$

$$\dot{V} = u(t), \quad V(0)=0, \tag{2.5.3}$$

with the target

$$V(T) \leq \bar{V} := \frac{\bar{Q}}{F_{max}} \tag{2.5.4}$$

and the criterion (2.5.1). Moreover, we know from Proposition 2.4.2 that under Hypothesis 2 the state constraint (2.2.7) is never saturated for any optimal solution.

2.5.1 Application of the Maximum Principle

Let us write the Hamiltonian associated to this optimal control problem:

$$\begin{aligned} &H(t,S,\lambda_S,\lambda_V,u) := \\ &\quad \lambda_S k_1\Big(k_2u-(\varphi(t)K_S(S)+(1-\varphi(t))K_R(S))\Big)+\lambda_V u+\lambda_0\varphi(t)K_S(S), \end{aligned} \tag{2.5.5}$$

and its adjoint equations:

$$\dot{\lambda}_S \in \varphi(t)\big(\lambda_S k_1-\lambda_0\big)\partial_C K_S(S(t))+(1-\varphi(t))\lambda_S k_1\partial_C K_R(S(t)), \tag{2.5.6}$$

$$\dot{\lambda}_V = 0, \tag{2.5.7}$$

where $\partial_C K_S$, $\partial_C K_R$ denote the Clarke generalized gradients of the Lipschitz maps K_S, K_R. Therefore, λ_V is constant. The (non-smooth) Maximum Principle of Pontryagin (see for instance [26]) states that for any optimal solution $S(\cdot)$, $V(\cdot)$, $u(\cdot)$, there exists an adjoint vector $\lambda(\cdot) = (\lambda_S(\cdot),\lambda_V(\cdot))$ which is an absolutely continuous solution of the adjoint system (2.5.6)-(2.5.7) and a scalar λ_0 equal to 0 or 1 such that

$$\lambda_0+|\lambda_S(t)|+|\lambda_V(t)| \neq 0, \quad t\in[0,T], \tag{2.5.8}$$

which satisfy the transversality condition

$$\lambda_S(T)=0, \quad \lambda_V \leq 0 \tag{2.5.9}$$

(remind that $S(T)$ is free and that $V(T)$ touches the boundary of the target $V \leq \bar{V}$ at terminal time by Proposition 2.4.2), along with the maximization condition

$$H(t, S(t), \lambda_S(t), \lambda_V(t), u(t)) = \max_{v \in [0,1]} H(t, S(t), \lambda_S(t), \lambda_V(t), v), \quad \text{a.e. } t \in [0, T]. \tag{2.5.10}$$

Defining the *switching function*

$$\phi(t) := \lambda_S(t) k_1 k_2 + \lambda_V, \tag{2.5.11}$$

the maximization (2.5.10) gives, for a.e. $t \in [0, T]$

$$\begin{cases} u(t) = 1 & \text{if } \phi(t) > 0, \\ u(t) \in [0,1] & \text{if } \phi(t) = 0, \\ u(t) = 0 & \text{if } \phi(t) < 0. \end{cases} \tag{2.5.12}$$

We first show that that an optimal solution cannot be abnormal.

Lemma 2.5.1. *For any optimal solution, one has $\lambda_0 = 1$.*

Proof. If $\lambda_0 = 0$, the only solution of (2.5.6) for the terminal condition (2.5.9) is $\lambda_S(t) = 0$ for $t \in [0, T]$. Moreover, the constant value of λ_V has to be negative to fulfill the conditions (2.5.9) and (2.5.8). This implies that $\phi(t)$ is negative for any $t \in [0, T]$ and by (2.5.12), one has $u(t) = 0$ for a.e. $t \in [0, T]$ i.e. $\underline{S}(\cdot)$ is the optimal trajectory. Let $\underline{t} \in [0, T]$ be such that $\underline{S}(\underline{t}) = S^\star$. Then the control $v(\cdot)$ defined by

$$\bar{v}(t) = \begin{cases} 0, & t \in [0, \underline{t}), \\ \dfrac{\bar{Q}}{T - \underline{t}}, & t \in [\underline{t}, T], \end{cases}$$

is admissible, and its associated solution $S_v(\cdot)$ verifies

$$S_v(t) = \underline{S}(t), \; t \in [0, \underline{t}), \quad S_v(t) > \underline{S}(t), \; t \in [\underline{t}, T]$$

which implies the inequality

$$\int_0^T \varphi(t) K_S(S_v(t))\, dt > \int_0^T \varphi(t) K_S(\underline{S}(t))\, dt,$$

and thus is a contradiction with the optimality of $\underline{S}(\cdot)$.

We prove now sign properties of the adjoint variables, that will play a crucial role in the following.

Proposition 2.5.1. *For any optimal solution, one has $\lambda_S(t) \geq 0$ for any $t \in [0, T]$. Moreover, one has $\lambda_V < 0$.*

Proof. Let us consider the set

$$E := \{t \in [0, T) \text{ s.t. } \lambda_S(t) < 0\}$$

and assume by contradiction that E is non-empty. As one has $S(t) > S_h$ for any $t \in [0, T]$ (cf Lemma 2.2.1), and the functions K_S and K_R are respectively non-decreasing and increasing on $[S_h, 1]$ (by Assumption 2.2.1a), one obtains from equation (2.5.6) that λ_S is non-increasing on E. Therefore, one has $\sup E = T$ and $\lambda_S(T) < 0$, which is a contradiction with the transversality condition (2.5.9).

If $\lambda_V = 0$, λ_S can be null on a time interval only if $K_S(S)$ is constant on this time interval, according to the adjoint equation (2.5.6). Then, for a.e. t such that $S(t) \in (S_w, S^\star)$, one has $\phi(t) > 0$ and thus $u(t) = 1$, which prevents the solution to go below $S^\star$, in contradiction with Hypothesis 2. We conclude that λ_V is negative.

Then, Proposition 2.5.1 and the transversality condition (2.5.9) imply that one has $\phi(T) < 0$, which gives straightforwardly the following property of the optimal solutions.

Corollary 2.5.1. *For any optimal solution, there exists $\bar{t} < T$ such that $u(t) = 0$ for a.e. $t \in [\bar{t}, T]$. Moreover, one has $S(T) < S^\star$.*

2.5.2 Study of singular arcs

Let us now study the possibilities of singular arcs (we recall that a singular arc is a part of an optimal trajectory such that the switching function ϕ remains equal to zero).

Proposition 2.5.2. *A singular arc on a closed interval I not reduced to a singleton satisfies $S(t) = \tilde{S}$, $t \in I$ where $\tilde{S}$ is a corner point of K_S or K_R that belongs to $(S_w, S^\star]$.*

Proof. We proceed in two steps. We first show that a singular arc can only occur at a corner point of the function K_S or K_R, studying the switching function.

Then, we show by contradiction that a singular arc can occur only at corner points above the plant wilting threshold S_w.

A singular arc occurs when the switching function ϕ is equal to zero on a closed interval I of non-null measure. This amounts to have λ_S constant equal to $\lambda_S^\star := -\lambda_v/(k_1k_2) > 0$ on such an interval. If K_S and K_R are differentiable at $S(t_1)$ with $t_1 \in I$, $K_S'(S(t))$ and $K_R'(S(t))$ are constant equal to $K_S'(S(t_1))$ and $K_R'(S(t_1))$ on a neighborhood $(t_1 - \epsilon, t_1 + \epsilon)$ of t_1 (as the functions K_S, K_R are piecewise linear by Assumption 2.2.1a). Then, from equation (2.5.6), one gets

$$\varphi(t)\left[(\lambda_S^\star k_1 - 1\Big)K_S'(S(t_1)) - \lambda_S^\star k_1 K_R'(S(t_1)\right] = -\lambda_S^\star k_1 K_R'(S(t_1)), \quad t \in I \cap (t_1 - \epsilon, t_1 + \epsilon). \tag{2.5.13}$$

From Lemma 2.2.1, one has $S(t_1) > S_h$ and thus $K_R'(S(t_1)) > 0$ (by Assumption 2.2.1a). Finally, as the function φ is strictly increasing, we deduce that (2.5.13) cannot be fulfilled. We deduce that a singular arc can occur only for constant $S = \tilde{S}$ that are non-differential points of K_S or K_R.

Let us now show that it can occur only at corner points above S_w. From Proposition 2.4.2, we know that a singular arc cannot be optimal at values $\tilde{S}$ above $S^\star$, and from Lemma 2.2.1, that it cannot occur for $\tilde{S} \leq S_h$. From equation (2.5.6), one should have

$$0 \in \varphi(t)\Big(\lambda_S^\star k_1 - \lambda_0\Big)\partial_C K_S(\tilde{S}) + (1 - \varphi(t))\lambda_S^\star k_1 \partial_C K_R(\tilde{S}) \text{ a.e. } t \in I. \tag{2.5.14}$$

At $\tilde{S} \in (S_h, S_w)$, one has $\partial_C K_S(\tilde{S}) = \{0\}$ and any element in $\partial_C K_R(\tilde{S})$ is positive. Therefore condition (2.5.14) cannot be fulfilled. We show now that a singular arc with $\tilde{S} = S_w$ cannot be part of an optimal solution. If it is, let $I = [t_1, t_2]$ and define

$$\delta_I := \int_I u(t)\, dt = \int_I \tilde{u}_{S_w}(t)\, dt < t_2 - t_1.$$

From Corollary 2.5.1, one has $t_2 < T$. If $S(t) \leq S_w$ for any $t > t_2$, one would have $B(T) = B(t_1)$. Consider then the control

$$\check{u}(t) = \begin{cases} u(t), & t \in [0, t_1), \\ 1, & t \in [t_1, t_1 + \delta_I), \\ 0, & t \in [t_1 + \delta_I, T], \end{cases}$$

and the associated solution $\check{S}(\cdot)$, $\check{B}(\cdot)$. This control would be admissible (i.e. $Q[\check{u}(\cdot)] \leq$

$Q[u(\cdot)] = \bar{Q}$), and one would have $\check{B}(T) > \check{B}(t_1 + \delta_I) > \check{B}(t_1) = B(T)$ (because $\check{S}(t) > S_w$ for $t \in [t_1, t_1 + \delta I]$). Then, the control $u(\cdot)$ would not be optimal. Therefore $S(\cdot)$ has to take values above S_w in the time interval $[t_2, T]$ and from what precedes for a.e. $t > t_2$ with $S(t) > S_w$, either $\dot{S}(t) > 0$ (with $u(t) = 1$) or $\dot{S}(t) < 0$ (with $u(t) = 0$) or $\dot{S}(t) = 0$ (with a singular control). Then, from Corollary 2.5.1 we deduce that there exists a sub-interval $J = [\bar{t}_1, \bar{t}_2] \subset (t_2, T]$ with $\bar{t}_1 < \bar{t}_2$ such that $S(t) > S_w$ for any $t \in J$ and $u(t) = 0$ for a.e. $t \in J$.

We now construct another control that gives the same biomass production but with a lower water consumption. Consider a control with $t^\dagger \in I$, defined as follows

$$u^\dagger(t) = \begin{cases} 0 & \text{if } t \in [t_1, t^\dagger), \\ 1 & \text{if } t \in [t^\dagger, t_2], \\ u(t) & \text{otherwise.} \end{cases}$$

Denote by $S^\dagger$, $B^\dagger$ the corresponding solution. For $t^\dagger = t_1$, one has $S^\dagger(t_2) > S_w$ and for $t^\dagger = t_2$, $S^\dagger(t_2) < S_w$. By the intermediate value theorem, there exists $t^{\dagger_0} \in (t_1, t_2)$ such that $S^{\dagger_0}(t_2) = S_w$. Then, one has $S^{\dagger_0}(t) < S_w$ at any $t \in (t_1, t_2)$ and $S^{\dagger_0}(t) = S(t)$ for $t \in [0, T] \setminus (t_1, t_2)$. Moreover one has $B^{\dagger_0}(t) = B(t)$ for any $t \in [0, T]$ (due to $\dot{B}(t) = \dot{B}^{\dagger_0}(t) = 0$ at $t \in (t_1, t_2)$). As one has $S^{\dagger_0}(t_1) = S^{\dagger_0}(t_2) = S_w$, one gets from the integration of equation (2.2.2) on I and K_R increasing on (S_h, S_w), the inequality

$$\int_I u^{\dagger_0}(t)\, dt = \frac{1}{k_2} \int_I (1 - \varphi(t)) K_R(S^{\dagger_0}(t))\, dt < \frac{1}{k_2} \int_I (1 - \varphi(t)) K_R(S_w)\, dt = \int_I u(t)\, dt.$$

Therefore, the control $u^{\dagger_0}(\cdot)$ gives the same cost $B^{\dagger_0}(T) = B(T)$ but with a lower water consumption i.e. $Q[u_0^\dagger(\cdot)] < Q[u(\cdot)] = \bar{Q}$.

Finally, we derive a new control that produces a better objective. Let $\delta = \bar{Q} - Q[u_0^\dagger(\cdot)] > 0$ and consider the control

$$u^\#(t) = \begin{cases} u^{\dagger_0}(t), & t \in [0, \bar{t}_1) \cup [\min(\bar{t}_2, \bar{t}_1 + \delta), T], \\ 1, & t \in [\bar{t}_1, \min(\bar{t}_2, \bar{t}_1 + \delta)), \end{cases}$$

which satisfies also the constraint (2.3.2). Denote $S^\#(\cdot)$, $B^\#(\cdot)$ the solution with the control $u^\#(\cdot)$. One has $S^\#(\bar{t}_1) = S(\bar{t}_1)$ and $B^\#(\bar{t}_1) = B(\bar{t}_1)$ and then $S^\#(t) > S(t)$ for any $t > \bar{t}_1$, which implies $B^\#(t) > B(t)$ for any $t > \bar{t}_1$ (because $K_S(S^\#(t)) > K_S((t))$ for $t \in J$). Therefore, the control $u(\cdot)$ cannot be optimal, which shows that

$\tilde{S} = S_w$ cannot be a singular arc.

2.5.3 The SMS strategy

For convenience, we denote sub-sets of corner points of K_S or K_R as below.

Definition 2.5.1. *For any $S \in (S_w, S^\star]$, let $\mathcal{C}(S)$ be the set of corner points $\tilde{S} \geq S$ in $(S_w, S^\star]$ and $n(S) = card\,\mathcal{C}(S)$. Define then the increasing sequence of corner points $\{\tilde{S}_i(S)\}_{i=1\ldots n(S)}$ such that $\mathcal{C}(S) = \{\tilde{S}_1(S), \ldots, \tilde{S}_{n(S)}(S)\}$.*

Note that for any $S \in (S_w, S^\star]$, the set $\mathcal{C}(S)$ is non empty. It contains at least $S^\star$ as the largest element, i.e. $\tilde{S}_{n(S)}(S) = S^\star$. We define now the *saturated multiple shots* (SMS) strategy.

Definition 2.5.2. *For $S_m \in (S_h, S^\star]$ and a sequence of non decreasing numbers $V_i \in (0, \bar{V}]$, $i \in \{1, \ldots, n(S_m)\}$ with at least one equal to $\bar{V}$, define the time-varying feedback control*

$$\psi^{SMS}_{S_m,\{V_i\}}(t, S, V) := \begin{cases} 0 & \textit{if } V = \bar{V} \textit{ or } S > S_m \textit{ with } V = 0, \\ \tilde{u}_S(t) & \textit{if } S = \tilde{S}_i(S_m) \textit{ for } i \in \{1, \ldots, n(S_m)\} \textit{ with } V < V_i, \\ 1 & \textit{otherwise.} \end{cases} \tag{2.5.15}$$

Note that this control strategy is admissible because it guarantees $V(T) \leq \bar{V}$. This strategy consists of starting the irrigation when the moisture S reaches an *irrigation trigger threshold* S_m, with one or several stages. If the humidity rate $S(t)$ reaches a level $\tilde{S}_i(S_m)$ for some $i \in \{1, \ldots, n(S_m)\}$ the flow rate is saturated to maintain S constant at this *level value* as long as the used volume $V(t)$ stays below the value V_i. This is what we call a "saturated shot". Note that if V_i is too small or if $\tilde{S}_i(S_m)$ cannot be reached, there is no saturation of the flow rate for this value i.e. the trajectory does not present a step at this value. The generated trajectory has then at most $n(S_m)$ increasing saturated shots. This is why we call this feedback control a "saturated multiple shots" (see Section 2.6 for an illustration). Remark also that once $S(\cdot)$ has reached S_m then $S(t) \leq S^\star$ for any future time. We now give our main result about the optimality of the SMS strategy.

Theorem 2.5.1. *Under Assumptions 2.2.1a, 2.2.2, 2.2.3, 2.2.4 and Hypothesis 2, there exists a value $S_m \in (S_h, S^\star]$ and a sequence of non decreasing numbers V_i, $i = 1, \ldots, n(S_m)$*

with at least one equal to $\bar{V}$ such that the SMS feedback (2.5.15) *with*

$$\int_0^{t_M} \psi^{SMS}_{S_m,\{V_i\}}(t, S(t), V(t))\, dt = \bar{V} \quad \textit{for some } t_M < T \tag{2.5.16}$$

is optimal.

Proof. Let $u(\cdot)$ be an optimal control, $S(\cdot), V(\cdot)$ the associated solution of (2.5.2), (2.5.3) and $\lambda_S(\cdot)$, λ_V the adjoint variables given by the Maximum Principle (see Section 2.5.1). Recall first that under Hypothesis 2, $S(\cdot)$ satisfies $S(t) \geq \underline{S}(t)$ for any $t \in [0, T]$ with $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$ (cf Proposition 2.4.2.ii).

As $u(t) = 0$ for a.e. $t \in [0, \underline{t}]$ (Proposition 2.4.2.i), the switching function $\phi(\cdot)$ has to be non-positive on $[0, \underline{t}]$, or equivalently one should have $\lambda_S(t) \leq \lambda_{S^\star} = -\lambda_v/(k_1 k_2) > 0$ for $t \in [0, \underline{t}]$. On the interval $[0, \underline{t}]$, $S(\cdot)$ is thus decreasing with $S(\underline{t}) = S^\star$. As K_R and K_S are respectively increasing and constant on the interval $[S^\star, 1]$, we deduce from (2.5.6) that $\lambda_S(t) < \lambda_{S^\star}$ for any $t \in [0, \underline{t})$.

Consider the set

$$C := \{t \in [\underline{t}, T] \text{ s.t. } \lambda_S(t) \geq \lambda_{S^\star}\}$$

which is of non empty interior (otherwise $u(t) = 0$ would be optimal for any $t \in [\underline{t}, T]$, which is not possible by Proposition 2.4.2.iii). Let us show that the set C is connected. If not, consider a time interval (t_1, t_2) in-between two consecutive connected components of C, that is such that $\lambda_S(t_1) = \lambda_S(t_2) = \lambda_{S^\star}$ and $\lambda_S(t) < \lambda_{S^\star}$ for any $t \in (t_1, t_2)$. Note that $\lambda_S(t) < \lambda_{S^\star}$ for $t \in (t_1, t_2)$ implies that $u(t) = 0$ for almost any $t \in (t_1, t_2)$ and consequently $S(\cdot)$ is decreasing on (t_1, t_2).

The function λ_S attains necessarily its minimum on (t_1, t_2), say at $\hat{t}$, and one has then

$$0 \in \varphi(\hat{t})(\lambda_S(\hat{t})k_1 - 1)\partial_C K_S(S(\hat{t})) + (1 - \varphi(\hat{t}))\lambda_S(\hat{t}))k_1 \partial_C K_R(S(\hat{t})). \tag{2.5.17}$$

If $S(\hat{t}) < S_w$, one has $\partial_C K_S(S(\hat{t})) = \{0\}$ and $\partial_C K_R(S(\hat{t})) \subset \mathbb{R}_+^\star$ (by Assumption 2.2.1a and Lemma 2.2.1), and thus (2.5.17) cannot be satisfied. So, one has necessarily $S(\hat{t}) \geq S_w$. Note that one has also $\lambda_S(\hat{t})k_1 - 1 < 0$.

As $S(\cdot)$ is decreasing on (t_1, t_2), $K_S'(S(t))$ and $K_R'(S(t))$ exist for almost any $t \in (t_1, t_2)$. The function K_S, resp. K_R, being concave, resp. convex, on $(S_w, S^\star)$, and as $S(t) \in (S_w, S^\star)$ for any $t \in (t_1, \hat{t})$, one has the property

$$
\begin{array}{l}
S(t) > S(\hat{t}) \;\Rightarrow \\
\left\{\xi \geq K_S'(S(t)) > 0,\ \forall \xi \in \partial_C K_S(S(\hat{t})) \text{ and } K_R'(S(t)) \geq \zeta > 0,\ \forall \zeta \in \partial_C K_R(S(\hat{t}))\right\},
\end{array}
\tag{2.5.18}
$$

for almost any $t \in (t_1, \hat{t})$. Now, $\dot{\lambda}_S(t)$ exists for a.e. $t \in (t_1, \hat{t})$ and one can write

$$
\begin{array}{rcl}
\dot{\lambda}_S(t) & = & \varphi(t)(\lambda_S(t)k_1 - 1)K_S'(S(t)) + (1-\varphi(t))\lambda_S(t))k_1 K_R'(S(t)) \\
& \geq & \varphi(t)(\lambda_S(\hat{t})k_1 - 1)K_S'(S(t)) + (1-\varphi(t))\lambda_S(\hat{t}))k_1 K_R'(S(t)),
\end{array}
$$

for almost any $t \in (t_1, \hat{t})$. Finally, as the function φ is increasing and positive, one obtains with (2.5.18) the inequality

$$
\dot{\lambda}_S(t) \geq \varphi(\hat{t})(\lambda_S(\hat{t})k_1 - 1)\xi + (1-\varphi(\hat{t}))\lambda_S(\hat{t}))k_1\zeta, \quad \forall \xi \in \partial_C K_S(S(\hat{t})),\ \forall \zeta \in \partial_C K_R(S(\hat{t})),
$$

for almost any $t \in (t_1, \hat{t})$. From (2.5.17), one gets $\dot{\lambda}_S(t) \geq 0$ for almost any $t \in (t_1, \hat{t})$, which contradicts $\lambda_S(\hat{t})$ being a minimum of λ_S on (t_1, t_2). The set C is thus connected.

The set C being non-empty connected and $\lambda_S(\cdot)$ continuous, C is a closed interval $[t_m, t_M]$ and $u(t) = 0$ for a.e. $t \notin C$. From Proposition 2.4.2 and Lemma 2.2.1, we have that $t_m \geq \underline{t}$ and $S_m = S(t_m)$ belongs to $(S_h, S^\star]$. Moreover, from Corollary 2.5.1, we have $t_M < T$. Then, at any $t \in C$, the switching function $\phi(t)$ is non-negative. Therefore, for a.e. $t \in C$, either $u(t) = 1$ or $u(t) = \tilde{u}_{\tilde{S}}(t)$ with $\tilde{S} \in (S_w, S^\star)$ a corner point (Proposition 2.5.2)). Consequently, $V(\cdot)$ is increasing on C and $S(\cdot)$ non decreasing on C, composed of increasing parts $u = 1$ and possibly singular ones $S = \tilde{S}_i$ with $\tilde{S}_i \in \mathcal{C}(S_m)$. As $V(\cdot)$ is increasing, the time t_i when the solution $S(\cdot)$ leaves $S = \tilde{S}_i$ is equivalently defined by the value $V_i = V(t_i)$ reached by the variable $V(\cdot)$. If $\tilde{S}_j$ is not reached, we can set by convention $V_j = \bar{V}$.

At $t = t_M$, Proposition 2.4.2 gives $S(t_M) \leq S^\star$ and $V(t_M) = \bar{V}$. Then, either one has $S(t_M) \notin \mathcal{C}(S_m)$ and $S^\star = \tilde{S}_{n(S_m)}$ is not reached, or $S(t_M) = \tilde{S}_j \in \mathcal{C}(S_m)$ with $j \in \{1, \ldots, n(S_m)\}$ and $V_j = \bar{V}$. In any case there exists $i \in \{1, \ldots, n(S_m)\}$ such that $V_i = \bar{V}$. Finally, one can easily check that $u(\cdot)$ fulfills $u(t) = \psi^{SMS}_{S_m,\{V_i\}}(t, S(t), V(t))$ for a.e. $t \in [0, T]$, and property iii of Proposition 2.4.2 imposes that the S_m and the V_i are such that the equality (2.5.16) is satisfied.

Let us underline that the structure of the feedback (2.5.15) does not depend on the shape of the radiation interception efficiency function φ, although the optimal switching times do depend on the values of this function.

2.5.4 The SOS strategy

When the functions K_S, K_R have no corner on the interval $(S_w, S^\star)$, as it is the case for Assumption 2.2.1b, the SMS strategy takes a simpler expression, that we define as a "saturated one shot" (SOS) strategy as follows.

Definition 2.5.3. *For $S_m \in (S_h, S^\star]$, define the time-varying feedback control*

$$\psi^{SOS}_{S_m}(t, S, V) := \begin{cases} 0 & \text{if } V = \bar{V} \text{ or } S > S_m \text{ with } V = 0, \\ \tilde{u}_{S^\star}(t) & \text{if } S = S^\star \text{ with } V < \bar{V}, \\ 1 & \text{otherwise.} \end{cases} \tag{2.5.19}$$

This strategy consists of irrigating crops at once when the humidity rate $S(t)$ gets equal to S_m. The water is delivered at the maximal flow rate ($u = 1$) as long as the humidity rate $S(\cdot)$ is below $S^\star$, or maintain it $S = S^\star$ (with the singular control $\tilde{u}_{S^\star}(\cdot)$), until the entire water budget $\bar{V}$ has been used up. This strategy is parameterized by the single value S_m or equivalently the time when irrigation starts. This explains the wording "saturated one shot". One has the following result about the optimality of this strategy.

Theorem 2.5.2. *Under Assumptions 2.2.1b, 2.2.2, 2.2.3, 2.2.4 and Hypothesis 2, there exists a value $S_m \in (S_h, S^\star]$ such that the SOS strategy* (2.5.19) *with*

$$\int_0^{t_M} \psi^{SOS}_{S_m}(t, S(t), V(t))\, dt = \bar{V} \quad \text{for some } t_M < T, \tag{2.5.20}$$

is optimal.

Proof. Under Assumption 2.2.1b, one has $n(S) = 1$ for any $S \in (S_h, S^\star)$ with $V_1 = \bar{V}$. Therefore, the SMS feedback (2.5.15) has only one parameter S_m which gives the simpler expression (2.5.19).

2.6 Numerical simulations and discussion

In this section, we compare numerically three irrigation strategies on two examples: the SMS and SOS previously introduced and the “One Shot” (OS) strategy which consists of delivering water at maximum flow rate during a single irrigation period at a triggering humidity level S_m.

Definition 2.6.1. *For $S_m \in (S_h, S^\star]$, we denote by t_S the irrigation triggering time associated to an humidity level S_m ($t_S = \underline{S}^{-1}(S_m)$) and define the OS open-loop control as follows :*

$$u^{OS}_{S_m}(t) := \begin{cases} 0 & \textit{if } t < t_S \textit{ or } t > \min(t_S + \bar{Q}/F_{max}, T)), \\ 1 & \textit{if } t \in [t_S, \min(t_S + \bar{Q}/F_{max}, T)). \end{cases}$$

The OS strategy is a pure "bang-bang" control that is implemented in open-loop (once the irrigation starts, it lasts $\bar{Q}/F_{max}$)). It represents a class of widely used irrigation strategies, typically when drip irrigation is not available. For this strategy, we shall look for the triggering humidity level S_m that give the best biomass production, to be compared with the biomass obtained with the best SMS and SOS strategies.

Theorem 2.5.1 has shown that the original optimization problem in infinite dimension is transformed into a simple finite dimensional optimization problems of parameters $(S_m, V_1, \dots, V_n)$. Therefore, the optimal SMS (and SOS) strategy have been numerically determined, looking for the best parameters $(S_m, V_1, \dots, V_n)$ in the set

$$\mathcal{P} := (S_w, S^\star] \times \{V \in \mathbb{R}^n;\, 0 \leq V_1 \leq V_2 \leq \cdots \leq V_n = \bar{V}\}$$

where $n \geq 1$ is the number of distinct corner points $\tilde{S}$ of the functions K_S or K_R in the interval $(S_w, S^\star]$. For each $(S_m, V_1, \dots, V_n)$ in $\mathcal{P}$, the SMS feedback (2.5.15) is well defined and generates a unique solution of equations (2.2.2)-(2.2.3), to which we associate the cost $B(T)$, even though the constraint (2.5.16) might not be satisfied. Indeed, when looking for values of the parameters in $\mathcal{P}$ that give the largest $B(T)$, we know that the optimal ones necessarily satisfy the equality (2.5.16), according to Proposition 2.4.2.

In order to compare different irrigation strategies by their triggering level, we also introduce the partially optimized control SMS*, parameterized by a humidity level S_m, which corresponds to the best SMS strategy for a given triggering level S_m. This strategy can thus be obtained numerically by the approach presented above but optimizing with respect to the only parameters $(V_1, \ldots, V_n)$. This allows us the compare the crop productions provided by the three strategies SMS, SOS, OS for the same triggering level S_m. For each of them, their optimal production is then obtained for their best value S_m.

The performances of these strategies are compared for two kinds of configurations: one with the simplest K_s, K_r functions that fulfill Assumption 2.2.1b, (see Fig. 3.1) and a second one when the function K_s exhibits a more complex shape (see Fig. 2.4). For illustrative purposes only, we have considered dimensionless parameters (by normalizing the units), functions φ in the family of $t \mapsto (t/T)^\alpha$ $(\alpha > 0)$ and functions f as logistic laws $B \mapsto rB(1 - \frac{B}{B_{max}})$ parameterized by (r, B_{max}, B_0). The optimal solutions have been verified with the Bocop-HJB solver [17] that provided with a very good accuracy the same optimal trajectories, but in open-loop.

2.6.1 Under Assumption 2.2.1b

We recall that under this assumption, the SMS and SOS strategies coincide. We present in Fig. 2.5 the simulations performed with irrigation strategies SOS and OS and with inputs data given in Table 2.1.

T	k_1	k_2	$S^\star$	S_w	S_h	F_{max}	$\bar{Q}$	α	r	B_{max}	S_0	B_0
1	2.1	5	0.7	0.4	0.2	1.2	0.1	4	25	1	1	0.0005

TABLE 2.1 – Normalized parameters used for the simulations under Assumption 2.2.1b

The best OS strategy was obtained for a triggering level $S_m = 0.338$ and produced a biomass $B(T) = 0.022$. The corresponding humidity dynamics are plotted in Fig. 2.5c. It can be seen that some values of S are above $S^\star$. It can be therefore concluded from the application of Proposition 2.4.2 that an OS strategy cannot be optimal. This is further illustrated by applying the SOS strategies for the same input data. We find that the optimal SOS strategy gives a final biomass $B(T) = 0.0388$ which is 77% higher than what gives the best OS strategy. The associated control is a bang-singular-bang (see Fig. 2.5b).

2.6.2 Under Assumption 2.2.1a without Assumption 2.2.1b

In this example, we consider that the function K_s presents an additional corner point S_c in between S_w and $S^\star$ (see Fig. 2.4). We present in Fig. 2.6 the simulations

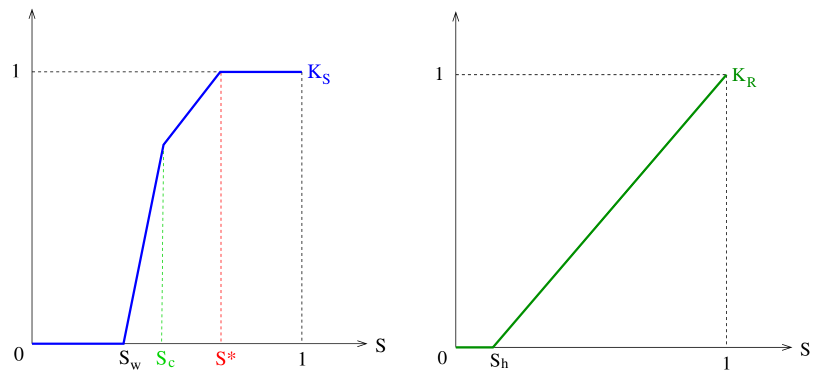


FIGURE 2.4 – Graphs of the functions K_S and K_R that fulfill Assumption 2.2.1a but not Assumption 2.2.1b

performed with irrigation strategies SMS, SOS and OS and with inputs data given in Table 2.2. The biomass levels obtained when optimizing these strategies were 0.111 for SMS, 0.094 for SOS and 0.052 for OS. The best production is obtained with the SMS strategy (as expected by Theorem 2.5.1) and corresponds to the highest triggering humidity level ($S = 0.43$) among the three tested strategies (Fig. 2.6a). This level corresponds to the earliest irrigation triggering time ($t_S = 0.392$, see Fig. 2.6b).

We have shown that the optimal solution may have a singular arc corresponding to intermediate corner points. When looking at the humidity profile in Fig. 2.6c, we can see that indeed two arcs do occur in this setting: one at $S = S_c$ and another one later at $S = S^\star$.

T	k_1	k_2	$S^\star$	S_w	S_h	F_{max}	$\bar{Q}$	α	r	B_{max}	S_0	B_0	S_c
1	2.5	5	0.7	0.4	0.2	1.2	0.09	3	25	1	1	0.0005	0.43

TABLE 2.2 – Normalized parameters used for the simulations under Assumption 2.2.1a.

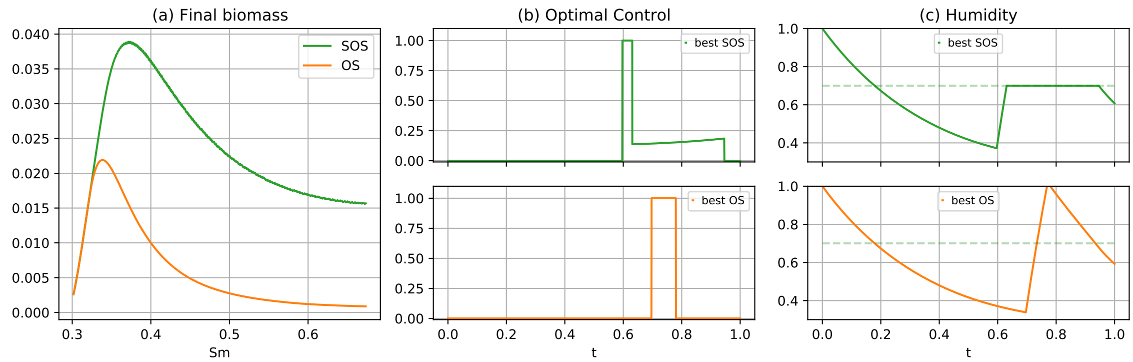


FIGURE 2.5 – Comparison of OS and SOS controls strategies with model parameters given in Table 2.1.

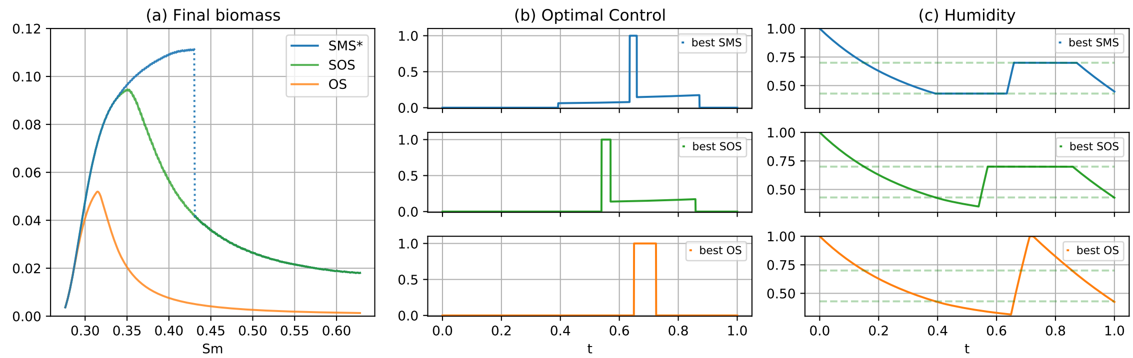


FIGURE 2.6 – Comparison of the SMS control with OS and SOS controls with K_S having three corner points. Model parameters used are given in Table 2.2.

.

2.6.3 Discussion

The structure of SMS strategies differs from the structure of solutions obtained by the authors in [64] with a comparable methodological approach but different model, criterion and constraint: their optimal irrigation policy consists of bringing the soil moisture from its initial level to an optimal target value as fast as possible and maintaining it until harvesting time when irrigation ceases. In a different context (with rainfall inputs and no water quota), the authors of [53] study on a simple irrigation model the optimal control minimizing the total quantity of water to ensure the soil humidity to remain above a given threshold S_{min}. They found an optimal control policy whose structure is an SOS strategy with $\bar{S} = S_{min}$ but with a triggering level equal to $\bar{S}$. We believe that these differences between these models, their criterion, and their optimal solutions are of interest and make

our mathematical analysis worth of interest.
Note finally that the practical implementation of an SMS (OR SOS) strategy on a real irrigation system requires an adaptive controller to maintain the humidity level constant at the values of the singular arcs (differently to the OS strategy), which needs the on-line measurement of the variable S. Moreover, differently to the OS strategy, it gradually changes the input flow rate during the singular arc phase. Therefore, the SMS (or SOS) strategy can be considered more "sophisticated" than the OS one, as it requires a humidity sensor for its concrete application.
Finally, this work allows to provide the best *crop-water production functions* (mapping water quotas to maximal biomass productions), often mentioned in the literature as a relevant tool for irrigation planning (see e.g. [22]). Other criterion could have been considered balancing between biomass production and water consumption, but that would require to choose weights in the mixed penalization or precise water pricing. Crop-water production functions provide more intrinsic information about the crop irrigation, the cost-benefit analysis being left to the decision-makers.

2.7 Conclusion

We have introduced a simple crop irrigation model in order to study optimal irrigation scheduling using mathematical analysis. We have first shown, using a comparison tool, that the state constraint of this model is never activated for the optimal control problem solutions.

Moreover, we have demonstrated that, under water scarcity, an optimal trajectory has to reach as fast as possible the domain for which the relative humidity is below or equal to the threshold of maximal crop transpiration, and then to maintain it in this domain until the harvesting time. However, due to water scarcity, it has to be strictly below this threshold at some stage. We have then proved that the optimal strategy consists of irrigating once but with possible multiple steps ("SMS strategy") and not necessarily with a single shot ("OS strategy") as commonly used in practice. Moreover, we have shown that when the water-stress functions do not present any corner in between their extreme values, the SMS strategy has at most one step which is necessary at the maximal crop transpiration threshold (that we called "SOS strategy"). The SOS strategy is simpler to apply than the SMS one (with more than one step) as the soil moisture has simply to be maintained at the transpiration threshold until the water quota is reached. A remarkable feature is that the structure of the optimal strategy does not depend on the radiation interception efficiency function (although optimal trigger threshold and step values do rely on this function). As the considered stress functions are piecewise linear, we have used the non-smooth Maximum Principle to obtain relatively simple strategies, relying on the single determination of singular arcs. Paradoxically, the consideration of smooth stress functions would have led to optimal strategies without a particular structure easy to characterize and to implement as the SMS (and SOS) ones.

We have then compared the three control strategies: the open-loop one-shot (OS), commonly used in practice, the feedback saturated one-shot (SOS), that could exhibit a singular arc, and the more sophisticated feedback strategy with multiple increasing shots (SMS), that could exhibit several singular arcs. We have shown numerically the superiority of this last strategy. We guess that the SMS strategy (that coincides with the SOS one for simple water-stress functions) could also be the best one in real situations. This would be a promising result since SMS/SOS irrigation schemes are not such intuitive control strategies, that could also be tested on simulations of more detailed models. This shall be the matter of future work.

MINIMAL TIME OF CRISIS CRITERION FOR THE IRRIGATION PROBLEM

This chapter corresponds to an on-going work

Contents

3.1 Introduction.

Optimal control has been used a good number of times in the literature to deal with irrigation scheduling (see for instance [21, 52, 53, 74, 64, 80]). Usually, the control is represented as the amount of water used in irrigation, and at least a trajectory is designated as the humidity of the soil. However, different criterion and constraints have been considered to deal with the limited availability of irrigation water. In [52, 53] for instance, the authors propose to minimize the total water used in irrigation under a constraint of soil moisture. In [21], the criterion considered is the final biomass production under a condition on the total water consumed. In the same spirit of the mentioned work, we propose an optimal control approach using a new criterion to deal with water constraints.

Each plant has a hydrological threshold need for optimal growth. In [21], it is shown that the final biomass produced is maximal if the soil moisture remains above the optimal stress threshold (noted S^*) when the amount of available water for irrigation allows it. Another way to approach this problem is to consider a constraint on the humidity by forcing it to stay above S^* all the time (i.e., $S(t) \in [S^*, 1]$ for all $t \in [0, T]$). However, if the water available for irrigation is not large enough this state constraint will not be satisfied all the time. Indeed, the trajectories are obliged to pass below S^* for a certain time with the null control once the used water reaches the imposed water quota. It is therefore interesting to look for a control law that would allow to minimize the time spent in this crisis domain: this is the problem of *minimal time of crisis*. This type of problem has been studied in the literature in a general and theoretical way (see. [7, 8, 9, 10, 11, 20]). In the present paper, we consider the minimal time of crisis problem applied to the crop model introduced in [21] in aim to describe an optimal irrigation scheduling in the context of water scarcity. Intuitively, one may believe that minimizing the time when the crops are stressed could conduct to maximizing the total biomass, as we have studied in the previous chapter. The objective of the present work is to see if these two strategies are equivalent or not.

The particular characteristics of the optimal control problem proposed here is twofold: On the one hand, the discontinuity of the integrand and the non-differentiability of the dynamics with respect to the state on the other hand. Combining the two difficulties prevents applying the classical results in optimal control, such as the regular Pontryagin maximum principle that gives necessary optimal conditions and makes the analytical approach difficult. In this work, we have begun to determine the optimal solution of the problem with numerical simulations using a dynamic programming approach provided by the program BocopHJB [17]. We then compare the optimal solutions of the minimal time of crisis problem and the ones of the maximized final biomass problem that has been proved in the previous work.

3.2 Problem statement.

The problem here consists in optimizing the irrigation scheduling by means of the optimal control. We consider the following simplified dynamical model of crop irrigation introduced in [21]:

$$\dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)), \quad S(0) = 1 \tag{3.2.1}$$

$$\dot{B} = k_3\varphi(t)K_S(S), \quad B(0) = 0 \tag{3.2.2}$$

$$\dot{V} = u(t), \quad V(0) = 0. \tag{3.2.3}$$

Here, $S(t)$ is relative soil humidity in the root zone and $B(t)$ is the crop biomass at time t in an interval $[0, T]$ representing the crop growth season, where 0 and T stand for sowing and harvesting dates. The control variable $u(t) = F(t)/F_{max} \in [0, 1]$ is the ratio of the input water flow rate $F(t)$ at time t over the maximal flow F_{max} that the irrigation allows. The functions K_S and K_R are piecewise linear non decreasing from $[0, 1]$ to $[0, 1]$ given by the following expressions:

$$K_S(S) = \begin{cases} 0 & S \in [0, S_w], \\ \dfrac{S - S_w}{S^\star - S_w} & S \in [S_w, S^\star], \\ 1 & S \in [S^\star, 1], \end{cases} \qquad K_R(S) = \begin{cases} 0 & S \in [0, S_h], \\ \dfrac{S - S_h}{1 - S_h} & S \in [S_h, 1], \end{cases} \tag{3.2.4}$$

Where $0 < S_h < S_w < S^\star < 1$. The S^* is the minimal threshold on the soil

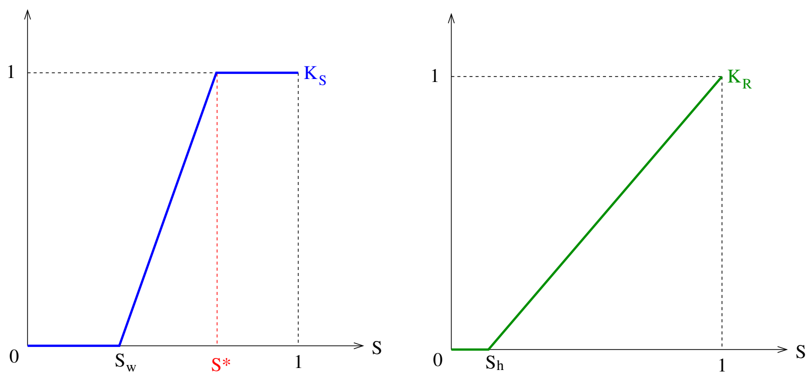


FIGURE 3.1 – Graphs of the functions K_S and K_R given by expressions (2.2.1)

humidity that gives the best biomass production. S_w represents the plant wilting point and S_h is the hydroscopic point. The function φ is C^1 increasing with $\varphi(0) \geq 0$ and $\varphi(T) \leq 1$. k_1, k_2 and k_3 are positive parameters with $k_2 \geq 1$.

The controlled system (3.2.1)-(3.2.2)-(3.2.3) was considered in the previous chapter [21] for an optimal control study. The optimal control problem consists in maximizing the final biomass $B(T)$ under a constraint on the total water used during irrigation $V(T) \leq \bar{V}$. Here, we consider a different criterion to optimize.

Let K be the subset defined by $K = \{S \in [0,1];\ S \geq S_{crisis}\}$ where $S_{crisis} \in [S_w, S^*]$ is a hydrological threshold of crops. We consider the optimal control problem applied to the controlled system (3.2.1)-(3.2.2)-(3.2.3):

$$\inf_{u \in \mathcal{U}} \int_0^T \mathbb{1}_{K^c}(S(t))\,\mathrm{d}t, \tag{3.2.5}$$

with the target

$$V(T) < \bar{V}, \tag{3.2.6}$$

where $\mathbb{1}_{K^c}(\cdot)$ is the characteristic function of the set K and $\bar{V} > 0$ is a given parameter that represents the total water available for irrigation. This optimal control problem is a Lagrange type of problem with a target and a discontinuous integrand and non-smooth dynamics with respect to the state. It is a highly difficult problem to deal with analytically. Consequently, a numerical approach appears to be more tractable to tackle this kind of problem. However, let us give a first analysis of the problem. First, we introduce the following definitions:

Definition 3.2.1. *Let us denote $\underline{S}(\cdot)$ the trajectory associated to the null control and let us define*

$$\underline{t} := \inf\{t \in [0,T] \ s.t.\ \underline{S}(t) < S^\star\}.$$

Let us also define the singular controls that allows to maintain $S(\cdot)$ constant at a given threshold. For any $\tilde{S} \in (0,1)$, define the control

$$\tilde{u}_{\tilde{S}}(t) := \frac{\varphi(t)K_S(\tilde{S}) + (1-\varphi(t))K_R(\tilde{S})}{k_2}, \quad t \in [0,T] \tag{3.2.7}$$

and for $\tilde{S} = S_{crisis}$, define

$$V_c := \int_{\underline{t}}^T \tilde{u}_{S_{crisis}}(t)\,dt.$$

One can quickly check that the following lemma holds.

Lemma 3.2.1.

(i) If $\underline{t} = T$, then any admissible control $u(\cdot)$ gives $\int_0^T \mathbb{1}_{K^c}(S(t))\,\mathrm{d}t = 0$. (The optimal control problem is straightforward)

(ii) If $\underline{t} < T$, then For any $\bar{V} \geq V_c$, the control

$$u(t) = \begin{cases} 0 & t \in [0, \underline{t}), \\ \tilde{u}_{S_{crisis}}(t) & t \in [\underline{t}, T], \end{cases} \tag{3.2.8}$$

gives $\int_0^T \mathbb{1}_{K^c}(S(t))\,\mathrm{d}t = 0$. *(The optimal control problem is straightforward)*

(iii) For any $\bar{V} < V_c$ *and admissible control* $u(\cdot)$ *satisfying the final constraint* (3.2.6), *one has* $\int_0^T \mathbb{1}_{K^c}(S(t))\,\mathrm{d}t > 0$. *(The optimal control problem is not straightforward)*

When $\underline{t} = T$ or $\bar{V} \geq V_c$, the optimal control problem (3.2.5) is trivial. We consider in the following that the next condition (which corresponds to water scarcity situations) holds.

Hypothesis 2. $\underline{t} < T$ *and* $\bar{V} < V_c$.

3.3 Numerical simulation and discussion.

3.3.1 Optimal solution structure.

Here, we use BocopHJB to treat the optimal control problem (3.2.5). The main feature of this solver is that it implements a global optimization method, moreover the optimal solution is using dynamic programming approach. As the problem is low dimensional and the discontinuous integrand prevents the direct methods to converge, this solver is more adapted to deal with the problem (3.2.5). We present in Fig. 3.2 the simulations result given by BocopHJB with inputs data provided in Table 3.1.

T	k_1	k_2	$S^\star$	S_w	S_h	F_{max}	$\bar{V}$	α	S_{crisis}
1	2.1	5	0.7	0.4	0.2	1.2	0.1	4	0.7

TABLE 3.1 – Normalized parameters used for the simulations of minimal time of crisis

The numerical tests with different value of the crisis threshold S_{crisis} show that an optimal trajectory consists of reaching as fast as possible the crisis threshold and maintaining S at this threshold as long the as water budget is not null (see figs. 3.2). This optimal structure with singular arcs have also been found in the literature using different criterion to optimize for irrigation scheduling (see for instance [20, 64]).

Note that the optimal trajectory S hits the set boundary K tangentially and then leaves in a non-transverse way; it's a one-sided crossing from K to K^c. The consideration of this type of transverse solution will be studied in a general way in Chapter 5 of this thesis dissertation.

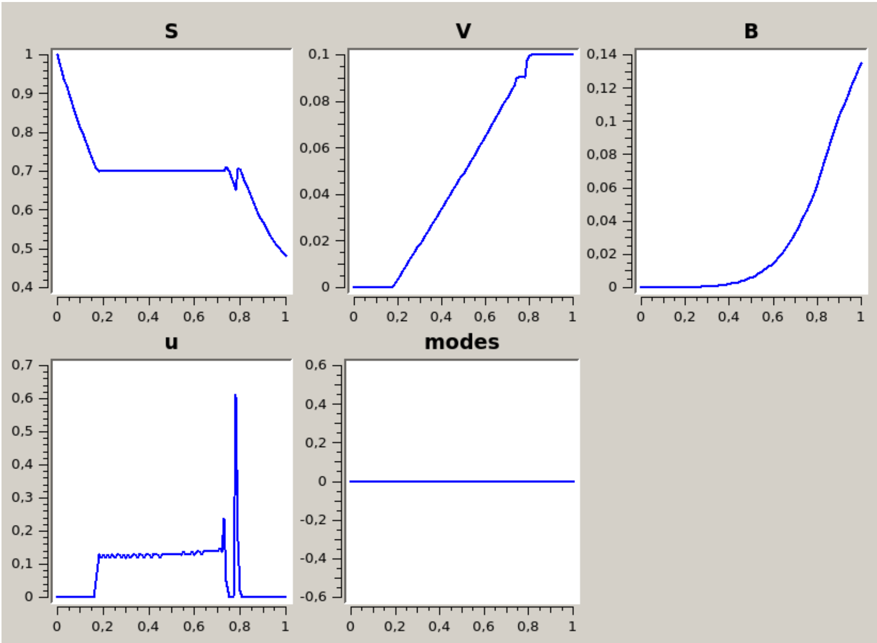


FIGURE 3.2 – $S_{crisis} = S^* = 0.7,\ \bar{V} = 0.1$

3.3.2 Comparison between different criterion.

In this subsection we make a comparison between the optimal solutions of the and minimal time of crisis problem (3.2.5) and the ones of the optimal control problem introduced in [21] which consists of maximizing the final biomass i.e.,

$$\max_u B(T).$$

Recall that an optimal solution to this problem corresponds to the so-called *SOS strategy*. A triggering threshold parametrizes this strategy that consists of joining as fast as possible the domain under the optimal level S^* with the control $u = 0$ until it reaches the triggering level, then enters the singular arc S^* with the control $u = 1$ and stay with the singular control until harvesting time.
Observe that the optimal solution of (3.2.5) given by BocopHJB can also be seen as an SOS strategy when $S_{crisis} \leq S^*$. The triggering threshold, in this case, corresponds to the crisis threshold S_{crisis} (see fig. 3.3). Nonetheless, this solution doesn't necessarily give the optimal biomass production as seen in Table 3.2 since the trig-

gering level is not the best one for maximizing the biomass.

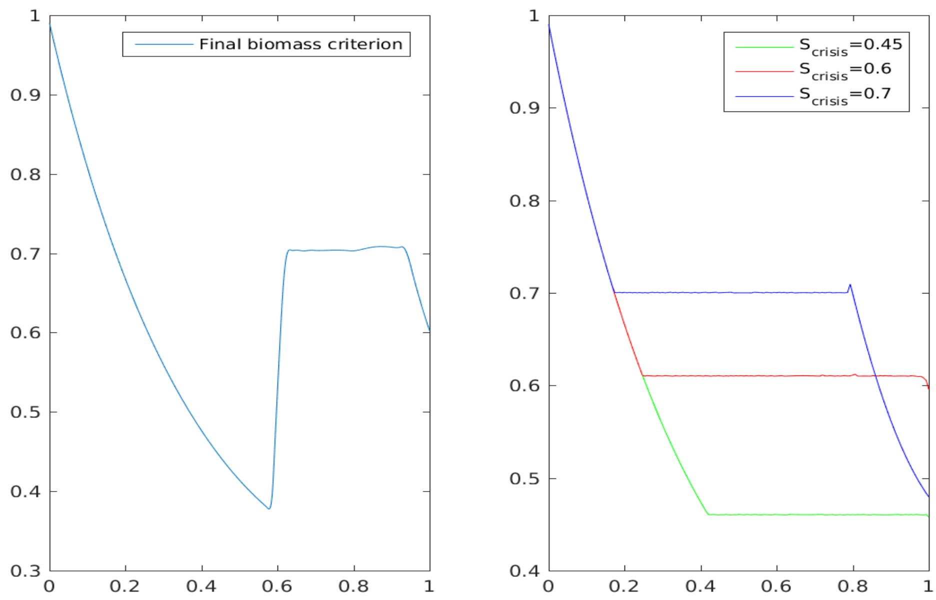


FIGURE 3.3 – Optimal trajectories in the case of maximizing biomass criterion (left) and minimizing the time of crisis (right) with different crisis threshold and same water target $\bar{V} = 0.1$.

Biomass criterion	$S_{crisis} = 0.7$	$S_{crisis} = 0.55$
0.16	0.13	0.09

TABLE 3.2 – Comparison between final biomass production using different criterion. The message is that minimizing the time crisis does not introduce a triggering time to be optimized, differently from maximizing the produced biomass. This latter problem is conducting a more sophisticated control strategy.

3.4 Conclusion.

In this work, we applied the minimal time of crisis problem to a crop model in order to find an optimal strategy that deals with the constraint of the total water available for irrigation. Due to the lack of regularity of the problem, our approach was purely numerical using the software BocopHJB. The numerical simulations show that an optimal solution consists of reaching as fast as possible the crisis threshold and staying on it until the water available is wholly consumed. This strategy is a particular case of the SOS strategy introduced in [20] that consists of maximizing the biomass while taking into account the constraint on the water.

Even though the optimal strategy of the minimal time of crisis doesn't necessarily give a maximized biomass, it remains a reliable irrigation scheduling, easy to be implemented (i.e. without requiring the knowledge of model parameters) that let the crop stay little time under the water stress when a constraint on the total amount of water has to be respected.

Troisième partie

Synthèse sur l’optimisation du temps de crise

PENALTY FUNCTION METHOD FOR THE MINIMAL TIME CRISIS PROBLEM

This chapter corresponds to the published paper:

K. BOUMAZA, T. BAYEN AND A. RAPAPORT. *Penalty function method for the minimal time crisis problem.* ESAIM: Proceedings and Surveys, EDP Sciences, Vol. 71, p. 21-32, August 2021. [20]

Contents

4.1 Introduction

The *minimal time of crisis problem* was introduced in [34] in the context of viability theory (see [3]). It consists in minimizing the time spent by a solution of a controlled dynamics outside a given closed set K (representing typically some constraints). It has been mainly studied in the context of ordinary differential equations (ODEs). Notice however that a similar approach has been proposed for linear parabolic partial differential equations (see [18, 16]) in connection with practical applications.

In the context of ODEs, the objective function of the time of crisis can be expressed via the indicator function of the complementary of the set K, which is discontinuous with respect to the state variable, therefore the Pontryagin Maximum Principle (PMP) cannot be directly applied to compute an optimal control. On the other hand, the application of the Hybrid Maximum Principle (HMP) [28] can be used to express necessary optimality conditions (see e.g. [8, 9, 10, 11, 41]), but under a certain "*transverse crossing condition*" of the set K (see also [41]). This condition requires that optimal trajectories enter and leave the set K non tangentially. In this work, our main aim is to present a different approach to the time crisis problem, approximating its solutions. This will allow us to bypass the discontinuity of the indicator function as well as the use of the transverse crossing condition.

Our methodology relies on the introduction of an additional control function and on the definition of an auxiliary optimal control problem with mixed state-control constraint, whose solutions exactly coincide with the ones of the time crisis problem. This new problem is then approximated thanks to a penalty function which is our main purpose in this paper.

The paper is structured as follows. In Section 4.2, we introduce an auxiliary optimal control problem where the discontinuity of the integrand with respect to the state is formulated using a new control with values in $\{\pm 1\}$ and a mixed state-control constraint. Next, we show that it is equivalent to the time crisis problem. Because of this new constraint, we then introduce a penalty method and study properties of the value function associated with this approximated problem as well as properties of the regularization arising from the penalty function. In Section 4.3, we prove convergence properties, namely that the sequence of value functions converges to the value function associated with the time crisis problem. As well, convergence of optimal solutions of the regularized problem to an optimal solution of the original problem is also provided (although the velocity set is non-convex). Finally, these convergence results are illustrated in Section 4.4 on two academic examples for which optimal trajectories can enter and leave the crisis set K tangentially.

4.2 A new formulation

4.2.1 Statement of the problem

Given a set $K \subset \mathbb{R}^n$, a positive number T, a set $U \subset \mathbb{R}^m$ and a map $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ that fulfill the following assumptions:

(H1) The set U is a non-empty compact subset of $\mathbb{R}^m$,

(H2) The map $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous w.r.t (x, u), locally Lipschitz w.r.t x and satisfies the linear growth condition there exist $c_1 > 0$ and $c_2 > 0$ such that for all $x \in \mathbb{R}$ and all $u \in U$, one has

$$|f(x, u)| \leq c_1|x| + c_2, \tag{4.2.1}$$

we consider for any $\tau \in [0, T]$ and $y \in \mathbb{R}^n$, the following optimal control problem ("minimal time crisis")

$$\inf_{u \in \mathcal{U}} \int_\tau^T \mathbb{1}_{K^c}(x_{u,\tau,y}(t))\, \mathrm{d}t, \tag{TC}$$

where $x_{u,\tau,y}(\cdot) : [\tau, T] \to \mathbb{R}^n$ (simply denoted by $x(\cdot)$ hereafter) is the unique solution to the Cauchy problem

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [\tau, T], \quad x(\tau) = y, \tag{4.2.2}$$

associated with a control $u \in \mathcal{U}$, the set of all measurable controls $u : [0, T] \to U$. We also assume the following hypotheses to be satisfied:

(H3) For any $x \in \mathbb{R}^n$, the set $F(x) := \{f(x, u) \ ; \ u \in U\}$ is a non-empty convex subset of $\mathbb{R}^n$.

(H4) The set K is a non-empty closed subset of $\mathbb{R}^n$ described by

$$K := \{x \in \mathbb{R}^n \ ; \ \varphi(x) \leq 0\},$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous function that takes value in $\mathbb{R}$.

Note that the existence of an optimal solution to this problem is standard (see, e.g., [9, Proposition 2.1]). Since the integrand defining (TC) is discontinuous, we cannot apply the Pontryagin Maximum Principle (PMP) that requires data to be Lipschitz continuous w.r.t. the variable x. There exist many ways to approximate the indicator function with a sequence of Lipschitz continuous functions (see, e.g., [66]) in such a way to obtain a sequence of optimal trajectories converging to an optimal solution of the original problem. With Lipschitz data, one can use different numerical techniques, such as direct methods, or the Hamilton-Jacobi-Bellman (HJB) equation (to obtain a sequence of Lipschitz continuous value functions, from which one can construct a sequence of optimal trajectories). If one approximates the indicator function with more regular functions, say C^1, then the (classical) PMP can be used to characterize a sequence of optimal trajectories. Here, we discuss another way to represent the discontinuity of the indicator function with the use of an additional control, taking advantage that in classical optimal control theory, control functions are naturally sought among measurable functions (thus discontinuous).

4.2.2 Formulation with mixed constraint

Let $\mathcal{V}$ be the set of all measurable controls $v : [0, T] \to \Omega$ where $\Omega := \{\pm 1\}$, and consider the mixed state-control constraint

$$v(t)\varphi(x(t)) - |\varphi(x(t))| = 0 \quad \text{a.e. } t \in [\tau, T], \tag{4.2.3}$$

where $x(\cdot)$ is any admissible solution, and $v(\cdot)$ any control function in $\mathcal{V}$. We define a new optimal control problem with mixed state-control constraint

$$\inf_{(u,v)\in\mathcal{U}\times\mathcal{V}} \int_\tau^T \left(\frac{1+v(t)}{2}\right) \mathrm{d}t \quad \text{subject to the constraint (4.2.3),} \tag{TCR}$$

for the controlled dynamics (4.2.2). Note that, given any admissible solution $(x(\cdot), u(\cdot), v(\cdot))$ satisfying the constraint (4.2.3), one has

$$v(t) = \text{sign}(\varphi(x(t))),$$

provided that $\varphi(x(t)) \neq 0$, and $v(t) \in \Omega$ if $\varphi(x(t)) = 0$. It is known that the lack of regularity of the integrand w.r.t. the state is bothersome, whereas it is common for an optimal control to be discontinuous w.r.t. t. In this new formulation, the lack of regularity of the integrand defining (TC) has been replaced by the addition of the new control variable v that only takes two values $+1$ and -1 together with the mixed state-control constraint (4.2.3). We now prove that problems (TC) and (TCR) are equivalent.

Lemma 4.2.1. *Let $u^\star \in \mathcal{U}$. Then $(x^\star, u^\star)$ is an optimal solution of* (TC) *if and only if $(x^\star, v^\star, u^\star)$ is an optimal triple for Problem* (TCR)*, where $v^\star \in \mathcal{V}$ is defined for any $t \in [\tau, T]$ as*

$$v^\star(t) := \begin{cases} \text{sign}(\varphi(x^\star(t))) & \text{if} \varphi(x^\star(t)) \neq 0, \\ -1 & \text{otherwise.} \end{cases} \tag{4.2.4}$$

Proof. *Notice first that for any admissible solution $x^\star$, the control $v^\star$ defined by* (4.2.4) *satisfies the constraint* (4.2.3) *and thus one has*

$$\int_\tau^T \left(\frac{1 + v^\star(t)}{2}\right) \mathrm{d}t = \int_\tau^T \mathbb{1}_{K^c}(x^\star(t))\, \mathrm{d}t\ .$$

Therefore, if $(x^\star, u^\star, v^\star)$ is an optimal triple for (TCR)*, then the pair $(x^\star, u^\star)$ is also optimal for* (TC)*. Conversely, let $(x^\star, u^\star)$ be an optimal solution of* (TC)*, and suppose by contradiction that there exists an optimal pair $(\bar{u}, \bar{v}) \in \mathcal{U} \times \mathcal{V}$ for Problem* (TCR) *such that*

$$\int_\tau^T \left(\frac{1 + \bar{v}(t)}{2}\right) \mathrm{d}t < \int_\tau^T \mathbb{1}_{K^c}(x^\star(t))\, \mathrm{d}t\ .$$

Let $\bar{x}$ be its associated trajectory. Since $(\bar{x}, \bar{v})$ satisfies the constraint (4.2.3)*, it follows that one has*

$$\bar{v}(t) = \text{sign}(\varphi(\bar{x}(t)))\ \textit{a.e.}\ t \in [\tau, T] \quad \textit{s.t.}\ \varphi(\bar{x}(t)) \neq 0\ .$$

Hence, we deduce that

$$\int_\tau^T \mathbb{1}_{K^c}(\bar{x}(t))\mathrm{d}t \leq \int_\tau^T \left(\frac{1 + \bar{v}(t)}{2}\right) \mathrm{d}t,$$

which contradicts the optimality of $x^\star$ and concludes the proof.

4.2.3 Approximation with a penalty function

The penalty method is a common technique in optimization to approximate constrained optimization problems with a sequence of unconstrained optimization problems [29, 55, 60, 78]. In this section, we apply a penalty approach to (TCR). Let us start by introducing the following penalty function $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$ associated with the constraint (4.2.3) as follows:

$$P(x, v) := \Big(v\varphi(x) - |\varphi(x)|\Big)^2,$$

and the following auxiliary optimal control problem defined by

$$\inf_{(u,v)\in\mathcal{U}\times\mathcal{V}^\#} \int_\tau^T \left(\frac{1+v(t)}{2} + nP(x(t), v(t))\right) \mathrm{d}t, \qquad (\mathrm{TCR}_n^\#)$$

where $\mathcal{V}^\#$ is the set of all admissible controls $v : [0, T] \to \mathrm{co}(\Omega)$, where $\mathrm{co}(\Omega) := [-1, 1]$, and $x(\cdot)$ is a solution of (4.2.2). Since the mapping

$$v \mapsto \frac{1+v}{2} + nP(x, v)$$

is convex for any $x \in \mathbb{R}^n$, the extended velocity set

$$\left\{ \begin{bmatrix} f(x, u) \\ \frac{1+v}{2} + nP(x, v) + w \end{bmatrix}, \ (u, v, w) \in U \times [-1, 1] \times \mathbb{R}_+ \right\}$$

is convex for any $x \in \mathbb{R}^n$. The existence of an optimal solution for Problem $(\mathrm{TCR}_n^\#)$ follows by a direct application of Filippov's Theorem [28], under Assumptions (H1)-(H2)-(H3)-(H4). Next, $V_n : [0, T) \times \mathbb{R}^n \to \mathbb{R}$ will stand for the value function associated with $(\mathrm{TCR}_n^\#)$. For each fixed $n \in \mathbb{N}$, Problem $(\mathrm{TCR}_n^\#)$ is a classical optimal control problem of Bolza type with Lipschitz bounded data, for which its value function V_n is then locally Lipschitz continuous over $[0, T) \times \mathbb{R}$ (see, e.g., [5]). In addition, V_n is the unique viscosity solution to the following HJB equation

$$\partial_t V(t, y) + \sup_{(u,v)\in U\times\mathrm{co}(\Omega)} H_n(x, \nabla_y V(t, y), u, v) = 0, \quad (t, y) \in [0, T] \times \mathbb{R}^n, \qquad (4.2.5)$$

with the boundary condition

$$V(T, y) = 0, \qquad y \in \mathbb{R}^n, \qquad (4.2.6)$$

where the Hamiltonian $H_n : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is defined as

$$H_n(x,p,u,v) := p \cdot f(x,u) - \Big(\frac{1+v}{2} + nP(x,v)\Big).$$

By maximizing the Hamiltonian w.r.t. v, the expression of a maximizer v_n is given by

$$v_n(x) = \max\left(-1, \mathrm{sign}(\varphi(x)) - \frac{1}{4n\varphi(x)^2}\right), \quad x \in \mathbb{R}^n,\ n \in \mathbb{N}^\star.$$

One can also check, thanks to the above expression of v_n, that the following inequality holds

$$\frac{1+v_n(x)}{2} + nP(x, v_n(x)) \leq 1, \quad x \in \mathbb{R}^n,\ n \in \mathbb{N}^\star.$$

Without any loss of generality, we can then write V_n as

$$V_n(\tau, y) = \inf_{(u,v)\in\mathcal{U}\times\mathcal{V}^\#} \int_\tau^T \min\left(1, \frac{1+v(t)}{2} + nP(x(t), v(t))\right) \mathrm{d}t.$$

Let us now introduce a slight variation of the previous optimal control problem in which controls $v(\cdot)$ are with values in Ω, and not in $\mathrm{co}(\Omega)$:

$$\inf_{(u,v)\in\mathcal{U}\times\mathcal{V}} \int_\tau^T \min\left(1, \frac{1+v(t)}{2} + nP(x(t), v(t))\right) \mathrm{d}t, \tag{TCR$_n$}$$

and for which, we denote by $\overline{V}_n$ the associated value function. By using a similar argumentation as above, one can show that $\overline{V}_n$ is the unique viscosity solution of the following HJB equation

$$\partial_t V(t,y) + \sup_{(u,v)\in U\times\Omega} \overline{H}_n(x, \nabla_y V(t,y), u, v) = 0, \quad (t,y) \in [0,T] \times \mathbb{R}^n, \tag{4.2.7}$$

with the boundary condition (4.2.6) and the Hamiltonian $\overline{H}_n$ defined as

$$\overline{H}_n(x,p,u,v) := p \cdot f(x,u) - \min\left(1, \frac{1+v}{2} + nP(x,v)\right).$$

Note that the extended velocity set associated with this optimal control problem is not convex, due the fact that we consider controls v taking values -1 or 1 only. Therefore, one cannot apply Filippov's Theorem to ensure the existence of an optimal solution. To investigate the behavior of (V_n), let $V : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ be the value function associated with the auxiliary Problem (TCR) defined as

$$V(\tau,y)=\inf_{(u,v)\in\mathcal{U}\times\mathcal{V}}\int_\tau^T\left(\frac{1+v(t)}{2}\right)\mathrm{d}t\quad\text{under constraint (4.2.3)}.$$

Proposition 4.2.1. *For each $n\in\mathbb{N}$ and for each $(\tau,y)\in[0,T)\times\mathbb{R}^n$, one has the following inequality*

$$V_n(\tau,y)\le\overline{V}_n(\tau,y)\le V(\tau,y). \tag{4.2.8}$$

Moreover, the problem (TCR_n) *admits an optimal solution for any $n\in\mathbb{N}$.* ***Proof.*** *Let $(x^\star,u^\star)$ be an optimal pair for Problem* (TC)*. According to Lemma 4.2.1, the triple $(x^\star,u^\star,v^\star)$, where $v^\star$ is defined by* (4.2.4)*, is optimal for Problem* (TCR)*. It follows that one has $P(x^\star(t),v^\star(t))=0$ for any $t\in[\tau,T]$, which gives $\overline{V}_n(\tau,y)\le V(\tau,y)$ for any n. Clearly, since Problem* (TCR_n) *is sought for the same criterion than Problem* $(\mathrm{TCR}_n^\#)$*, but for a smaller set of control functions, we get the inequality $V_n\le\overline{V}_n$ for any $n\in\mathbb{N}$.*

Consider now Problem (TCR_n)*. By maximizing the Hamiltonian $\overline{H}_n$ w.r.t. v (with values ±1 only), one obtains the expression of a maximizer $\overline{v}_n$, for each $n\in\mathbb{N}$, as follows*

$$\overline{v}_n(x)=\begin{cases}1 & \text{if } \varphi(x)>\dfrac{1}{2\sqrt{n}},\\ -1 & \text{otherwise.}\end{cases}$$

By replacing v into (TCR_n) *with the expression of $\overline{v}_n(x)$, one obtains the optimal control problem*

$$\inf_{u\in\mathcal{U}}\int_\tau^T Q_n(\varphi(x(t)))\,\mathrm{d}t, \tag{TCR'_n}$$

with

$$Q_n(z):=\begin{cases}0 & \text{if } z\le 0,\\ 4nz^2 & \text{if } 0<z\le\dfrac{1}{2\sqrt{n}},\\ 1 & \text{if } \dfrac{1}{2\sqrt{n}}<z,\end{cases} \tag{4.2.9}$$

which is classical Bolza problem with Lipschitz data, for which the value function is the unique viscosity solution of the HJB equation (4.2.7) *with boundary condition* (4.2.6)*. By uniqueness of solutions of* (4.2.7)-(4.2.6) *in the class of Lipschitz functions, we deduce that its value function coincides with $\overline{V}_n$ for any $n\in\mathbb{N}$. Moreover, Problem* (TCR'_n) *admits an optimal pair (x_n,u_n), thanks to Filippov's Theorem (under Assumptions (H1) to (H4)). Then, the triple (x_n,u_n,v_n), where $v_n(t):=\bar{v}_n(x_n(t))$ for any $t\in[\tau,T]$, is optimal for Problem* (TCR_n)*.*

4.2.4 Discussion in terms of regularization of the indicator function

We give next some properties of this approach in terms of regularization of the indicator of K^c defining the time crisis function. Indeed, Problems $(\mathrm{TCR}_n^\#)$ and (TCR_n) amount to consider two different regularizations of the indicator function, considering that an optimal control function $v(\cdot)$ has to maximize the corresponding Hamiltonian at almost any t, and that its maximizing expression can be merely replaced in the integrand. One can straightforwardly show the following results.

Proposition 4.2.2. *Define the function*

$$Q_n^\#(z) := \begin{cases} 0 & \text{if } z \le 0, \\ 4nz^2 & \text{if } 0 < z \le \dfrac{1}{2\sqrt{2n}}, \\ 1 - \dfrac{1}{16nz^2} & \text{if } \dfrac{1}{2\sqrt{2n}} < z. \end{cases}$$

Then, one has for Problem $(\mathrm{TCR}_n^\#)$

$$\max_{v\in[-1,1]} -\left(\frac{1+v}{2} + nP(x,v)\right) = -Q_n^\#(\varphi(x)),$$

and for Problem (TCR_n)*, one has*

$$\max_{v\in\{-1,1\}} -\left(\min\left(1, \frac{1+v}{2} + nP(x,v)\right)\right) = -Q_n(\varphi(x))$$

where $Q_n(\cdot)$ is defined in (4.2.9).

At this step, one can observe the following properties:

1. One has $Q_n^\#(\varphi(x)) \le Q_n(\varphi(x)) \le \mathbb{1}_{K^c}(x)$ for any x and $Q_n(\varphi(x)) = \mathbb{1}_{K^c}(x)$ when $\varphi(x) \notin [0, \frac{1}{2\sqrt{n}}]$, while $Q_n^\#(\varphi(x)) < \mathbb{1}_{K^c}(x)$ for any $x \in K^c$ (see Fig. 4.1). Therefore, the criterion of Problem (TCR_n) gives a much better estimate of the time crisis than $(\mathrm{TCR}_n^\#)$ for any admissible trajectory.

2. $Q_n^\#$ is differentiable. Provided that φ is differentiable, one can then use the (regular) PMP to characterize optimal pairs (x_n, u_n) for Problem $(\mathrm{TCR}_n^\#)$

considering the single control u. On the opposite, Q_n is not differentiable at $z = \frac{1}{2\sqrt{n}}$. However, it is possible to use the PMP provided that the Problem (TCR_n) is considered as a problem with two controls u and v, since the function P is differentiable w.r.t. x.

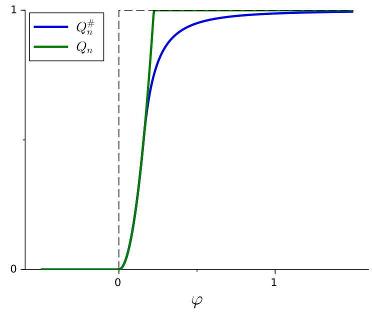


FIGURE 4.1 – Example of graphs of the functions $Q_n^\#$ and Q_n (for $n = 5$).

These facts justify the interest of considering Problem (TCR_n) (with two controls) instead of $(\mathrm{TCR}_n^\#)$, as an approximation procedure of Problem (TC), strengthened by the fact that this problem admits optimal solutions despite the lack of convexity of its augmented velocity set. However, the consideration of Problem $(\mathrm{TCR}_n^\#)$ has been useful to justify the choice of the bounded penalization $\min\left(1, \frac{1+v}{2} + nP(x,v)\right)$ instead of the unbounded one given by $\frac{1+v}{2} + nP(x,v)$ in the criterion. From a purely numerical viewpoint, having to consider only two possible values for the control v can be an advantage if one considers numerical schemes based on dynamic programming, as we did in examples in Section 4.4.

4.3 Convergence results

We now provide convergence results of solutions to the penalized optimal control problems (namely (TCR_n) and $(\mathrm{TCR}_n^\#)$) to an optimal solution of (TC).

Proposition 4.3.1. *The functions V_n, resp. $\overline{V}_n$, converge pointwise to the function V in $[0,T]\times\mathbb{R}^n$. Moreover, any optimal sequence x_n, resp. $\overline{x}_n$, for Problem* $(\mathrm{TCR}_n^\#)$, *resp.* (TCR_n), *converges, up to a sub-sequence, uniformly to an optimal solution $x^\star$ of Problem* (TC), *and their derivatives weakly to $\dot{x}^\star$ in $L^2(\tau,T;\mathbb{R}^n)$.*

Proof.

Since for each $(\tau, y) \in [0, T) \times \mathbb{R}^n$, the sequence $(V_n(\tau, y))_n$ is non-decreasing, bounded above, and Lipschitz continuous, it converges pointwise to some function $V_\infty(\tau, y) \leq V(\tau, y)$. It can be also observed that Problem ($\text{TCR}_n^\#$) can be equivalently rewritten as a Mayer problem in $\mathbb{R}^{n+2}$:

$$V_n(\tau, y) = \inf_{(u,v)\in\mathcal{U}\times\mathcal{V}^\#} l(T) + np(T),$$

subject to the augmented dynamics

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)), \quad x(\tau) = y \\ \dot{l}(t) &= \dfrac{1 + v(t)}{2}, \quad l(\tau) = 0 \\ \dot{p}(t) &= \Big(v(t)\varphi(x(t)) - |\varphi(x(t))|\Big)^2, \quad p(\tau) = 0 \end{cases} \tag{4.3.1}$$

Under hypotheses (H1) up to (H4), Filippov's Theorem gives the existence of an optimal solution (x_n, l_n, p_n) associated with a pair of controls $(u_n, v_n) \in \mathcal{U} \times \mathcal{V}^\#$, for any $n \in \mathbb{N}$. Then, from the standard compactness properties of trajectories (see, e.g., [28, Theorem 1.11]), there exists a sub-sequence, also denoted (x_n, l_n, p_n), and a pair of controls $(u^\star, v^\star) \in \mathcal{U} \times \mathcal{V}$ such that (x_n, l_n, p_n) uniformly converges to a solution of (4.3.1) denoted by $(x^\star, l^\star, p^\star)$ and associated with the control $(u^\star, v^\star)$. In addition, the sequence $(\dot{x}_n, \dot{l}_n, \dot{p}_n)$ weakly converges to $(\dot{x}^\star, \dot{l}^\star, \dot{p}^\star)$ in $L^2(\tau, T; \mathbb{R}^n)$. Let us now show that the pair $(x^\star, u^\star)$ is optimal.

First, note that one has

$$0 \leq l_n(T) + np_n(T) \leq V(\tau, y), \quad n \in \mathbb{N}$$

where l_n, p_n are non-negative functions. Therefore, $p_n(T)$ has to converge to 0 when n tends to $+\infty$, which implies $p^\star(T) = 0$. Since $p^\star$ is absolutely continuous with $p^\star(\tau) = 0$ and satisfies $\dot{p}^\star \geq 0$ a.e., we deduce that the function $p^\star$ is identically null. Then, one has the equality

$$p^\star(T) - p^\star(\tau) = \int_\tau^T \Big(v^\star(t)\varphi(x^\star(t)) - |\varphi(x^\star(t))|\Big)^2 \mathrm{d}t = 0,$$

from which we deduce

$$v^\star(t)\varphi(x^\star(t)) - |\varphi(x^\star(t))| = 0 \quad \text{a.e. } t \in [\tau, T].$$

Thus, $(x^\star, v^\star)$ satisfies the constraint (4.2.3) and we conclude that one has

$$V_\infty(\tau, y) = l^\star(T) = \int_\tau^T \frac{1+v^\star(t)}{2}\,\mathrm{d}t \geq V(\tau, y).$$

(by definition of V) which proves the equality $V_\infty(\tau, y) = V(\tau, y)$ and that $(x^\star, u^\star)$ is optimal for Problem (TC).

Consider now Problem (TCR_n). From Proposition 4.2.1, we obtain that $\overline{V}_n$ converges pointwise to the function V as well, and the existence of a sequence of optimal trajectories $\overline{x}_n$ for Problem (TCR_n), with associated controls $(\overline{u}_n, \overline{v}_n)$ in $\mathcal{U} \times \mathcal{V} \subset \mathcal{U} \times \mathcal{V}^\#$. Let now $(\bar{l}_n, \overline{p}_n)$ be the solution of (4.3.1) associated with those controls. Using again the compactness properties of trajectories of the solutions of (4.3.1) for controls in $\mathcal{U} \times \mathcal{V}^\#$, we obtain the uniform convergence of $(\overline{x}_n, \bar{l}_n, \overline{p}_n)$, up to a sub-sequence, to a certain [1] $(x^\star, l^\star, p^\star)$ and we conclude as before that $x^\star$ is optimal for Problem (TC).

Remark 4.3.1. *The "strong-weak" compactness of the set of trajectories does not in general provide the convergence of controls. Nevertheless, here, one can see that optimal controls* (v_n) *for Problem* $(\mathrm{TCR}_n^\#)$ *or Problem* (TCR_n) *converge a.e. to* $v^\star$ *defined in* (4.2.4) *(up to a sub-sequence). Indeed, since* $\dot{p}_n$ *weakly converges to zero and* $\dot{p}_n \geq 0$ *a.e., it converges a.e. to zero (up to a sub-sequence). It follows that* $\int_\tau^T (1 + v_n(t))/2\,\mathrm{d}t \to \int_\tau^T (1 + v^\star(t))/2\,\mathrm{d}t = V(\tau, y)$*, and from Lemma 4.2.1,* $v^\star$ *is then given by* (4.2.4).

4.4 Numerical examples

We provide two examples illustrating numerically the convergence of approximated optimal solutions (to the penalized problem) to an optimal solution of the minimal time crisis problem. These two examples have the particularity that the optimal trajectories enter and leave tangentially to the set K, which does not allow the use of the HMP. Moreover, in the second example, the optimal trajectory stays on the boundary of K for a non-null duration. As controlled dynamics, we

1. Note that the limits of $(x_n, l_n, p_n, u_n, v_n)$ and $(\bar{x}_n, \bar{l}_n, \bar{p}_n, \bar{u}_n, \bar{v}_n)$ in the appropriate topology may not be the same.

consider the planar system, as in [9]

$$\begin{cases} \dot{x}_1(t) &= -x_2(t)(2+u(t)), \\ \dot{x}_2(t) &= x_1(t)(2+u(t)), \end{cases} \tag{4.4.1}$$

with initial condition $(x_1(0), x_2(0)) = (0, 1)$ and $u(t)$ taking values in $[-1, 1]$, but here with different sets K. The function φ (defining K) is given by:

Example 1: $\varphi(x_1, x_2) = x_1^2 + 4\min(0, x_2)^2 - 1$

Example 2: $\varphi(x_1, x_2) = x_1^2 + \max(-\frac{1}{2}, x_2, 2x_2)^2 - 1$

In the first example, the initial condition lies inside the set K whereas it is outside of K in the second one. Both examples highlight the possibility of entering or leaving the set K tangentially (see Fig. 4.2). In the present context, we introduce

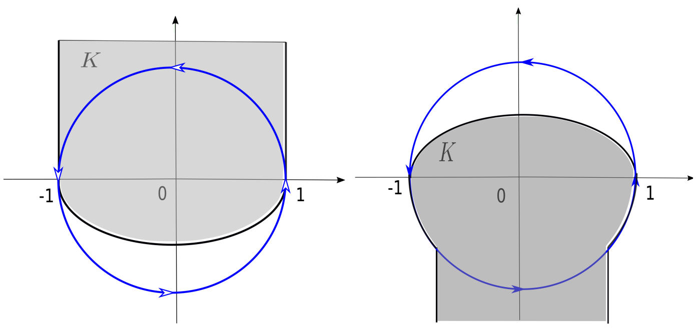


FIGURE 4.2 – Illustration of a trajectory entering and leaving the set K tangentially for the first example (left) and the second (right).

the so-called *myopic strategy* (see [9, 11]) defined as the feedback

$$u[x] = \begin{cases} +1 & \text{if } x \notin K, \\ -1 & \text{if } x \in K. \end{cases} \tag{4.4.2}$$

Roughly speaking, taking $u = +1$ outside K drives the state as fast as possible inside K whereas taking $u = -1$ in K allows the system to spent as much time as possible in K. This feedback clearly minimizes the time spent by the trajectory outside the set K in both example. The corresponding trajectories and time crisis can then serve as a test of numerical methods solving Problem ($\mathrm{TCR}_n^\#$) or Problem (TCR_n) for different values of n. We have used here the `BocopHJB` software (see [72, 17]) which solves the HJB equation (but other numerical methods could be used) to compare the optimal solution of Problem (TC) with the approximated solutions.

For numerical purposes we took $\tau = 0$, $T = 5$ in both examples. Numerical results are depicted in Fig. 4.3, Fig. 4.4. We can see the convergence of the approximated trajectories to the myopic solution, which illustrates our convergence results. One can also observe the convergence of controls u_n, although we are not able to prove it here. As expected, the solution of Problem (TCR_n) is closer to the optimal solution, compared to the solution of Problem $(\mathrm{TCR}_n^{\#})$ (in particular in the first example). Finally, optimal values for the various costs are reported in Table 4.1. This highlights the interest of choosing $v \in \{\pm 1\}$ despite the lack of convexity.

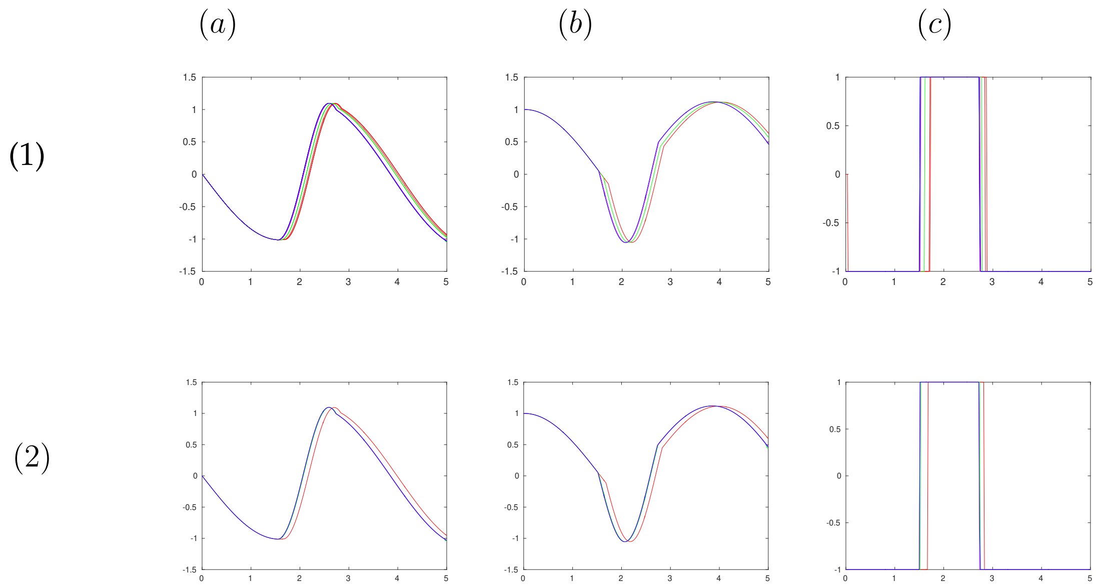
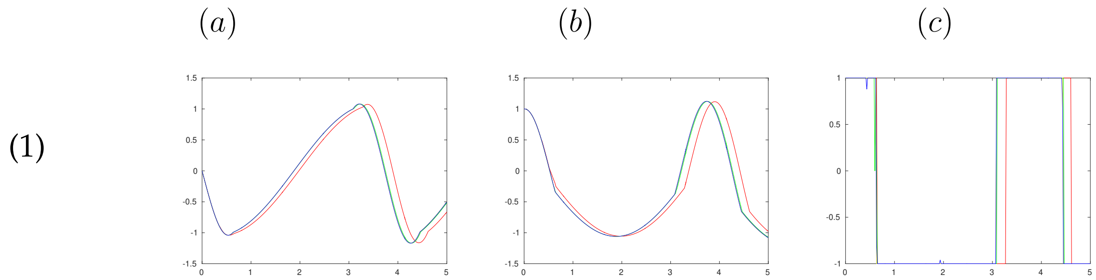


FIGURE 4.3 – Example 1. (1): Problem $(\mathrm{TCR}_n^{\#})$, (2): Problem (TCR_n) (a), (b): trajectories $x_{1,n}$, $x_{2,n}$, (c): controls u_n (— $n = 10$, — $n = 100$, — $n = 1000$, — myopic solution)



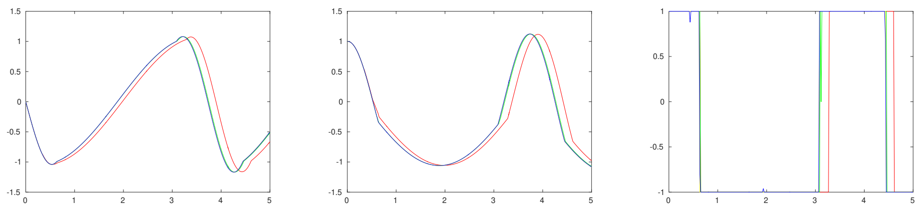
(2) 

FIGURE 4.4 – Example 2. (1): Problem ($\mathrm{TCR}_n^\#$), (2): Problem (TCR_n) (a), (b): trajectories $x_{1,n}$, $x_{2,n}$, (c): controls u_n (— $n = 10$, — $n = 100$, — $n = 1000$, — myopic solution)

(1)

	$\mathrm{TCR}_n^\#$	TCR_n	TC
$n = 10$	1.076	1.13126	1.38333
$n = 100$	1.19337	1.26485	1.38333
$n = 1000$	1.32697	1.3516	1.38333

(2)

	$\mathrm{TCR}_n^\#$	TCR_n	TC
$n = 10$	1.72829	1.7746	1.98333
$n = 100$	1.90914	1.96223	1.98333
$n = 1000$	1.95686	1.96378	1.98333

TABLE 4.1 – Costs of the optimal trajectories. (1): Example 1. (2): Example 2.

4.5 Conclusion

In this note, we have proposed a new smoothing procedure for the minimal time crisis problem by introducing an additional control v with values in $\{\pm 1\}$, replacing the original problem with discontinuous integrand by a regular optimal control problem involving a mixed state-control constraint. We have then used a penalty method to avoid to deal with this constraint. This has led us to the study of a sequence of unconstrained optimal control problems. We have proved the convergence of the value function and optimal trajectories of the regularized problem to the solution to the original problem. Our numerical examples illustrate these results and validate the good performance of the proposed method. We observed that considering an additional control with only two possible values is quite efficient from a numerical view point, despite the lack of convexity of the augmented velocity set. This regularization technique authorizes trajectories to leave and enter a set K without requiring a transverse condition on optimal paths nor convexity of the set K. In a future work, we shall investigate necessary optimality conditions of (TC) using this approach.

CHAPITRE 5.0

NECESSARY OPTIMALITY CONDITION FOR THE MINIMAL TIME CRISIS RELAXING TRANSVERSE CONDITION VIA REGULARIZATION

This chapter corresponds to the paper under review, minor revision

T. BAYEN, K. BOUMAZA AND A. RAPAPORT, *A new proof of optimality conditions for the time of crisis via regularization*, 2021. [7]

Contents

5.1 Introduction

This paper proposes a novel approach to derive optimality conditions for the so-called *time of crisis problem* [9] as well as (new) sufficient conditions ensuring the well-posedness of this method. Such conditions will slightly differ of those available in the literature that involve the behavior of an (a priori unknown) optimal path.

Originally, the time of crisis problem was introduced in [34], and it consists in minimizing the total time spent by a solution of a control system outside a given constraint set. It is of particular interest whenever it is not possible to maintain the system in such a set. In that case, alternative strategies consist in finding a control policy such that the associated solution spends the minimum of time outside the constraint set. The time of crisis arises in the context of viability theory [3, 4], see, *e.g.*, a case study in ecology in [11], and more generally whenever one is unable to maintain a controlled dynamics within a prescribed constraint set over a time window.

From a theoretical point of view, the formulation of the time of crisis involves a discontinuous function w.r.t. the state, namely the indicator function of the constraint set. The integrand is then equal to 0 or 1 depending if the state of the system lies inside or outside the constrained set (hence, it can be viewed as a particular instance of a hybrid optimal control problem). In particular, the *Pontryagin Maximum Principle* (PMP), see [58], cannot be applied straightforwardly to derive necessary conditions. Various approaches have been proposed in the literature to study this issue and we now wish to give an overview of the available methods in order to highlight the differences with our approach.

In the first paper about the time of crisis (see [34]), the optimal control problem was tackled via a dynamic programming approach only. The question of necessary optimality conditions has been investigated more recently in [9] using the so-called *hybrid maximum principle* (HMP) which is an extension of the PMP adapted to hybrid systems (see [38, 41, 26]). In [9], the authors provide necessary conditions by a direct application of this principle that requires a so-called *transverse hypothesis* on optimal trajectories which is as follows: any optimal solution crosses the boundary of the constraint set transversely. As in [41], this hypothesis is crucial for the derivation of necessary conditions (in particular, for an accurate definition of the jump of the covector at a crossing time). Thanks to this hypothesis, it is also shown in [9, 10] that extremals of a regularized [1] optimal control problem converge, up to a sub-sequence, to an extremal of the time of crisis problem. The methodology is in line with [41] using properties of variation vectors. Note that first and second order conditions have also been derived in [8] using the PMP on a reformulation of the time of crisis problem obtained via an augmentation technique (in the spirit of [30, 31, 32]). Let us also point out [2, 66] in which an approximation technique is introduced in order to obtain necessary conditions.

1. In [9, 11], the regularization technique is based on the Moreau-Yosida approximation of the indicator function, provided the constrained set to be convex.

The study of convergence of regularized extremals relies on a similar transverse assumption on optimal paths as in the framework of the HMP. In addition, the approximated optimal control problem involves the (unknown) optimal solution. It is worth noticing that this approach is of interest for obtaining optimality conditions via the use of the PMP on a smooth problem, however, it is not usable from a numerical point of view since the sequence of approximated problems itself involves the optimal solution.

Let us now describe more into details the content of this paper. As we have recalled, the time of crisis can be viewed as an application of the HMP on a discontinuous problem w.r.t. the state. The HMP available in the literature is a powerful tool, but its application requires an optimal solution to satisfy a transverse assumption related to the boundary of the constraint set (see [41]). In practice, this condition is hardly possible to check if one does not know in advance the optimal solution or estimates it with enough accuracy. That is why, we can wonder whether or not it is possible to derive optimality conditions for discontinuous integrands w.r.t. the state without the use of this hypothesis. Doing so, we introduce a sequence of regular optimal control problems whose integral cost approximates the time of crisis (this is made possible using mollifiers). Our contribution is twofold:

- We propose in the context of the time of crisis problem a sufficient condition for the derivation of necessary optimality conditions (see our Theorem 5.4.1). This condition relies on the data of the problem and on a sequence of approximated solutions, that can be easily checked.
- Necessary conditions for the time of crisis are derived under this condition which also covers the transverse case (recovering optimality conditions that can be alternatively obtained using the HMP in this case).

It is worth noticing that our alternative condition (see Theorem 5.4.1) does not involve the knowledge of the velocity of the optimal solution at a crossing time. As a byproduct of this approach (and in contrast with [2, 66] for instance), the sequence of approximated optimal control problems allows to approach an optimal solution of the original problem with a solution associated with a regular optimal control problem, that can be solved numerically with existing efficient methods.

The paper is organized as follows. In Section 5.2, we introduce the time of crisis problem and the regularization scheme, and we also apply the PMP on the regularized optimal control problem. In section 5.3, we study properties of a sequence of approximated solutions (namely, the integral of the approximated solution computed along the mollifier). This sequence will be crucial to introduce an auxiliary hypothesis in Section 5.5 that will allow us to derive optimality conditions for the time of crisis problem. Section 5.4 provides optimality conditions for the time of crisis under the hypothesis that the suitable sequence is bounded (and so, without a transverse hypothesis which is the novelty here). A sufficient condition involving this sequence is presented in Section 5.5. The last section is devoted to a complete study of an illustrative example that highlights the convergence of the sequence of adjoint vectors to a discontinuous covector having a jump at each crossing time. This phenomenon is not only depicted in a transverse situation, but also when a trajectory hits the boundary of the constrained set tangentially (and leaves the set non-tangentially).

5.2 Definitions and regularization of the time crisis problem

5.2.1 Notations and main hypotheses

Throughout the paper, $m, n \geq 1$ are integers and $T > 0$. Let us introduce the following notations:

- For $x, y \in \mathbb{R}^n$, $|x|$ and $x \cdot y$, denote the euclidean norm of x and the scalar product between x and y. If A is a square matrix of dimension n, $\|A\|$ stands for the norm of a A and its transpose is written $A^\top$.
- Given a mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, we denote respectively by $D_x f(x,u)$, $D_u f(x,u)$ the Jacobian matrix of f w.r.t. variables x and u at some point $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. The notations $\nabla\varphi(x)$, $\partial_{x_i}\varphi(x)$ $D_{xx}\varphi(x)$ denote the gradient, the partial derivative w.r.t. x_i, and the Hessian of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at some point $x \in \mathbb{R}^n$.
- Given two integers $k, p \geq 1$ and a function $w : [0,T] \to \mathbb{R}^k$, the notation $\|w\|_{L^p(I\,;\mathbb{R}^k)}$ will stand for the L^p-norm of w over some time interval $I \subset [0,T]$.

- If $g \in L^\infty([0,T]\,;\mathbb{R}^s)$, $s \in \mathbb{N}^*$, we denote by $g(t^\pm)$ the right and left limits (when it exists) of g at point t. In the same way, we shall denote by $\dot{g}(t^\pm)$ the right and left derivative of a scalar function g (when it exists).

In the sequel, we consider two non-empty subsets U and K of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, as well as two mappings $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (the dynamics). Throughout the paper, we suppose that the following hypotheses are satisfied:

(H1) The set U is compact and f is of class C^1 with linear growth, *i.e.*, there is $c \geq 0$ such that for every $(x,u) \in \mathbb{R}^n \times U$, one has $|f(x,u)| \leq c(|x|+1)$.

(H2) For every $(x,p) \in \mathbb{R}^{2n}$, the set

$$\bigcup_{u \in U} \begin{bmatrix} f(x,u) \\ -D_x f(x,u)^\top p \end{bmatrix}$$

is a non-empty compact[2] convex subset of $\mathbb{R}^{2n}$.

(H3) We suppose that φ is of class C^2, that the set K is with non-empty interior and is the 0-sub-level set of φ:

$$K = \{x \in \mathbb{R}^n \; ; \; \varphi(x) \leq 0\}.$$

(H4) For every $x \in \partial K$ (the boundary of K), one has $\varphi(x) = 0$ and $\nabla\varphi(x) \neq 0$.

Note that (H2) is fulfilled whenever the dynamics f is affine w.r.t. to the control u and U is convex.

5.2.2 The time of crisis

Throughout the paper, we consider the admissible control set

$$\mathcal{U} := \{u : [0,T] \to U \; ; \; u \in L^\infty([0,T]\,;U)\}.$$

2. The compactness property actually follows from the continuity of f and $D_x f$ and the compactness of U.

Given an initial condition $x_0 \in \mathbb{R}^n$, the *minimal time crisis problem* [3] (over $[0,T]$) is defined as

$$\inf_{u\in\mathcal{U}} \int_0^T \mathbb{1}_{K^c}(x_u(t))\,\mathrm{d}t, \tag{5.2.1}$$

where $\mathbb{1}_{K^c}$ denotes the characteristic function of the complement of K, *i.e.*, $\mathbb{1}_{K^c}(x) = 1$ if $x \notin K$, $\mathbb{1}_{K^c}(x) = 0$ if $x \in K$, and $x_u(\cdot)$ is the unique (global) solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) & = f(x(t),u(t)) \quad \text{a.e. } t\in[0,T], \\ x(0) & = x_0. \end{cases} \tag{5.2.2}$$

Under the previous assumptions, the existence of an optimal control for (5.2.1) follows easily from the lower semi-continuity of $\mathbb{1}_{K^c}$ (see, *e.g.*, [9, Proposition 2.1] for more details). An important feature for studying optimality conditions of (5.2.1) is the notion of crossing time that we recall below.

Definition 5.2.1. *Given a solution $x(\cdot)$ of* (5.2.2)*, let us define the absolutely function ρ as:*

$$\rho(t) := \varphi(x(t)), \quad t\in[0,T]. \tag{5.2.3}$$

(i) *A crossing time from K to K^c, resp. from K^c to K, is a time $\tau \in (0,T)$ for which there is $\eta > 0$ with $[\tau-\eta,\tau+\eta] \subset [0,T]$ such that $x(t) \in K$, resp. $x(t) \in K^c$, for $t \in [\tau-\eta,\tau]$ and $x(t) \in K^c$, resp. $x(t) \in K$, for $t \in (\tau,\tau+\eta]$. We shall say that a crossing time is "outward" if x crosses ∂K from K to K^c, and "inward" if it crosses from K^c to K.*

(ii) *A crossing time τ is transverse if moreover the function ρ admits non-null left and right limits, i.e.,*

$$\dot{\rho}(\tau^{\pm}) = \lim_{t\to\tau^{\pm}} \nabla\varphi(x(t))\cdot\dot{x}(t) \neq 0, \tag{5.2.4}$$

(negative for an outward crossing time, positive for an inward crossing time.)

Remark 5.2.1. *Definition 5.2.1 (i) is equivalent to say that τ is an isolated root of ρ such that the map $t \mapsto \rho(t)(t-\tau)$ is locally of constant sign (positive from from K to K^c, negative from from K^c to K).*

Assumption 5.2.1. *Any optimal solution x^* of* (5.2.1) *has a finite number $r \geq 1$ of (alternated) crossing times $(\tau_i)_{1\leq i\leq r}$ such that*

$$0 < \tau_1 < \tau_2 < \cdots < \tau_r < T.$$

3. To shorten, we also write *time of crisis* or *time crisis problem*.

Throughout the paper, we suppose that an optimal solution x^* of (5.2.1) has a finite number $r \geq 1$ of (alternated) crossing times $(\tau_i)_{1\leq i\leq r}$ such that

$$0 < \tau_1 < \tau_2 < \cdots < \tau_r < T.$$

In particular, we will not consider trajectories that may cross the boundary of K an infinite number of times over $[0, T]$ (such as chattering [81]).

5.2.3 Regularization scheme

The approach developed in the present paper is an approximation procedure of Problem (5.2.1) with a sequence of regular problems that can be solved with standard optimality conditions (such as the PMP) or existing numerical tools. It will allow us to recover the conditions obtained for instance in [9] using the HMP [41]. As mentioned in the introduction, other authors considered regularization of problems similar to (5.2.1), but requiring an a priori knowledge of an optimal control [2, 66], which therefore cannot be used as a practical approximation, in contrast with our approach. In addition, we shall see that the sufficient condition for the derivation of optimality conditions that we obtain in Section 5.5 does not involve the assumption that each crossing time of an optimal solution is transverse, as it is required in the HMP. Instead, this condition relies only on the boundedness in L^1 of a suitable sequence related to the regularized problem, that can be tested numerically.

Let us now introduce a regularized scheme associated with (5.2.1). Doing so, we consider approximations of the Heaviside step function [4], denoted here G, by a sequence of functions that fulfill the following properties.

(H5) For any $n \in \mathbb{N}$, the map $G_n : \mathbb{R} \to [0, 1]$ is C^2 and non decreasing. The sequence $(G_n)_n$ is such that

$$\forall \sigma \in \mathbb{R}, \quad \lim_{n\to+\infty} G_n(\sigma) = G(\sigma).$$

Moreover, there exists two sequences of numbers $(a_n)_n \in \mathbb{R}^n_-$ and $(b_n)_n \in \mathbb{R}^n_+$

4. We define G as the function such that $G(\sigma) = 0$, resp. $G(\sigma) = 1$ whenever $\sigma \leq 0$, resp. $\sigma > 0$.

resp. increasing and decreasing and of limit 0 when $n \to +\infty$ such that:

$$\forall n \in \mathbb{N}, \begin{cases} G_n(\sigma) = 0 & \text{if} \quad \sigma \leq a_n, \\ G_n(\sigma) = 1 & \text{if} \quad \sigma \geq b_n. \end{cases}$$

In view of its definition, each function G_n is Lipschitz continuous, and its Lipschitz constant L_n is such that $L_n \to +\infty$ whenever $n \to +\infty$. Note also that the sequence $(h_n)_n$ defined as

$$h_n := G_n',$$

is C^1 a mollifier, *i.e.*, for every $n \in \mathbb{N}$, the support of h_n is contained in $[a_n, b_n]$ and one has $\int_{\mathbb{R}} h_n(\sigma)\mathrm{d}\sigma = 1$ for all $n \in \mathbb{N}$. In addition, one has $\sup_{\sigma\in\mathbb{R}} |h_n'(\sigma)| \to +\infty$ whenever $n \to +\infty$. A typical example of such a sequence of functions is given by:

$$G_n(\sigma) := \begin{cases} 0 & \text{if } \sigma \leq -\frac{1}{2n}, \\ 3\Big(n\sigma + \frac{1}{2}\Big)^2 - 2\Big(n\sigma + \frac{1}{2}\Big)^3 & \text{if } \sigma \in \Big(-\frac{1}{2n}, \frac{1}{2n}\Big), \\ 1 & \text{if } \sigma \geq \frac{1}{2n}, \end{cases} \tag{5.2.5}$$

(see Fig. 5.1). In the sequel, we consider the following sequence of optimal control

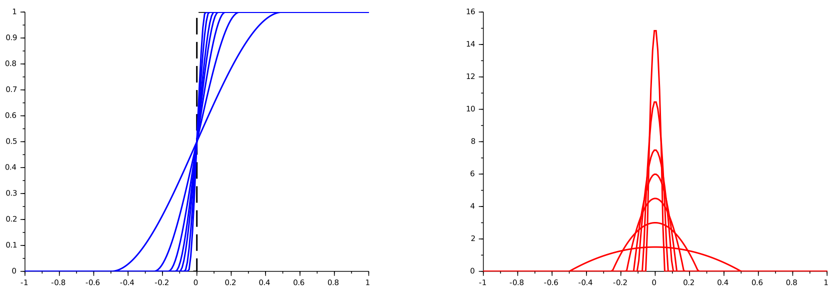


FIGURE 5.1 – Graphs of functions G_n given by (5.2.5) for $n \in \{1, ..., 10\}$ (Fig. left) and of their derivatives h_n (Fig. right).

problems

$$\inf_{u\in\mathcal{U}} \int_0^T G_n(\varphi(x_u(t)))\,\mathrm{d}t, \tag{5.2.6}$$

where $x_u(\cdot)$ is the unique solution to (5.2.2). The existence of an optimal control of (5.2.6) is straightforward using Filippov's existence Theorem [75] under our assumptions. Hereafter, we denote by (x_n, u_n) an optimal pair of (5.2.6). Following [7], we can show that, up to a sub-sequence, the sequence x_n converges strongly-weakly[5] to an optimal pair (x^*, u^*) of (5.2.1). Let us stress that (u_n) may not converge point-wise to u^*. However, more can be said about this point whenever the system is affine w.r.t. the control and under additional assumptions (such as the absence of singular arcs), see *e.g.* [10, Remark 4.3] or [65].

5.2.4 Optimality conditions for the regularized problem

We are now in a position to apply the Pontryagin Maximum Principle on Problem (5.2.6). Let $H_n : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the Hamiltonian[6] associated with (5.2.6) defined as:

$$H_n(x, p, u) = p \cdot f(x, u) - G_n(\varphi(x)).$$

Since (x_n, u_n) is optimal for (5.2.6), there is an absolutely continuous map $p_n : [0, T] \to \mathbb{R}^n$ satisfying the adjoint equation $\dot{p}_n = -\frac{\partial H_n}{\partial x}$ almost everywhere and $p_n(T) = 0$, that is,

$$\left|\begin{array}{ll} \dot{p_n}(t) & = -D_x f(x_n(t), u_n(t))^\top p_n(t) + h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t)) \quad \text{a.e. } t \in [0, T], \\ p_n(T) & = 0, \end{array}\right. \tag{5.2.7}$$

as well as the Hamiltonian condition which can be written

$$\forall u \in U, \quad p_n(t) \cdot f(x_n(t), u) \leq p_n(t) \cdot f(x_n(t), u_n(t)) \quad \text{a.e. } t \in [0, T]. \tag{5.2.8}$$

A triple (x_n, p_n, u_n) satisfying (5.2.2)-(5.2.7)-(5.2.8) is called an extremal (recall that only normal extremals occur here as there is no terminal condition). Let us observe that the problem is autonomous, therefore, the Hamiltonian is conserved along any extremal. For every $n \in \mathbb{N}$, there is $\tilde{H}_n \in \mathbb{R}$ such that:

$$\tilde{H}_n = H_n(x_n(t), p_n(t), u_n(t)) = p_n(t) \cdot f(x_n(t), u_n(t)) - G_n(\varphi(x_n(t))) \quad \text{a.e. } t \in [0, T].$$

Remark 5.2.2. *Since $(x_n)_n$ strongly-weakly converges to x^* which has r crossing times, it can be observed that $t \mapsto h_n(\varphi(x_n(t)))$ takes arbitrarily large values in* (5.2.7)*. Hence, we can expect the sequence $(\dot{p}_n)_n$ to be unbounded in $L^\infty([0, T]\,;\, \mathbb{R}^n)$. We shall see that whenever every crossing time of x^* is transverse, then $(p_n)_n$ is indeed bounded in $L^\infty([0, T]\,;\, \mathbb{R}^n)$*

5. This means that $(x_n)_n$ uniformly converges to x^* over $[0, T]$ and that $(\dot{x}_n)_n$ weakly converges to $\dot{x}^*$ in $L^2([0, T]\,;\, \mathbb{R}^n)$.

6. Since no terminal constraint appear in (5.2.6), one can directly take $p^0 = -1$ for the multiplier associated to the objective function G_n, *i.e.*, only normal extremals occur.

even though this is not the case for $(\dot{p}_n)_n$*. This is actually the main difficulty for studying the behavior of* (5.2.7) *and for passing to the limit when* $n \to +\infty$*.*

The boundedness of the sequence $(p_n)_n$ is related to the behavior of the sequence $(I_n)_n$ defined as

$$I_n := \int_0^T h_n(\varphi(x_n(t)))\,\mathrm{d}t. \tag{5.2.9}$$

As this integral will play a crucial role in the establishment of optimality conditions, passing to the limit into (5.2.7) when $n \to +\infty$, we devote the next session to the analysis of its properties.

5.3 Properties of the sequence of integrals $(I_n)_n$

We start by proving that $(I_n)_n$ is bounded if and only if $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\,\mathbb{R}^n)$. Recall that the limiting path x^* has a finite number of crossing times $(\tau_i)_{1\le i\le r}$ (recall Assumption 5.2.1). Set

$$\delta := \min_{0\le i\le r}(\tau_{i+1} - \tau_i),$$

with the convention $\tau_0 := 0$, $\tau_{r+1} := T$, and define the subsets

$$\mathcal{I}_\eta := \bigcup_{1\le i\le r}[\tau_i - \eta, \tau_i + \eta] \quad ; \quad \mathcal{J}_\eta := [0,T]\backslash\mathcal{I}_\eta.$$

The following property will be used at several places.

Property 5.3.1. *For all* $\eta \in (0,\delta)$*, there is* $N \in \mathbb{N}$ *such that for all* $n \ge N$ *and* $t \in \mathcal{J}_\eta$*, one has* $h_n(\varphi(x_n(t))) = 0$*.*

Proof. Take $\eta \in (0,\delta)$. Since $(x_n)_n$ uniformly converges to x^* and as $\varphi(x^*(t)) = 0$ if and only if $t \in \{\tau_1, ..., \tau_r\}$, there are $N_1 \in \mathbb{N}$ and $\gamma > 0$ such that

$$\forall n \ge N_1,\ \forall t \in \mathcal{J}_\eta, \quad |\varphi(x_n(t))| \ge \gamma.$$

Now, recall that both sequences $(a_n)_n$, $(b_n)_n$ defining the the support of h_n converge to zero, whence the result.

Next, we have the following equivalence between the boundedness of $(p_n)_n$ and $(I_n)_n$.

Proposition 5.3.1. *The sequence $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ if and only if $(I_n)_n$ is bounded in $\mathbb{R}_+$.*

Proof. Since $p_n(T)=0$ for every $n\in\mathbb{N}$, the boundedness of $(p_n)_n$ easily follows from the boundedness of $(I_n)_n$ using Gronwall's Lemma and the fact that $(x_n)_n$ and $(u_n)_n$ are bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ and $L^\infty([0,T]\,;\mathbb{R}^m)$ respectively.

Let us now assume that $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. By (5.2.7) one has

$$\int_0^T h_n(\varphi(x_n(t)))|\nabla\varphi(x_n(t))|^2 \mathrm{d}t = \int_0^T \dot p_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t + \int_0^T D_x f(x_n(t),u_n(t))^\top p_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t. \tag{5.3.1}$$

From (H4) and using the uniform convergence of $(x_n)_n$ to x^*, there are $\eta\in(0,\delta)$, $N_1\in\mathbb{N}$, and $\gamma'>0$ such that

$$\forall t\in\mathcal{I}_\eta,\ \forall n\geq N_1,\quad |\nabla\varphi(x_n(t))|^2\geq\gamma'.$$

From property 5.3.1, there is $N\in\mathbb{N}$ such that for every $n\geq N$ and every $t\in\mathcal{J}_\eta$, one has $h_n(\varphi(x_n(t)))=0$. It follows that for every $n\geq N_2:=\max(N,N_1)$, one has:

$$\int_0^T h_n(\varphi(x_n(t)))|\nabla\varphi(x_n(t))|^2\,\mathrm{d}t\geq\gamma'\int_{\mathcal{I}_\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t. \tag{5.3.2}$$

Now, from the hypotheses on f and φ, there is $C\geq 0$ such that:

$$\forall n\in\mathbb{N},\quad \forall t\in[0,T],\quad |D_x f(x_n(t),u_n(t))^\top p_n(t)\cdot\nabla\varphi(x_n(t))|\leq C.$$

By an integration by parts, we obtain using the terminal condition $p_n(T)=0$ for every $n\in\mathbb{N}$:

$$\int_0^T \dot p_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t = -p_n(0)\nabla\varphi(x_0)-\int_0^T p_n(t)\cdot D_{xx}\varphi(x_n(t))\dot x_n(t)\,\mathrm{d}t.$$

As the sequence $(x_n)_n$, $(\dot x_n)_n$, and $(p_n)_n$ are uniformly bounded, we deduce that there is $C'\geq 0$ such that for every $n\in\mathbb{N}$, one has

$$\left|\int_0^T \dot p_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t\right|\leq |p_n(0)||\nabla\varphi(x_0)|+\int_0^T \|D_{xx}\varphi(x_n(t))\|\,|p_n(t)|\,|\dot x_n(t)|\,\mathrm{d}t\leq C' \tag{5.3.3}$$

Combining (5.3.1)-(5.3.2)-(5.3.3) then implies

$$\forall n\geq N_2,\quad \gamma'\int_{\mathcal{I}_\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t\leq CT+C'.$$

We have thus proved that the sequence $\left(\int_{\mathcal{I}_\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t\right)_n$ is bounded. Since $h_n(\varphi(x_n(t))) = 0$ for every $t \in \mathcal{J}_\eta$ and every $n \geq N$, the result follows.

This proposition will be used later on to obtain optimality conditions for (5.2.1) by letting $n \to +\infty$ in (5.2.7) and by reasoning by contradiction supposing that $(p_n)_n$ is unbounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. We now aim at studying convergence properties of the sequence $(I_n)_n$. For $1 \leq i \leq r$ and $n \in \mathbb{N}$, define the integrals

$$\tilde{I}_{i,n}(\eta) := \int_{\tau_i-\eta}^{\tau_i+\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t, \quad \eta \in (0,\delta).$$

Lemma 5.3.1. *Suppose that $(I_n)_n$ is bounded. Then, for any $i \in \{1,...,r\}$, there exists $\ell_i \in \mathbb{R}_+$ such that for every $\eta \in (0,\delta]$, $\tilde{I}_{i,n}(\eta) \to \ell_i$ whenever $n \to +\infty$.*

Proof. Fix $i \in \{1,...,r\}$. Since $(I_n)_n$ is bounded so is $(\tilde{I}_{i,n}(\eta))_n$ for $\eta \in (0,\delta]$, hence, we may assume that there exists $\ell_i \in \mathbb{R}_+$ such that (up to a sub-sequence), $\tilde{I}_{i,n}(\delta) \to \ell_i$ when $n \to +\infty$. We can then write

$$\tilde{I}_{i,n}(\delta) - \tilde{I}_{i,n}(\eta) = \int_{\tau_i-\delta}^{\tau_i-\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t + \int_{\tau_i+\eta}^{\tau_i+\delta} h_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

Let then

$$\gamma_\eta := \min_{t\in[\tau_i-\delta,\tau_i-\eta]\cup[\tau_i+\eta,\tau_i+\delta]} |\varphi(x^*(t))| > 0.$$

Recall that $(x_n)_n$ uniformly converges to x^* when $n \to +\infty$. Thus, there exists $N \in \mathbb{N}$ such that:

$$\forall n \geq N,\ \forall t \in [\tau_i-\delta,\tau_i-\eta] \cup [\tau_i+\eta,\tau_i+\delta], \quad |\varphi(x_n(t))| \geq \frac{\gamma_\eta}{2}.$$

Now, both sequence $(a_n)_n$ and $(b_n)_n$ converge to zero, hence there exists $N' \geq N$ such that

$$\forall n \geq N',\ [a_n,b_n] \subset \left[-\frac{\gamma_\eta}{2},\frac{\gamma_\eta}{2}\right].$$

Because the support of h_n is contained in $[a_n,b_n]$, we conclude that

$$\forall n \geq N',\ \tilde{I}_{i,n}(\delta) = \tilde{I}_{i,n}(\eta).$$

This proves that $\tilde{I}_{i,n}(\eta) \to \ell_i$ when $n \to +\infty$. Since $\eta \in (0,\delta]$ is arbitrary, the result follows.

Thanks to this lemma, we can now show the following result which provides a kernel-type property [7] of (I_n) and that will be crucial hereafter to study the convergence of $(p_n)_n$.

Proposition 5.3.2. *If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a function of class C^1 and $(I_n)_n$ is bounded, then:*

$$\forall \varepsilon > 0,\ \exists \eta_i \in (0, \delta],\ \exists N \in \mathbb{N},\ \forall n \geq N, \quad \left| \int_{\tau_i - \eta_i}^{\tau_i + \eta_i} h_n(\varphi(x_n(t))) g(x_n(t))\,\mathrm{d}t - \ell_i g(x^*(\tau_i)) \right| \leq \varepsilon. \tag{5.3.4}$$

In addition, η_i goes to zero as $\varepsilon \downarrow 0$.

Proof. For $\eta \in (0, \delta]$, one can write:

$$\begin{aligned} \int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t))) g(x_n(t))\,\mathrm{d}t - \ell_i g(x^*(\tau_i)) &= \underbrace{\int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t)))[g(x_n(t)) - g(x^*(t))]\,\mathrm{d}t}_{\Lambda_n^1(\eta)} \\ &\quad + \underbrace{\int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t)))[g(x^*(t)) - g(x^*(\tau_i))]\,\mathrm{d}t}_{\Lambda_n^2(\eta)} \\ &\quad + (\tilde{I}_{i,n}(\eta) - \ell_i) g(x^*(\tau_i)). \end{aligned}$$

Let then $\varepsilon > 0$ be fixed and set $M := \sup_n I_n$. By continuity of x^* at $t = \tau_i$, there exists $\eta_i > 0$ such that

$$|g(x^*(t)) - g(x^*(\tau_i))| \leq \frac{\varepsilon}{3M}, \quad t \in [\tau_i - \eta_i, \tau_i + \eta_i]. \tag{5.3.5}$$

Without any loss of generality, one can choose η_i such that it goes to zero as ε tends to 0 because $t \mapsto (g \circ x^*)(t)$ is Lipschitz continuous over $[0, T]$ (since g is of class C^1 and $\dot{x}^*$ is bounded in $L^\infty([0, T]\,; \mathbb{R}^n)$).

Since for every $\eta \in (0, \eta_i]$ one has

$$\begin{aligned} \forall n \in \mathbb{N},\ |\Lambda_n^2(\eta)| &\leq \int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t))) |g(x^*(t)) - g(x^*(\tau_i))|\,\mathrm{d}t \\ &\leq \int_{\tau_i - \eta_i}^{\tau_i + \eta_i} h_n(\varphi(x_n(t))) |g(x^*(t)) - g(x^*(\tau_i))|\,\mathrm{d}t, \end{aligned}$$

we deduce that

$$\forall n \in \mathbb{N},\ \forall \eta \in (0, \eta_i],\ |\Lambda_n^2(\eta)| \leq \frac{\varepsilon}{3}. \tag{5.3.6}$$

7. We refer to a classical property which asserts that given a sequence of mollifier $(f_n)_n$ defined over $[0, 1]$ and a continuous function $g : [0, 1] \to \mathbb{R}$, then $\int_0^1 f_n(t) g(t)\,\mathrm{d}t \to g(0)$ when n goes to infinity.

Now, the sequence $(x_n)_n$ uniformly converges to x^* over $[0,T]$. Hence, there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, one has $|g(x_n(t)) - g(x^*(t))| \leq \frac{\varepsilon}{3M}$ for every $t \in [0,T]$. This gives us the following property:

$$\exists N' \in \mathbb{N},\ \forall n \geq N',\ \forall \eta \in (0,\delta],\ |\Lambda_n^1(\eta)| \leq \frac{\varepsilon}{3}. \tag{5.3.7}$$

The last step is to apply Lemma 5.3.1 for $\eta = \eta_i$ which provides the next property:

$$\exists N'' \in \mathbb{N},\ \forall n \geq N'',\ |(\tilde{I}_{i,n}(\eta_i) - \ell_i)g(x^*(\tau_i))| \leq \frac{\varepsilon}{3}. \tag{5.3.8}$$

Let us set $N := \max(N', N'')$. Combining (5.3.6), (5.3.7) with $\eta = \eta_i$, and (5.3.8) then gives (5.3.4).

Let us underline that for an arbitrary sequence $(x_n)_n$ satisfying (5.2.2), which (up to a sub-sequence) converges strongly-weakly to a solution $\bar{x}$ of (5.2.2), then, the associated sequence of integrals $(I_n)_n$ is not necessarily bounded even if the limiting trajectory $\bar{x}$ has a transverse crossing time (see example below).

Example 5.3.1. *Consider the scalar dynamics*

$$\dot{x}(t) = u(t) \quad \text{a.e. } t \in [0,2],$$

where $u(t) \in [0,1]$, together with the set $K := \mathbb{R}_-$ (associated with $\varphi(x) := x$). As nominal path, we consider $\bar{x}(t) := t-1$, $t \in [0,2]$. Observe that the function $\bar{\rho}(t) = \varphi(\bar{x}(t))$ is differentiable with $\bar{\rho}'(t) = 1 > 0$ for every $t \in [0,2]$, thus $\tau = 1$ is a transverse crossing time. For this example, we suppose for convenience that the mollifier $(h_n)_n$ is such that $a_n = 0$ and $b_n > 0$ for every $n \in \mathbb{N}$. Let us denote by $c_n \in (a_n, b_n)$ the unique point at which h_n achieves its maximum (one has $h_n(c_n) \to +\infty$ whenever $n \to +\infty$). Next, we consider the sequence of absolutely continuous function $(x_n)_n$ defined as

$$\begin{cases} x_n(t) = & \frac{1+c_n}{1-\xi_n} t - 1 & t \in [0, 1-\xi_n], \\ x_n(t) = & c_n & t \in [1-\xi_n, 1+\xi_n'], \\ x_n(t) = & t - 1 - \zeta_n' & t \in [1+\xi_n', 2], \end{cases} \tag{5.3.9}$$

where $(\zeta_n)_n$, $(\zeta_n')_n$ converge to 0 when $n \to +\infty$, $(\xi_n)_n$ and $(\xi_n')_n$ are with values in $(0,1)$, and:

$$\forall n \in \mathbb{N}, \quad \zeta_n - \xi_n = c_n, \quad \xi_n' - \zeta_n' = c_n. \tag{5.3.10}$$

Equality (5.3.10) *guarantees the continuity of x_n at $t = 1-\xi_n$ and at $t = 1+\xi_n'$ for all $n \in \mathbb{N}$. Sequences $(\xi_n)_n$ and $(\xi_n')_n$ will be made more precise hereafter and* (5.3.10) *allows* *to uniquely define the sequences $(\zeta_n)_n$ and $(\zeta'_n)_n$.*

Suppose now that $(\xi_n)_n$ and $(\xi'_n)_n$ are chosen such that $(\xi_n+\xi'_n)h_n(c_n)\to+\infty$ whenever $n\to+\infty$. Then, we we can easily see that $I_n\to+\infty$. Indeed, I_n can be written $I_n=I_n^1+I_n^2+I_n^3$ with

$$I_n^1 := \int_0^{1-\xi_n} h_n(\varphi(x_n(t)))\,\mathrm{d}t\ ;\ I_n^2 := \int_{1-\xi_n}^{1+\xi'_n} h_n(\varphi(x_n(t)))\,\mathrm{d}t\ ;\ I_n^3 := \int_{1+\xi'_n}^{2} h_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

Recall that $\varphi(x)=x$, thus by changing the integration variable t into $s:=t-1+\zeta_n$, resp. $s:=t-1-\zeta'_n$ in I_n^1, resp. in I_n^3, we obtain that for every $n\in\mathbb{N}$, one has $I_n^1\le\frac{1+c_n}{1-\xi_n}$ and $I_n^3\le 1$. We deduce that $(I_n^3)_n$ is bounded and so is $(I_n^1)_n$ because $(c_n)_n$ and $(\xi_n)_n$ converge to zero. Now, for all $n\in\mathbb{N}$, we have

$$I_n^2 = h_n(c_n)(\xi_n+\xi'_n),$$

which shows that $I_n\to+\infty$. In addition, we easily check that the sequence $(x_n)_n$ strongly-weakly converges to $\bar{x}$. First, one has

$$\sup_{t\in[0,T]}|x_n(t)-\bar{x}(t)|\le\max(\zeta_n,\zeta'_n,c_n)\to 0,$$

whenever $n\to+\infty$. Second, one can verify that $(\dot{x}_n)$ converges a.e. to $\dot{\bar{x}}$ (actually for every $t\in[0,2]\backslash\{1\}$) which is enough to ensure the weak convergence of $(\dot{x}_n)_n$ to $\dot{\bar{x}}$ in $L^2([0,2]\,;\mathbb{R})$.

Nevertheless, we shall see later on that this phenomenon does not occur whenever $(x_n)_n$ is a minimizing sequence obtained from (5.2.6). This is due to the application of Pontryagin's Principle that provides additional properties on the extremal (x_n,p_n,u_n) preventing $(I_n)_n$ to blow up whenever every crossing time of the optimal path x^* is transverse.

5.4 Optimality conditions for the time crisis problem

In this section, we give necessary optimality conditions without the HMP, *i.e.*, by passing to the limit into the state-adjoint system satisfied by the extremal (x_n,p_n,u_n). Let us start by giving a definition of a covector associated with Problem (5.2.1). Doing so, we define the Hamiltonian $H:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ associated with

(5.2.1) as

$$H(x,p,u) = p \cdot f(x,u) - \mathbb{1}_{K^c}(x).$$

Definition 5.4.1. *Given a solution (x^*, u^*) of* (5.2.1) *with r crossing times, we say that a piecewise absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ is a covector associated to x^* if p is absolutely continuous on $[0,T]\backslash\{\tau_1, ..., \tau_r\}$ and satisfies the following conditions.*

- *The function p fulfills the backward adjoint equation:*

$$\left| \begin{array}{ll} \dot{p}(t) & = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in [0,T], \\ p(T) & = 0. \end{array} \right. \tag{5.4.1}$$

- *The Hamiltonian condition is fulfilled almost everywhere:*

$$\forall u \in U,\ p(t) \cdot f(x^*(t), u) \leq p(t) \cdot f(x^*(t), u^*(t)) \quad \text{a.e. } t \in [0,T]. \tag{5.4.2}$$

- *At every crossing time, p admits left and right limits, i.e.,*

$$\forall i \in \{1, ..., r\},\ \exists\, p(\tau_i^\pm) := \lim_{t \to \tau_i^\pm} p(t). \tag{5.4.3}$$

- *There exist r positive numbers $\ell_1, ..., \ell_r$ such that:*

$$\forall i \in \{1, ..., r\}, \quad p(\tau_i^+) - p(\tau_i^-) = \ell_i \nabla\varphi(x^*(\tau_i)). \tag{5.4.4}$$

- *The Hamiltonian is constant almost everywhere over $[0,T]$, i.e., there is $\tilde{H} \in \mathbb{R}$ such that*

$$\tilde{H} = H(x(t), p(t), u(t)) = \max_{u \in U} H(x(t), p(t), u) = -\mathbb{1}_{K^c}(x^*(T)) \quad \text{a.e. } t \in [0,T]. \tag{5.4.5}$$

We call extremal of (5.2.1) *any triple (x^*, p, u^*) satisfying* (5.2.2),(5.4.1)-(5.4.2), (5.4.4) *and* (5.4.5).

Remark 5.4.1. *Equality* (5.4.4) *amounts to say that p has a jump at $t = \tau_i$ in the normal cone to the set K (meaning here that the jump is in the direction of $\nabla\varphi(x(\tau_i))$, being assumed that K has a smooth boundary).*

To establish optimality conditions for (5.2.1), we start by proving the convergence of $(p_n)_n$. Doing so, we proceed step by step.

Lemma 5.4.1. *Suppose that $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ and let $1 \leq$*

$i \leq r$, $\eta \in (0, \delta]$. Then, up to a sub-sequence, the pair $(x_n, p_n)_n$ strongly-weakly converges over $[\tau_i + \eta, \tau_{i+1} - \eta]$ to a solution of

$$\left| \begin{array}{ll} \dot{x}^*(t) & = f(x^*(t), u^*(t)) \\ \dot{p}(t) & = -D_x f(x^*(t), u^*(t))^\top p(t) \end{array} \right. \qquad \text{a.e. } t \in [\tau_i + \eta, \tau_{i+1} - \eta].$$

Proof. In view of Property 5.3.1, for n large enough, the pair (x_n, p_n) satisfies the system

$$\left| \begin{array}{ll} \dot{x}_n(t) & = f(x_n(t), u_n(t)) \\ \dot{p}_n(t) & = -D_x f(x_n(t), u_n(t))^\top p_n(t) \end{array} \right. \qquad \text{a.e. } t \in [\tau_i + \eta, \tau_{i+1} - \eta].$$

Let $t_0 \in [\tau_i + \eta, \tau_{i+1} - \eta]$. Since $(p_n)_n$ is bounded, we may assume that $(p_n(t_0))_n$ converges (up to a sub-sequence). Because $(x_n)_n$ uniformly converges to x^*, we deduce that $x_n(t_0) \to x^*(t_0)$. Using (H1)-(H2), the result of compactness of solutions of a control system (see, *e.g.*, [28, Theorem 1.11]) yields the result.

Next, we show that $(p_n)_n$ strongly-weakly converges over $[0, T]\backslash\{\tau_1, ..., \tau_r\}$.

Lemma 5.4.2. *Suppose that $(p_n)_n$ is uniformly bounded in $L^\infty([0, T]\,; \mathbb{R}^n)$. Then, there exists a function $p : [0, T] \to \mathbb{R}^n$ absolutely continuous on $[0, T]\backslash\{\tau_1, ..., \tau_r\}$ satisfying*

$$\dot{p}(t) = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in [0, T], \tag{5.4.6}$$

such that for every $\eta \in (0, \delta)$, $(p_n)_n$ strongly-weakly converges to p over $\mathcal{J}_\eta$.

Proof. Let $\eta \in (0, \delta)$. By using the previous lemma, for every $1 \leq i \leq r$, we obtain the existence of an absolutely continuous function p_η defined over $\mathcal{J}_\eta$ and satisfying (5.4.6) over $\mathcal{J}_\eta$. We now argue that p_η does not depend on η by considering a sequence of positive numbers $(\eta_k)_k$ such that $\eta_k \downarrow 0$ which allows us to define p_{η_k} for every $k \in \mathbb{N}$, as previously (over $\mathcal{J}_{\eta_k}$). We then obtain $p_{\eta_{k+1}}(t) = p_{\eta_k}(t)$ for every $t \in \mathcal{J}_{\eta_k}$ because $\mathcal{J}_{\eta_k} \subset \mathcal{J}_{\eta_{k+1}}$ for every $k \in \mathbb{N}$. This shows that we can then define a function $p : [0, T]\backslash\{\tau_1, ..., \tau_r\} \to \mathbb{R}^n$ without any ambiguity by the equality $p = p_\eta$ on every set $\mathcal{J}_\eta$, $\eta \in (0, \delta]$. By construction, p does not depend on η, which is as wanted.

Let us now address the question of the constancy of the Hamiltonian along (x^*, p, u^*) and the Hamiltonian condition (5.4.2).

Lemma 5.4.3. *Suppose that $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. Then, the function p satisfies the Hamiltonian condition* (5.4.2) *and* (5.4.5).

Proof. Recall that x^* has r crossing times τ_i, $i = 1,...,r$. Let $0 \leq i \leq r$ and $t_0 \in (\tau_i, \tau_{i+1})$ be a Lebesgue point of the measurable function $t \mapsto p(t)\cdot \dot{x}^*(t)$. From (5.2.8), we obtain for any $u \in U$ and $\nu > 0$ (small enough):

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot f(x_n(t),u)\,\mathrm{d}t \leq \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot \dot{x}_n(t)\,\mathrm{d}t.$$

Now, $(p_n)_n$ and $(x_n)_n$ uniformly converge over $[t_0, t_0+\nu]$ to p and x^* respectively. In addition, $(\dot{x}_n)_n$ weakly converges to $\dot{x}^*$ over $[t_0, t_0+\nu]$. It follows that

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot f(x^*(t),u)\,\mathrm{d}t \leq \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot \dot{x}^*(t)\,\mathrm{d}t.$$

Letting $\nu \downarrow 0$ then gives

$$p(t_0)\cdot f(x(t_0),u) \leq p(t_0)\cdot \dot{x}^*(t_0).$$

Since $u \in U$ is arbitrary and almost every point of $[0,T]$ is a Lebesgue point of $t \mapsto p(t)\cdot\dot{x}^*(t)$, we obtain (5.4.2).

We proceed similarly to show the constancy of H over time, *i.e.*, for showing (5.4.5). Since the Hamiltonian H_n is autonomous for any $n \in \mathbb{N}$, one has

$$\tilde{H}_n := \max_{u\in U} H_n(x_n(t),p_n(t),u) = -G_n(\varphi(x_n(T)) \quad \text{a.e. } t \in [0,T],$$

and as $G_n(\varphi(x_n(T)) \in [0,1]$, $(\tilde{H}_n)_n$ converges, up to a sub-sequence, to some constant $\tilde{H} \in [-1,0]$. Let $i \in \{0,...,r\}$ and again, let $t_0 \in (\tau_i,\tau_{i+1})$ be a Lebesgue point of $t \mapsto p(t)\cdot\dot{x}^*(t)$. For $\nu > 0$ small enough, one has:

$$\tilde{H}_n = \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot\dot{x}_n(t)\,\mathrm{d}t - \frac{1}{\nu}\int_{t_0}^{t_0+\nu} G_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

By letting $n \to +\infty$, we see that

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot\dot{x}_n(t)\,\mathrm{d}t \to \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot\dot{x}^*(t)\,\mathrm{d}t.$$

By the uniform convergence of $(x_n)_n$ over $[t_0, t_0+\nu] \subset (\tau_i, \tau_{i+1})$, we also have

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} G_n(\varphi(x_n(t)))\,\mathrm{d}t \to \frac{1}{\nu}\int_{t_0}^{t_0+\nu} \mathbb{1}_{K^c}(x^*(t))\,\mathrm{d}t,$$

when $n \to +\infty$ (this follows using the dominated convergence Theorem). We can then conclude that

$$\tilde{H} = \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot \dot{x}^*(t)\,\mathrm{d}t - \frac{1}{\nu}\int_{t_0}^{t_0+\nu} \mathbb{1}_{K^c}(x^*(t))\,\mathrm{d}t.$$

Now, letting $\nu \downarrow 0$ (recall that $t_0 \in (\tau_i, \tau_{i+1})$ for $i = 1, ..., r$) gives

$$\tilde{H} = p(t_0)\cdot \dot{x}(t_0) - \mathbb{1}_{K^c}(x(t_0)).$$

Since almost every point $t_0 \in [0,T]$ is a Lebesgue point of the map $t \mapsto p(t)\cdot \dot{x}^*(t)$, one has then

$$H(x^*(t), p(t), u^*(t)) = \tilde{H} \quad \text{a.e. } t \in [0,T].$$

By the Hamiltonian condition (5.4.2) and the continuity property of the map $t \mapsto \max_{u\in U} H(x^*(t), p(t), u)$ over $[0,T]\backslash\{\tau_1, ..., \tau_r\}$, (5.4.5) is thus fulfilled (recall that $p(T) = 0$ and thus $\tilde{H} = -\mathbb{1}_{K^c}(x(T))$).

Thanks to the previous lemma, we can now give the main result of this section, namely that (x^*, p, u^*) is an extremal of Problem (5.2.1).

Theorem 5.4.1. *Suppose that the sequence of integrals $(I_n)_n$ is bounded or that the sequence $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. Then, there exists a non-null covector $p : [0,T] \to \mathbb{R}^n$ associated with x^* in the sense of Definition 5.4.1. In addition, the sequence $(p_n)_n$ strongly-weakly converges to p over every compact subset of $[0,T]\backslash\{\tau_1, ..., \tau_r\}$.*

Proof. Recall from Proposition 5.3.1 that having the sequence of integrals $(I_n)_n$ bounded or the sequence $(p_n)_n$ uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ are equivalent. The existence of a function $p : [0,T]\backslash\{\tau_1, ..., \tau_r\} \to \mathbb{R}^n$ satisfying (5.4.1)-(5.4.2)-(5.4.5) follows from the previous lemma. Thanks to Lemma 5.4.2, we also obtain the desired strong-weak convergence of $(p_n)_n$ on every subset $\mathcal{I}_\eta$ (with $0 < \eta < \delta$) of $[0,T]$, and thus on every compact subset of $[0,T]\backslash\{\tau_1, ..., \tau_r\}$. Note that one has $p(T) = 0$ because $p_n(T) = 0$ for all $n \in \mathbb{N}$ and $p(T) = \lim_{n\to+\infty} p_n(T)$ (recall that $x^*(T) \notin \partial K$ since $\tau_r < T$ is the last crossing time).

Let us now show (5.4.3). Since $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$, there is $R \geq 0$ such that for every $n \in \mathbb{N}$ one has $\|p_n\|_{L^\infty([0,T]\,;\,\mathbb{R}^n)} \leq R$. Since for every $t \in [0,T]\backslash\{\tau_1,...,\tau_r\}$, one has $p(t) = \lim_{n\to+\infty} p_n(t)$, we deduce that $\|p\|_{L^\infty([0,T]\,;\,\mathbb{R}^n)} \leq R$. Now, fix $1 \leq i \leq r$ and observe that

$$\dot{p}(t) = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in (\tau_i, \tau_{i+1}).$$

Given $t_1, t_2 \in (\tau_i, \tau_{i+1})$, we can thus write:

$$|p(t_2) - p(t_1)| = \left| \int_{t_1}^{t_2} -D_x f(x^*(t), u^*(t))^\top p(t)\,\mathrm{d}t \right| \leq A|t_1 - t_2|,$$

where $A := R \times \sup_{t\in[0,T]} |D_x f(x^*(t), u^*(t))^\top p(t)|$. This inequality shows that $p(\cdot)$ satisfies the Cauchy criterion at $t = \tau_i^+$ which proves that the right limit $p(\tau_i^+)$ exists. Similarly, one obtains the existence of a left limit $p(\tau_i^-)$. We can repeat this argumentation at every crossing time τ_i which gives (5.4.3).

Let us now prove the jump formula (5.4.4). Doing so, consider a sequence $(\varepsilon_k)_k$ such that $\varepsilon_k \downarrow 0$ and let us apply Proposition 5.3.2 with $\nabla\varphi$ in place of g (φ being of class C^2). For every $k \in \mathbb{N}$, there exist $\eta_k \in (0, \delta]$ and $N_k \in \mathbb{N}$ such that for every $n \geq N_k$, one has

$$\left| \int_{\tau_i-\eta_k}^{\tau_i+\eta_k} h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t))\,\mathrm{d}t - \ell_i \nabla\varphi(x^*(\tau_i)) \right| \leq \varepsilon_k.$$

Notice that from Proposition 5.3.2, one has $\eta_k \to 0$ when $k \to +\infty$. Integrating (5.2.7) over $[\tau_i - \eta_k, \tau_i + \eta_k]$ yields

$$\begin{aligned} \forall n \in \mathbb{N},\ p_n(\tau_i + \eta_k) - p_n(\tau_i - \eta_k) = &\int_{\tau_i-\eta_k}^{\tau_i+\eta_k} -D_x f(x_n(t), u_n(t))^\top p_n(t)\,\mathrm{d}t \\ &+ \int_{\tau_i-\eta_k}^{\tau_i+\eta_k} h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t))\,\mathrm{d}t. \end{aligned}$$

Now, $t \mapsto D_x f(x_n(t), u_n(t))p_n(t)$ is uniformly bounded over $[0,T]$ (say by a constant $B \geq 0$). It follows that

$$\forall n \geq N_k,\ |p_n(\tau_i + \eta_k) - p_n(\tau_i - \eta_k) - \ell_i \nabla\varphi(x^*(\tau_i))| \leq 2B\eta_k + \varepsilon_k.$$

First, we let n goes to infinity (k being fixed) which gives:

$$\forall k \in \mathbb{N}, \quad |p(\tau_i + \eta_k) - p(\tau_i - \eta_k) - \ell_i \nabla\varphi(x^*(\tau_i))| \leq 2B\eta_k + \varepsilon_k.$$

Now, we let $k \to +\infty$ observing that $p(\tau_i \pm \eta_k) \to p(\tau_i^{\pm})$ and we obtain

$$p(\tau_i^+) - p(\tau_i^-) = \ell_i \nabla\varphi(x^*(\tau_i)),$$

which is the desired property.

The last step is to show that for every $1 \leq i \leq r$, one has $\ell_i \neq 0$. Consider the map

$$h(t) := \max_{u\in U} H(x^*(t), p(t), u) \quad t \in [0,T]\backslash\{\tau_1, ..., \tau_r\},$$

which is continuous on each time interval (τ_{i-1}, τ_i). As p admits left and right limits at each τ_i, so is h. Consider $i \in \{1, ..., r\}$ such that x^* crosses ∂K from K to K^c at $t = \tau_i$. One has then

$$h(\tau_i^-) = \max_{u\in U} p(\tau_i^-) \cdot f(x^*(t), u), \quad h(\tau_i^+) = \max_{u\in U} p(\tau_i^+) \cdot f(x^*(t), u) - 1.$$

If $\ell_i = 0$, one has $p(\tau_i^+) = p(\tau_i^-)$ and thus $h(\tau_i^+) - h(\tau_i^-) = -1$, which contradicts property (5.4.5). Similarly, if x^* crosses ∂K from K^c to K at τ_i with $\ell_i = 0$, one gets $h(\tau_i^+) - h(\tau_i^-) = 1$ and again a contradiction with (5.4.5).

Let us stress that this result does not involve the transverse assumption (H') to be found below in Section 5.5) which is about the transversality of x^*. Using the constancy of the Hamiltonian along an extremal, the jump formula can be also written as follows (see also [41]).

Corollary 5.4.1. *Assume that the sequence of integrals $(I_n)_n$ is bounded. If $\dot{x}^*$ admits left and right derivative at a crossing time τ_i, $i \in \{1, ..., r\}$ such that $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) \neq 0$ or $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0$, then the jump of the covector p at τ_i can be written as:*

$$p(\tau_i^+) = p(\tau_i^-) + \frac{\delta_i + p(\tau_i^-) \cdot (\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+)} \nabla\varphi(x^*(\tau_i)), \quad \textit{if } \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0, \tag{5.4.7}$$

or

$$p(\tau_i^-) = p(\tau_i^+) - \frac{\delta_i + p(\tau_i^+) \cdot (\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-)} \nabla\varphi(x^*(\tau_i)), \quad \textit{if } \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) \neq 0, \tag{5.4.8}$$

where $\delta_i = +1$ resp. $\delta_i = -1$ if the crossing time τ_i is outward, resp. inward.

Proof. Let us write the conservation of the Hamiltonian in a right and left neighborhood of τ_i. If τ_i is an outward crossing time, one has then:

$$p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) - 1 = p(\tau_i^+)\dot{x}^*(\tau_i^+), \tag{5.4.9}$$

and $p(\tau_i^+) = p(\tau_i^-) + \ell_i \nabla\varphi(x^*(\tau_i))$. If $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0$, replacing $p(\tau_i^+)$ by this last expression in equation (5.4.9) raises

$$\ell_i = \frac{1 + p(\tau_i^-) \cdot (\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+)}.$$

Similarly, if $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) \neq 0$, replacing $p(\tau_i^-)$ by $p(\tau_i^+) - \ell_i \nabla\varphi(x^*(\tau_i))$ in equation (5.4.9) gives the equivalent expression

$$\ell_i = \frac{1 + p(\tau_i^+) \cdot (\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-)}.$$

If τ_i is an inward crossing time, one can easily check that this amounts to replace 1 by -1 in (5.4.9).

Remark 5.4.2. *When every crossing time τ_i of the optimal solution x^* is transverse, one can use the expression* (5.4.8) *to determine explicitly the jumps of p, recursively from the terminal time T in a univocal way. Therefore, for any choice of sequences $(a_n)_n$, $(b_n)_n$ such that $(I_n)_n$ is bounded and $(x_n)_n$ converges to x^* (up to a sub-sequence), one obtains the same numbers ℓ_i. If not, the jumps are determined only implicitly from conditions* (5.4.7) *and* (5.4.8).

5.5 Sufficient conditions for the boundedness of the sequence $(I_n)_n$

The aim of this section is to give sufficient conditions on the system that ensure the boundedness of $(I_n)_n$. In that case, necessary optimality conditions for an optimal path are guaranteed by Theorem (5.4.1). Given an optimal solution (x^*, u^*) of Problem 5.2.1, let us introduce the following hypothesis (in the spirit of the Hybrid Maximum Principle that requires also a transverse assumption [41]).

(H′) Every crossing time of x^* is transverse.

This hypothesis excludes the cases where the optimal solution x^* hits the boundary of K tangentially, *i.e.*,

$$\lim_{t\to\tau^+} \nabla\varphi(x(t))\cdot\dot{x}(t) = 0 \quad \text{or} \quad \lim_{t\to\tau^-} \nabla\varphi(x(t))\cdot\dot{x}(t) = 0, \tag{5.5.1}$$

at a crossing time τ. Actually, several cases could appear depending if both scalar products are zero in (5.5.1) or only one. As far as we know, the derivation of necessary conditions in this case has been very little considered in the literature (except in [6]) and remains a thorough open question. We shall see in Section 5.6 an example of optimal paths that possess non-transverse crossing times.

5.5.1 The transverse case

We start by the following result which covers the case where every crossing times of the optimal solution x^* is transverse.

Proposition 5.5.1. *Under Hypothesis* (H'), *the sequence* $(I_n)_n$ *is bounded.*

Proof. Suppose by contradiction that $(I_n)_n$ is unbounded. Extracting a sequence if necessary, we may assume that $I_n \to +\infty$ whenever $n \to +\infty$. Observe that the function $q_n := \frac{p_n}{I_n}$ satisfies the differential system

$$\left|\begin{array}{ll} \dot{q_n}(t) & = -D_x f(x_n(t),u_n(t))^\top q_n(t) + \tilde{h}_n(\varphi(x_n(t)))\nabla\varphi(x_n(t)) \quad \text{a.e. } t\in[0,T], \\ q_n(T) & = 0, \end{array}\right.$$

where $\tilde{h}_n(\sigma) := \frac{h_n(\sigma)}{I_n}$, $\sigma\in\mathbb{R}$. By this change of variable, one has obviously

$$\forall n\in\mathbb{N},\ \int_0^T \tilde{h}_n(\varphi(x_n(t)))\,\mathrm{d}t = 1, \tag{5.5.2}$$

so, Proposition 5.3.1 implies that the sequence $(q_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. One can then repeat the same argumentation than in the proof of Theorem 5.4.1 on the sequence $(q_n)_n$, excepted the last point about the value of the Hamiltonian. Indeed, as the covector p_n has been renormalized, we have no longer the value of the Hamiltonian equal to $-G_n(\varphi(x_n(T))$. However, we obtain that there exists a piecewise absolutely continuous function $q:[0,T]\backslash\{\tau_1,...,\tau_r\}\to\mathbb{R}^n$ satisfying the following properties:

- Up to a sub-sequence, $(q_n)_n$ converges to q on every compact set of $[0,T]\backslash\{\tau_1,...,\tau_r\}$.

- The function q satisfies

$$\left|\begin{array}{ll} \dot{q}(t) & = -D_x f(x^*(t),u^*(t))^\top q(t) \quad \text{a.e. } t\in[0,T], \\ q(T) & = 0. \end{array}\right.$$

- The Hamiltonian condition is fulfilled almost everywhere:

$$\forall u\in U,\ q(t)\cdot f(x^*(t),u)\le q(t)\cdot f(x^*(t),u^*(t)) \quad \text{a.e. } t\in[0,T]. \tag{5.5.3}$$

- For every $1\le i\le r$, q admits a limit at $\tau_i^{\pm}$, *i.e.*, there exists $\lim_{t\to\tau_i^{\pm}} q(t)$.
- At every crossing time τ_i, $1\le i\le r$, there exists $\tilde{\ell}_i\in\mathbb{R}$ such that

$$q(\tau_i^+)-q(\tau_i^-)=\tilde{\ell}_i\nabla\varphi(x^*(\tau_i)). \tag{5.5.4}$$

Here, we can no longer guarantee that each scalar $\tilde{\ell}_i$ is non null. Nevertheless, Proposition 5.3.2 implies that for every $1\le i\le r$ and every $\eta\in(0,\delta]$ one has

$$\tilde{\ell}_i=\lim_{n\to+\infty}\int_{\tau_i-\eta}^{\tau_i+\eta}\tilde{h}_n(\varphi(x_n(t)))\,\mathrm{d}t. \tag{5.5.5}$$

Observe that

$$\forall n\in\mathbb{N},\ \frac{\tilde{H}_n}{I_n}=q_n(t)\cdot f(x_n(t),u_n(t))-\frac{G_n(\varphi(x_n(t)))}{I_n} \quad \text{a.e. } t\in[0,T].$$

For every $t\in[0,T]$, one has $\frac{\tilde{H}_n}{I_n}\to 0$ and $-\frac{G_n(\varphi(x_n(t)))}{I_n}\to 0$ when $n\to\infty$ because $\tilde{H}_n\in[-1,0]$ and $G_n(\varphi(x_n(t)))\in[0,1]$ for every $n\in\mathbb{N}$ and every $t\in[0,T]$. We then obtain that the covector q satisfies:

$$q(t)\cdot f(x^*(t),u^*(t))=0 \quad \text{a.e. } t\in[0,T], \tag{5.5.6}$$

by considering Lebesgue points of the map $t\mapsto q(t)\cdot f(x^*(t),u^*(t))$ and repeating exactly the same argumentation as in the proof of Lemma 5.4.3.

We claim now that $q\not\equiv 0$, *i.e.*, $q(\cdot)$ is non-null over $[0,T]$. To show this property, it is enough to prove that there exists $1\le i\le r$ such that $\tilde{\ell}_i\neq 0$. Suppose then by contradiction that for every $1\le i\le r$, one has $\tilde{\ell}_i=0$. By using Lemma 5.3.1 with

$\tilde{h}_n$ in place of h_n, we obtain that

$$\forall i \in \{1, ..., r\}, \forall \eta \in (0, \delta], \lim_{n \to +\infty} \int_{\tau_i - \eta}^{\tau_i + \eta} \tilde{h}_n(\varphi(x_n(t))) \, \mathrm{d}t = 0, \tag{5.5.7}$$

where $\eta \in (0, \delta]$ is fixed. From Property 5.3.1 one also has

$$\lim_{n \to +\infty} \int_{J_\eta} h_n(\varphi(x_n(t))) \, \mathrm{d}t = \lim_{n \to +\infty} \int_{J_\eta} \tilde{h}_n(\varphi(x_n(t))) \, \mathrm{d}t = 0. \tag{5.5.8}$$

Combining (5.5.7) and (5.5.8) gives us a contradiction with (5.5.2), thus we have obtained that there is $1 \leq i \leq r$ such that $\tilde{\ell}_i \neq 0$.

To conclude the proof of the proposition, we will finally exhibit a contradiction involving optimality conditions (5.5.4) and (5.5.6) satisfied by the covector q. Fix $1 \leq i \leq r$ such that $\tilde{\ell}_i \neq 0$. First, using (5.5.3)-(5.5.6) at $t = \tau_i^-$, one has

$$q(\tau_i^-) \cdot \dot{x}^*(\tau_i^+) \leq q(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = 0.$$

Using (5.5.4), we get

$$q(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) - q(\tau_i^-) \cdot \dot{x}^*(\tau_i^+) = \tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+).$$

It follows that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) = -q(\tau_i^-) \cdot \dot{x}^*(\tau_i^+) \geq 0$$

Because $\tilde{\ell}_i \neq 0$ and $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0$, we deduce that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) > 0. \tag{5.5.9}$$

We now proceed with the same reasoning using (5.5.3)-(5.5.6) at $t = \tau_i^+$. We obtain

$$q(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) \leq q(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = 0.$$

Again, thanks to (5.5.4), one has

$$q(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) - q(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = \tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-).$$

It follows that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) = q(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) \leq 0.$$

Using again that $\tilde{\ell}_i \neq 0$ and $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0$, we deduce that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) < 0. \tag{5.5.10}$$

We claim that (5.5.9)-(5.5.10) is a contradiction. Indeed, because $\tilde{\ell}_i$ is non-zero, (5.5.9)-(5.5.10) imply that the scalar products $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-)$ and $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+)$ are of opposite sign. Because at $t = \tau_i$, the trajectory crosses the boundary of K, we necessarily have that

$$\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) > 0 \quad \text{and} \quad \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) > 0,$$

if τ_i is an outward crossing time, or

$$\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) < 0 \quad \text{and} \quad \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) < 0,$$

if it is inward. This contradiction completes the proof of the proposition and shows that $(I_n)_n$ is necessarily a bounded sequence.

As a consequence, we recover the HMP of the literature in the transverse case, as stated below.

Corollary 5.5.1. *Under Hypothesis* (H')*, there exists a non-null covector* $p : [0, T] \to \mathbb{R}^n$ *associated with* x^* *in the sense of Definition 5.4.1.*

Proof. Under Hypothesis (H'), the sequence $(I_n)_n$ is bounded, thanks to the previous proposition, and so is $(p_n)_n$ in $L^\infty([0, T] \; ; \; \mathbb{R}^n)$ from Proposition 5.3.1. The result follows from Theorem 5.4.1.

5.5.2 A sufficient condition on the approximating sequence

Note that condition (H') is expressed on the optimal solution $x^\star$, which is usually not known a priori. Instead, we give now conditions on the subsequence $(x_n)_n$, and not on the limiting solution x^*, that ensure the boundedness of the sequence of integrals $(I_n)_n$. For $n \in \mathbb{N}$, define the absolutely continuous function ρ_n as:

$$\rho_n(t) := \varphi(x_n(t)) \quad t \in [0, T].$$

We know that for each $n \in \mathbb{N}$, x_n is differentiable almost everywhere over $[0,T]$ and thus ρ_n as well, so that:

$$\dot{\rho}_n(t) = \nabla\varphi(x_n(t))\dot{x}_n(t) \quad \text{a.e. } t \in [0,T].$$

In addition, $(\rho_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R})$ thanks to (H1) and (H3). Therefore, for $i \in \{1,...,r\}$ and $n \in \mathbb{N}$, we can define:

$$l^+_{i,n} := \limsup_{h\to 0} \operatorname*{ess\,sup}_{t\in[\tau_i-h,\tau_i+h]} \dot{\rho}_n(t) \quad ; \quad l^-_{i,n} := \liminf_{h\to 0} \operatorname*{ess\,inf}_{t\in[\tau_i-h,\tau_i+h]} \dot{\rho}_n(t).$$

Remark 5.5.1. *In many optimal control problems, optimal solutions x_n are piecewise C^1, and thus the function ρ_n admits left and right derivatives at any $t \in (0,T)$. Then, the definition of the numbers $l^\pm_{i,n}$ simply involves the maximum and minimum of $\dot{\rho}_n(\tau_i^\pm)$.*

Proposition 5.5.2. *If for every $1 \le i \le r$, one has*

$$\begin{aligned} &\liminf_{n\to+\infty} l^-_{i,n} > 0 \text{ if } \tau_i \text{ is an outward crossing time, or} \\ &\limsup_{n\to+\infty} l^+_{i,n} < 0 \text{ if } \tau_i \text{ is an inward crossing time,} \end{aligned} \tag{5.5.11}$$

then the sequence $(I_n)_n$ is bounded.

Proof. For $i = 1,...,r$, set

$$\underline{l}_i = \liminf_{n\to+\infty} l^-_{i,n}, \quad \bar{l}_i = \limsup_{n\to+\infty} l^+_{i,n},$$

and define the numbers η_i^-, η_i^+ ($i = 1,...,r$) as

$$\eta_i^+ = \eta_{i+1}^- := \frac{1}{2}(\tau_{i+1} - \tau_i) \quad 1 \le i \le r-1,$$

together with $\eta_1^- := \tau_1$ and $\eta_r^+ := T - \tau_r$. As well, for $i \in \{1,...,r\}$, let us define the integrals:

$$\tilde{I}^-_{i,n}(\eta) := \int_{\tau_i-\eta}^{\tau_i} h_n(\varphi(x_n(t))\,\mathrm{d}t \quad ; \quad \tilde{I}^+_{i,n}(\eta) := \int_{\tau_i}^{\tau_i+\eta} h_n(\varphi(x_n(t))\,\mathrm{d}t.$$

One has clearly

$$I_n = \sum_{i=1}^{r} \left[\tilde{I}^-_{i,n}(\eta_i^-) + \tilde{I}^+_{i,n}(\eta_i^+)\right].$$

We show now that for any $i \in \{1, ..., r\}$ such that τ_i is an outward crossing time, the sequence $(\tilde{I}_{i,n}^-(\eta_i^-))_n$ is bounded. Observe first that one has:

$$\begin{aligned} \underline{l}_i \tilde{I}_{i,n}^-(\eta_i^-) &= \int_{\tau_i - \eta_i^-}^{\tau_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t + \int_{\tau_i - \eta_i^-}^{\tau_i} h_n(\rho_n(t))\dot{\rho}_n(t)\,\mathrm{d}t \\ &= \int_{\tau_i - \eta_i^-}^{\tau_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t + G_n(\varphi(x_n(\tau_i))) - G_n(\varphi(x_n(\tau_i - \eta_i^-))). \end{aligned} \tag{5.5.12}$$

Let $\varepsilon \in (0, \underline{l}_i)$. We claim that there exist $\zeta_i \in (0, \eta_i^-)$ and $N > 0$ such that

$$\forall n \geq N,\ \underline{l}_i < \dot{\rho}_n(t) + \varepsilon \quad \text{a.e. } t \in [\tau_i - \zeta_i, \tau_i]. \tag{5.5.13}$$

Indeed, by definition of $\underline{l}_i$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ one has

$$\underline{l}_i \leq l_{i,n}^- + \frac{\varepsilon}{2}.$$

We also have for every $h > 0$ sufficiently small:

$$\forall n \in \mathbb{N},\ \operatorname*{ess\,inf}_{s \in [\tau_i - h, \tau_i + h]} \dot{\rho}_n(s) \leq \dot{\rho}_n(t) \quad \text{a.e. } t \in [\tau_i - h, \tau_i + h].$$

By definition of $l_{i,n}^-$, we deduce that there exists $\zeta_i \in (0, \eta_i^-)$ such that

$$l_{i,n}^- \leq \operatorname*{ess\,inf}_{s \in [\tau_i - \zeta_i, \tau_i + \zeta_i]} \dot{\rho}_n(s) + \frac{\varepsilon}{2},$$

and thus, we obtain (5.5.13). One can then write

$$\begin{aligned} \int_{\tau_i - \eta_i^-}^{\tau_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t &< \varepsilon \int_{\tau_i - \zeta_i}^{\tau_i} h_n(\rho_n(t))\,\mathrm{d}t + \int_{\tau_i - \eta_i^-}^{\tau_i - \zeta_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t \\ &< \varepsilon \tilde{I}_{i,n}^-(\eta_i^-) + \int_{\tau_i - \eta_i^-}^{\tau_i - \zeta_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t. \end{aligned} \tag{5.5.14}$$

Observe now that the scalar μ defined as

$$\mu := \min_{t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i]} \varphi(x^*(t)),$$

is such that $\mu < 0$ since $x^*(t)$ belongs to K for $t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i]$. From the uniform convergence of the sequence $(x_n)_n$ to x^* over $[0, T]$ and since φ is continuous, there exists $N' \geq N$ such that one has

$$\varphi(x_n(t)) = \rho_n(t) < -\mu/2,$$

for any $n > N'$ and every $t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i]$. Then, from the convergence of the sequence of negative numbers $(a_n)_n$ to 0 (recall (H5)), there exists $\bar{N} \geq N'$ such that one has $a_n > -\mu/2$ for any $n > \bar{N}$. Since the support of h_n is contained in $[a_n, b_n]$, we deduce that

$$\forall n > \bar{N},\ \forall t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i],\ h_n(\rho_n(t)) = 0. \tag{5.5.15}$$

Finally, from (5.5.12), (5.5.14) and (5.5.15), one obtains

$$\forall n > \bar{N},\ \underline{l}_i \tilde{I}_{i,n}^-(\eta_i^-) < \varepsilon \tilde{I}_{i,n}^-(\eta_i^-) + 1.$$

This shows that the sequence $(\tilde{I}_{i,n}^-(\eta_i^-))_n$ is bounded. For $i \in \{1, ..., r\}$ such that τ_i is an inward crossing time, we proceed in the same way to show that the sequence $(\tilde{I}_{i,n}^-(\eta_i^-))_n$ is bounded, considering the number $\bar{l}_i < 0$. The proof of boundedness of the sequences $(\tilde{I}_{i,n}^+(\eta_i^+))_n$ is analogous.

From a practical point of view, x^* and its crossing times τ_i are known only approximately. Whenever ρ_n admits left and right derivatives, condition (5.5.11) is merely guaranteed whenever $(\dot{\rho}_n(t^{\pm}))_n$ is bounded from below by a positive number, or bounded from above by a negative number, locally at each crossing time τ_i, $1 \leq i \leq r$.

5.5.3 A reciprocal property

We have seen previously that under (H'), the sequence $(I_n)_n$ is bounded as well as under condition (5.5.11) which is a sufficient condition (expressed) on the approximated sequence x_n. Thanks to these conditions, we obtained optimality conditions for an optimal solution x^* (under the assumption that it has a finite number of crossing times). We now would like to address the converse question, namely, what can be said about x^* whenever the sequence $(I_n)_n$ is bounded? In that case, we can prove the following result.

Proposition 5.5.3. *Suppose that the sequence $(I_n)_n$ is bounded and let τ_i be a crossing time of x^* such that $\dot{x}^*(\tau_i^{\pm})$ exist. Then, if τ_i is an outward, resp. inward, crossing time, one has $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x(t_i)) \neq 0$, resp. $\dot{x}^*(t_i^-) \cdot \nabla\varphi(x(t_i)) \neq 0$.*

Proof. Since $(I_n)_n$ is bounded, there exists a (non-null) covector p as in Definition 5.4.1 (see Theorem 5.4.1). Consider now an outward crossing time τ_i. The conditions satisfied by the extremal (x^*, p, u^*) imply a jump of p at $t = \tau_i$ and the constancy of the Hamiltonian as in Definition 5.4.1. These conditions becomes

$$\left| \begin{array}{l} p(\tau_i^+) = p(\tau_i^-) + \ell_i \nabla\varphi(x^*(t_i)), \\ p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) - 1. \end{array} \right.$$

The Hamiltonian condition at $t = \tau_i$ also gives us the following inequalities:

$$\left| \begin{array}{l} p(\tau_i^-) \cdot f(x^*(\tau_i), u) \leq p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-), \\ p(\tau_i^+) \cdot f(x^*(\tau_i), u) \leq p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+), \end{array} \right.$$

for every $u \in U$. Suppose by contradiction that $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) = 0$. It follows that:

$$p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = p(\tau_i^-) \cdot \dot{x}^*(\tau_i^+)$$

Using the Hamiltonian condition, we obtain

$$p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = p(\tau_i^-) \cdot \dot{x}^*(\tau_i^+) \leq p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) - 1,$$

which is a contradiction. It follows that $\dot{x}^*(t_i^+) \cdot \nabla\varphi(x^*(t_i)) \neq 0$ as was to be proved.

In the case where τ_i is an inward crossing time, we proceed in the same way supposing by contradiction that $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) = 0$. In that case, the constancy of the Hamiltonian gives us

$$p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) - 1 = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+).$$

By a similar computation, we obtain using that $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) = 0$:

$$p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) \leq p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) - 1.$$

This is a contradiction, which ends the proof.

This proposition shows that if $(I_n)_n$ is bounded, then, at every crossing time τ_i of x^* for which $\dot{x}(\tau_i^\pm)$ exists, the trajectory is always transverse "at the exterior" of K, *i.e.*, we can consider the following cases:

- *case 1* : if τ_i is an outward crossing time, then $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) > 0$;

- *case* 2 : if τ_i is an inward crossing time, then $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) < 0$.

In both cases, we can only say that $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) \geq 0$ (case 1) or $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) \leq 0$ (case 2). If this scalar product is null, we shall say that the crossing time is *one-sided transverse*. In other words, if τ_i is one-sided transverse, then, one has

$$\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) = 0 \quad \text{and} \quad \dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) > 0 \tag{5.5.16}$$

in case 1 and

$$\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) > 0 \quad \text{and} \quad \dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) = 0, \tag{5.5.17}$$

in case 2. Case 1 will be illustrated in Section 5.6. The previous proposition also allows us to find an interesting property about the sequence $(p_n)_n$ whenever there is $i \in \{1, ..., r\}$ such that $\dot{x}^*(\tau_i^{\pm})$ exist and

$$\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) = \dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) = 0. \tag{5.5.18}$$

In that case, we shall say that the crossing time τ_i is *two-sided non transverse* (*i.e.*, the trajectory x^* enters and leaves K at τ_i tangentially).

Corollary 5.5.2. *Suppose that there exists $i \in \{1, ..., r\}$ such that τ_i is a two-sided non transverse crossing time of x^*. Then, the sequence $(p_n)_n$ is not bounded in $L^\infty([0, T]\ ;\ \mathbb{R}^n)$.*

Proof. Suppose by contradiction that $(p_n)_n$ is bounded in $L^\infty([0, T]\ ;\ \mathbb{R}^n)$. Proposition 5.3.1 then implies that the sequence $(I_n)_n$ is bounded. By Proposition 5.5.3, we obtain that either $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) \neq 0$ or $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) \neq 0$ depending if τ_i is an inward or an outward crossing time. This contradicts (5.5.18) which ends the proof.

Remark 5.5.2. *When any optimal solution x^* fulfills Hypothesis* (H'), *then the sequence $(I_n)_n$ is bounded for any choice of the sequences $(a_n)_n$, $(b_n)_n$, accordingly to Proposition 5.5.1. If not, one cannot guarantee a priori that the sequence of $(I_n)_n$ is bounded. However, we show in the example of section 5.6 that for the one-sided transverse case, a right choice of $(a_n)_n$, $(b_n)_n$ allows to obtain the boundedness of $(I_n)_n$ and thus the necessary optimality conditions.*

5.6 Example of an optimal path with a non-transverse crossing time

In this section, we provide an example for which an optimal solution of the time of crisis is non transverse. But still, for this solution, we can write optimality conditions using Theorem 5.4.1 even though the optimal path possesses a non-transverse crossing time.

We consider the time crisis problem over a finite interval $[0, T]$

$$\min_{u(\cdot)} \int_0^T \mathbb{1}_{K^c}(x(t))\, \mathrm{d}t, \tag{5.6.1}$$

for the controlled dynamics $x(\cdot)$ in the plane

$$\begin{cases} \dot{x}_1 &= 2 - u, & x_1(0) = 0, \\ \dot{x}_2 &= 4\cos\left(\frac{\pi}{2}x_1\right)^2 + 4u\sin\left(\frac{\pi}{2}x_1\right)^2, & x_2(0) = 0, \end{cases} \qquad u(\cdot) \in [0, 1], \tag{5.6.2}$$

and the set

$$K := \{x \in \mathbb{R}^2 \; ; \; x_2 \in (-\infty, 1] \cup [5, +\infty)\},$$

that can be written as $K = \{x \in \mathbb{R}^2 \, ; \, \varphi(x_2) \leq 0\}$ where

$$\varphi(y) := -(y-1)(y-5), \quad y \in \mathbb{R}.$$

Let us define the *myopic* [8] feedback strategy as

$$\psi^\star(x) := \begin{cases} 0, & x \in K \\ 1, & x \notin K \end{cases} \tag{5.6.3}$$

One can straightforwardly check that the solution $x^\star$ generated by the myopic

8. The terminology "myopic" means that the control strategy only considers the set K and its complementary and does not involve the dynamics.

strategy is given by the following expression

$$x_1^\star(t) = \begin{cases} 2t & t \in [0, \tau_1^\star], \\ t + \frac{1}{2} & t \in [\tau_1^\star, \tau_2^\star], \\ 2t - 1 & t \geq \tau_2^\star, \end{cases} \quad x_2^\star(t) = \begin{cases} 2t + \frac{1}{\pi}\sin(2\pi t) & t \in [0, \tau_1^\star], \\ 4t - 1 & t \in [\tau_1^\star, \tau_2^\star], \\ 2t + \frac{1}{\pi}\sin(\pi(2t-1)) + 2 & t \geq \tau_2^\star, \end{cases} \tag{5.6.4}$$

with $\tau_1^\star = \frac{1}{2}$ and $\tau_1^\star = \frac{3}{2}$. Note that $x^\star(t) \in K^c$ for $t \in (\tau_1^\star, \tau_2^\star)$ and that $\tau_2^\star - \tau_1^\star = 1$ (see Fig. 5.2). Also, we set

$$u^\star(t) := \psi(x^\star(t)) \quad \text{a.e. } t \in [0, T]. \tag{5.6.5}$$

We shall next prove that $u^\star$ is optimal for any $T > 0$. Before proving this fact, we

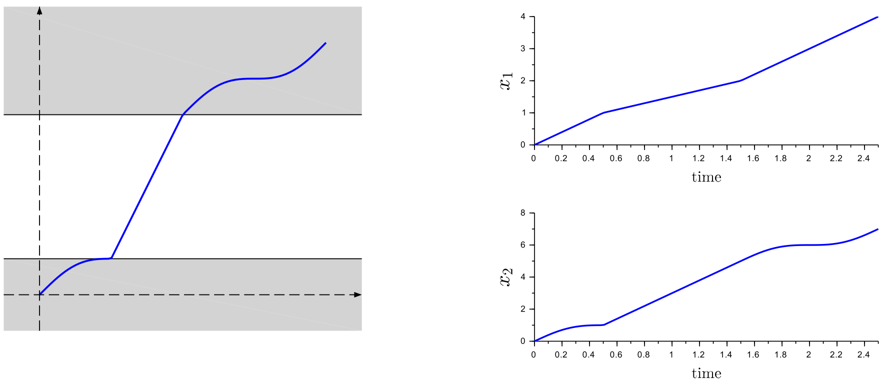


FIGURE 5.2 – Trajectory $x^\star$ provided by the myopic strategy starting at $(0, 0)$ (the set K is depicted in gray). The first crossing time is non-transverse (it is one-sided transverse of case 1, see (5.5.16)) whereas the second one is transverse.

give a useful property related to the controlled dynamics (5.6.4).

Lemma 5.6.1. *For any fixed $t_f > 0$, the control $u(\cdot) = 0$ is optimal for the auxiliary optimal control problem*

$$\min_{u(\cdot)} x_2(t_f),$$

where $x(\cdot)$ denotes the unique solution of (5.6.2) *associated with an admissible control $u(\cdot)$.*

Proof. We are in a position to apply the PMP on this Mayer optimal control problem. The Hamiltonian associated with this problem is

$$H_{aux}(x,p,u) := p_1(2-u) + 4p_2\left(\cos\left(\frac{\pi}{2}x_1\right)^2 + u\sin\left(\frac{\pi}{2}x_1\right)^2\right),$$

and the adjoint equations are

$$\begin{cases} \dot{p}_1 = -\partial_{x_1}H_{aux} = 2p_2\pi\sin(\pi x_1)(1-u) & p_1(t_f) = 0, \\ \dot{p}_2 = -\partial_{x_2}H_{aux} = 0 & p_2(t_f) = -1. \end{cases}$$

Notice that x_1 is always increasing. The switching function is (since p_2 is constant):

$$\phi(t) := -p_1(t) - 4\sin\left(\frac{\pi}{2}x_1(t)\right)^2,$$

and a straightforward computation gives

$$\dot{\phi}(t) = -2\pi\sin(\pi x_1(t)).$$

Since x_1 is non constant with bounded derivative, we deduce that no singular arc occurs along an optimal path and that the number of switching times is at most finite. In addition, if $u = 1$ a.e. on some time interval $[t_f - \delta, t_f]$ was optimal, one should have $p_1 = 0$ and thus $\phi < 0$ a.e. on this interval, which is a contradiction with the maximization of the Hamiltonian w.r.t. u. Thus, integrating backward $\dot{x}_1$, $\dot{p}_1$ from t_f with the control $u = 0$ gives $x_1(t) = 2(t - t_f) + x_1(t_f)$, $p_1(t) = \cos(\pi x_1(t)) - \cos(\pi x_1(t_f))$. We can conclude that ϕ satisfies

$$\phi(t) = \cos(\pi x_1(t)) + \cos(\pi x_1(t_f)) - 2, \quad t \in [0, t_f].$$

Thus, $\phi(t)$ is negative for a.e. $t \leq t_f$ whence the result.

Let then $\tilde{x}$ be the unique solution of (5.6.2) with $u = 0$. One can straightforwardly check that

$$\tilde{x}_1(t) = 2t, \quad \tilde{x}_2(t) = 2t + \frac{1}{\pi}\sin(2\pi t),$$

for $t \geq 0$ and that one has $\tilde{x}_2(\frac{5}{2}) = 5$. Lemma 5.6.1 also implies that any admissible path of (5.6.2) satisfies

$$\forall t \in [0,T], \quad x_2(t) \geq \tilde{x}_2(t) \tag{5.6.6}$$

(to show this property it is enough to argue by contradiction and to use Lemma 5.6.1). We now have the following optimality result about (5.6.1).

Proposition 5.6.1. *For any fixed $T > 0$, an optimal control of* (5.6.1) *is given by* (5.6.3).

Proof. Observe that any admissible solution x of (5.6.2) is such that $x_2(\cdot)$ is increasing. Suppose first that $T \leq \tau_1^\star$. Then the time crisis of $x^\star$ over $[0, T]$ (that is $\int_0^T \mathbb{1}_{K^c}(x^*(t))\,\mathrm{d}t$) is null and $x^\star$ is thus optimal. Suppose now that $T > \tau_1^\star$. Then, the time crisis of $x^\star$ over $[0, T]$ is $\min(1, T - \tau_1^\star)$. Take now an admissible solution x of (5.6.2). From (5.6.6), we deduce that there exists $\tau_1 \leq \tau_1^\star$ such that $x_2(\tau_1) = 1$. Moreover, one has $\dot{x}_2(t) \leq 4$ for a.e. t, which implies $x_2(\tau_1 + 1) \leq 5$. Therefore the time crisis of x over $[0, T]$ is $\min(1, T - \tau_1)$, and as $\tau_1 \leq \tau_1^\star$, we conclude that $\min(1, T - \tau_1) \geq \min(1, T - \tau_1^\star)$ implying the result.

Thanks to this result, we deduce the following properties.

- For $T > \frac{5}{2}$, any admissible solution x satisfies $x_2(T) \geq \tilde{x}_2(T) > \tilde{x}_2(\frac{5}{2}) = 5$, and thus has to cross the boundary of K exactly two times at $\tau_1 < \tau_2$ such that $x_2(\tau_1) = 1$, $x_2(\tau_2) = 5$.
- From (5.6.4), the crossing time $\tau_1^\star$ of the optimal solution $x^\star$ is non transverse because one has $\dot{x}_2(\tau_1^{\star-}) = 0$ (and $\dot{x}_2(\tau_1^{\star+}) = 4$), while the crossing time $\tau_2^\star$ is transverse with $\dot{x}_2(\tau_2^{\star-}) = \dot{x}_2(\tau_2^{\star+}) = 4$.

Therefore, Corollary 5.5.1 does not apply, but we shall next see that Theorem 5.4.1 can be applied. We then need to verify that the sequence $(I_n)_n$ is bounded. Doing so, we consider the regularized problem $\mathcal{P}_n$ for the following sequence of functions $(G_n)_n$ that fulfills Hypothesis H5 (see Fig. 5.3):

$$G_n(\sigma) := \begin{cases} 0 & \text{if } \sigma \leq 0, \\ 3(n\sigma)^2 - 2(n\sigma)^3 & \text{if } 0 < \sigma < \frac{1}{n}, \\ 1 & \text{if } \sigma \geq \frac{1}{n}. \end{cases}$$

Take $T > 0$ and let $x(\cdot)$ be an admissible solution of (5.6.2). From the expression of G_n, one has for every $n \in \mathbb{N}^*$:

$$J_n(x) := \int_0^T G_n(\varphi(x_2(t)))\,\mathrm{d}t = \int_{\tau_1}^{\tau_2} G_n(\varphi(x_2(t)))\,\mathrm{d}t.$$

Since the map $t \mapsto x_2(t)$ is increasing and $\dot{x}_2(t) \leq 4$ for a.e. $t \geq 0$, one obtains the inequality

$$J_n(x(\cdot)) \geq \frac{1}{4} \int_1^5 G_n(\varphi(x_2))\,\mathrm{d}x_2,$$

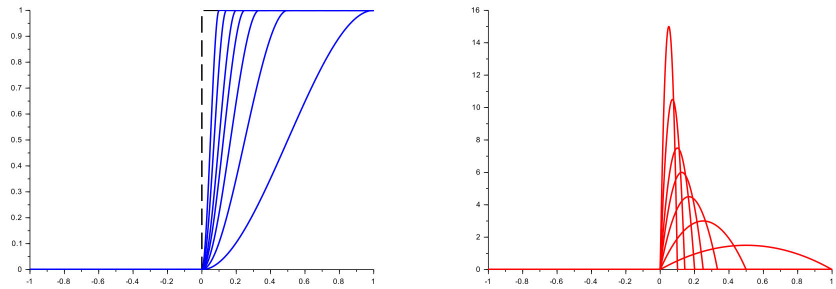


FIGURE 5.3 – Graphs of the functions G_n for $n \in \{1, ..., 10\}$ (Fig. left) and of their derivatives (Fig. right)

with equality whenever $x = x^\star$. We thus conclude that $x^\star$ is also optimal for the regularized problem over $[0, T]$ (with $T > \frac{5}{2}$) for every $n \in \mathbb{N}^*$. We now continue the study of this example showing that Theorem 5.4.1 can be applied. The next step consists in showing that the sequence of integrals

$$I_n := \int_0^T G_n'(\varphi(x_2^\star(t)))\,\mathrm{d}t$$

is bounded. Observe that one has

$$I_n = \int_{\tau_1^\star}^{\tau_2^\star} G_n'(\varphi(x_2^\star(t)))\,\mathrm{d}t = \int_{\frac{1}{2}}^{\frac{3}{2}} G_n'(-(4t-2)(4t-6))\,\mathrm{d}t.$$

We now proceed to a change of variable in the integral I_n. Consider the function $t \mapsto z(t) = -(4t-2)(4t-6)$ which is increasing from 0 to 4 for $t \in [\frac{1}{2}, 1]$ with $z'(t) = 8\sqrt{4 - z(t)}$, and decreasing from 4 to 0 for $t \in [1, \frac{3}{2}]$ with $z'(t) = -8\sqrt{4 - z(t)}$. One can then write

$$I_n = \int_0^4 \frac{G_n'(z)}{8\sqrt{4-z}}\,\mathrm{d}z + \int_4^0 \frac{G_n'(z)}{-8\sqrt{4-z}}\,\mathrm{d}z = \frac{1}{4}\int_0^4 \frac{G_n'(z)}{\sqrt{4-z}}\,\mathrm{d}z.$$

Note that for any $n \in \mathbb{N}^\star$, one has $G_n'(z) = 0$ for $z \geq 1$ which gives

$$\int_0^4 \frac{G_n'(z)}{\sqrt{4-z}}\,\mathrm{d}z = \int_0^1 \frac{G_n'(z)}{\sqrt{4-z}}\,\mathrm{d}z \leq \frac{1}{\sqrt{3}}\int_0^1 G_n'(z)\,\mathrm{d}z = \frac{G_n(1) - G_n(0)}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

We deduce that the sequence $(I_n)_n$ is bounded as wanted, so Theorem 5.4.1 does apply for $x^\star$ even if $\tau_1^\star$ is non-transverse. Let us next verify the existence of a covector associated with $(x^\star, u^\star)$ as in Definition 5.4.1 (recall that its existence is provided by Theorem 5.4.1). Doing so, recall that $u^\star$ is given by (5.6.5), that $x^\star$ has two crossing times $\tau_1^\star$, $\tau_2^\star$ such that $0 < \tau_1^\star < \tau_2^\star < T$. Also, let H be the Hamiltonian associated with (5.6.1):

$$H(x,p,u) := p_1(2-u) + 4p_2\left(\cos\left(\frac{\pi}{2}x_1\right)^2 + u\sin\left(\frac{\pi}{2}x_1\right)^2\right) - \mathbb{1}_{K^c}(x) \tag{5.6.7}$$

The adjoint system writes

$$\begin{cases} \dot{p}_1 = -\partial_{x_1}H = 2p_2\pi\sin(\pi x_1^\star(t))(1-u^\star(t)) & p_1(T) = 0, \\ \dot{p}_2 = -\partial_{x_2}H = 0 & p_2(T) = 0, \end{cases} \tag{5.6.8}$$

and jumps of p at crossing times $\tau_i^\star$ must satisfy

$$\begin{cases} p_1(\tau_i^{\star+}) - p_1(\tau_i^{\star-}) = 0, \\ p_2(\tau_i^{\star+}) - p_2(\tau_i^{\star-}) = l_i\varphi'(x_2(\tau_i^\star)) \text{ with } l_i > 0. \end{cases} \tag{5.6.9}$$

From (5.6.8), we first get $p = 0$ over $(\tau_2^\star, T]$. Note that the Hamiltonian (5.6.7) is equal to 0 on this interval. At the crossing time $\tau_2^\star$, the adjoint p_2 jumps from $p_2(\tau_2^{\star-}) > 0$ to $p_2(\tau_2^{\star+}) = 0$ accordingly to (5.6.9). Over the interval $(\tau_1^\star, \tau_2^\star)$, p_2 is constant equal to $p_2(\tau_2^{\star-})$ and $p_1 = 0$ because $u^\star = 1$ on this interval. The value $p_2(\tau_2^{\star-}) = \frac{1}{4}$ is obtained from the conservation of the Hamiltonian (5.6.7) which has also to be null on $(\tau_1^\star, \tau_2^\star)$.

At the crossing time $\tau_1^\star$, p_2 jumps from $p_2(\tau_1^{\star-}) < p_2(\tau_1^{\star+})$ to $p_2(\tau_1^{\star+}) = \frac{1}{4}$ accordingly to (5.6.9). Then, on the interval $[0, \tau_1^\star)$, $p = 0$ is a solution to the adjoint system that satisfying additionally the jump condition and the conservation of the Hamiltonian.

We can then conclude that the covector p is given over $[0,T]$ by the expression

$$p_1(t) = 0,\ t \in [0,T] \quad ; \quad p_2(t) = \begin{cases} 0, & t \in [0, \tau_1^\star), \\ \frac{1}{4}, & t \in (\tau_1^\star, \tau_2^\star), \\ 0, & t \in (\tau_1^\star, T]. \end{cases} \tag{5.6.10}$$

Finally, one can straightforwardly check that the control $u^\star$ verifies the maximization of the Hamiltonian (5.6.7) almost everywhere over $[0,T]$. Consequently, we have checked that the triple $(x^\star, p, u^\star)$ satisfies the necessary optimality conditions given by Theorem 5.4.1.

We conclude this section by giving an alternative approach showing that the sequence $(p_n)_n$ of covectors associated with the regularized problem $\mathcal{P}_n$ (whose optimal solution is $x^\star$) is bounded. Doing so, fix $n \in \mathbb{N}$. The Hamiltonian associated with $\mathcal{P}_n$ becomes

$$H_n(x, p_n, u) := p_{1,n}(2-u) + 4p_{2,n}\left(\cos\left(\frac{\pi}{2}x_1\right)^2 + u\sin\left(\frac{\pi}{2}x_1\right)^2\right) - G_n(\varphi(x_2)), \tag{5.6.11}$$

and p_n is solution to the adjoint system

$$\begin{cases} \dot{p}_{n,1} = -\partial_{x_1}H_n = 2p_{n,2}\pi\sin(\pi x_1^\star(t))(1-u^\star(t)) & p_{n,1}(T) = 0, \\ \dot{p}_{n,2} = -\partial_{x_2}H_n = G_n'(\varphi(x_2^\star(t)))\varphi'(x_2^\star(t)) & p_{n,2}(T) = 0. \end{cases} \tag{5.6.12}$$

Over the interval $[\tau_2^\star, T]$, one has $G_n(\varphi(x_2^\star)) = 0$ and thus $p_n \equiv 0$ over this interval. Over the interval $(\tau_1^\star, \tau_2^\star)$, one has $u^\star = 1$ and we deduce that $p_{n,1} = 0$ over this interval. One also has $\dot{x}_2^\star = 4$ and thus

$$p_{n,2}(t) = -\int_t^{\tau_2^\star} G_n'(\varphi(x_2(s)))\varphi'(x_2(s))\,\mathrm{d}s = -\frac{1}{4}G_n(\varphi(x_2^\star(t))) \quad t \in (\tau_1^\star, \tau_2^\star).$$

Finally, over the interval $[0, \tau_1^\star]$, one has again $G_n(\varphi(x_2^\star)) \equiv 0$ and $p_{n,2}$ is thus constant over this interval equal to $p_{n,2}(\tau_2^\star) = 0$. We deduce that $p_n \equiv 0$ over $[0, \tau_1^\star]$. We can then conclude that p_n is given by:

$$p_{n,1}(t) = 0 \; t \in [0,T] \quad ; \quad p_{n,2}(t) = \begin{cases} 0 & t \in [0, \tau_1^\star), \\ \frac{1}{4}G_n(\varphi(4t-1)) & t \in (\tau_1^\star, \tau_2^\star), \\ 0 & t \in (\tau_1^\star, T]. \end{cases}$$

A simple consequence of this computation is that $(p_n)_n$ is bounded in $L^\infty([0,T]\ ;\ \mathbb{R}^2)$ and converges pointwise to the piecewise constant function p given in (5.6.10) (see Fig. 5.4).

5.7 Conclusion

In this work, we have developed an approach based on a sequence of approximate optimal control problems. The proof of boundedness of the sequence of covectors that we have proposed here presents some analogy with materials of the

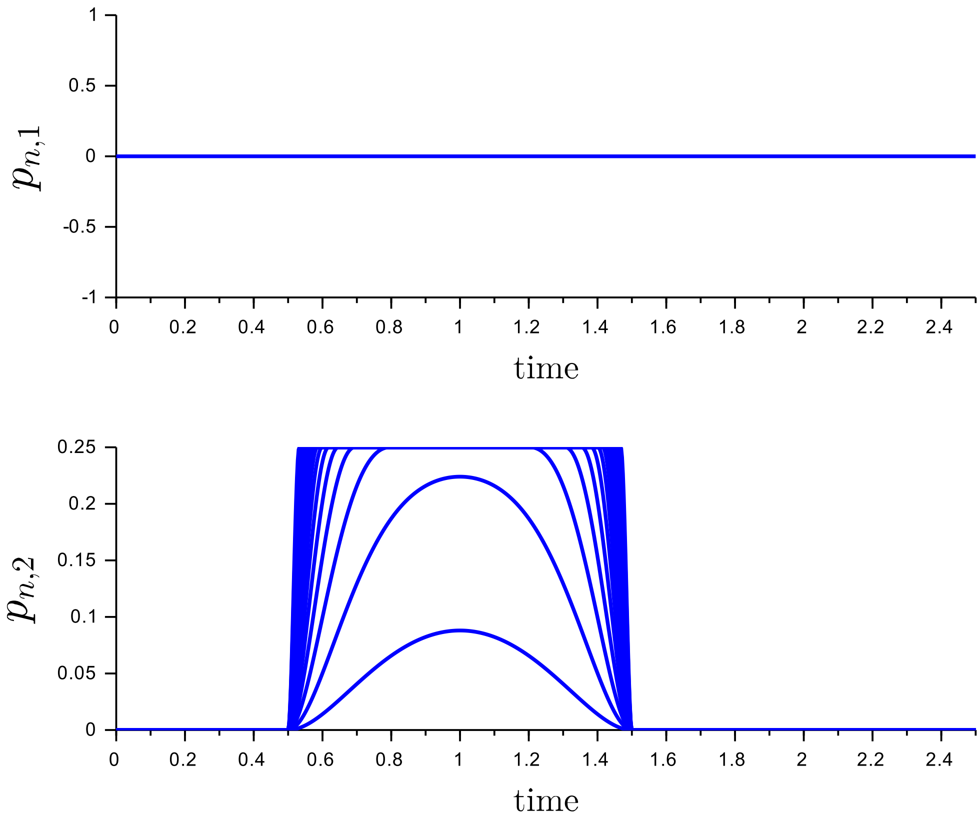


FIGURE 5.4 – Illustration of the convergence of (p_n) to the piecewise constant p.

work [41]. However, we have treated the question of boundedness via a suitable sequence of integrals $(I_n)_n$, and without the use of the Hybrid Maximum Principle (that we obtain indeed as a byproduct of this approach). As well, we also did not need to introduce needle-like perturbations and variation vectors to derive necessary optimality conditions. The main feature is that we obtain necessary optimality conditions for the time crisis problem under a more general condition than requiring transverse crossing times for the optimal trajectory. This condition related to the sequence of integrals $(I_n)_n$ does not involve the optimal path $x^\star$ neither its velocity. Besides, it allows to deal with non-transverse optimal paths at the boundary of the constraint set as we saw in Section 5.6.

Our methodology presents several interests from a numerical point of view since we introduced a regularized optimal control problem in place of a discontinuous problem (which is more delicate to handle numerically). In addition, we introduced an auxiliary condition to guarantee necessary conditions for an optimal solution x^*. This condition involves an approximated sequence which can be tested numerically and not a solution x^* to the initial optimal control problem (that is unknown a priori). Once this condition is checked (*i.e.*, the boundedness of the sequence $(I_n)_n$), one can then write the necessary optimality conditions which provides an exact characterization of the limit $x^\star$.

Some interesting issues that are out of the scope of this paper could be the matter of future works. In particular, one would like to better grasp the link between the boundedness of the sequence $(I_n)_n$ and the behavior of x^* at a crossing time τ (*i.e.*, to find a characterization of the boundedness of $(I_n)_n$ in terms of geometrical properties of the optimal path x^* at the crossing times), and to deal with optimal paths that could cross the constrained set tangentially on both sides. The methodology that has been developed in this paper could also be used to obtain an extension of necessary optimality conditions for more general hybrid problems whenever an optimal path has a so-called one-sided transverse crossing time, *i.e.*, is tangent to the constrained set only on one side.

Quatrième partie

Conclusion générale

CHAPITRE 6.0

CONCLUSIONS AND PERSPECTIVES

In this thesis we encountered different types of singularities: non-smooth dynamics, discontinuous integrand, which gave rise to original developments: SOS/SMS strategies for irrigation and optimal one-sided transverse trajectories for minimal time of crisis. In this overall conclusion, we begin by looking at the results of the research conducted. Next, we provide a number of perspectives on the basis of the results obtained to be undertaken in the field of the optimal control theory and crop modeling.

Results summary. In Chapter 2, we have introduced a simplified crop model based on a non-autonomous two-dimensional control system that is Lipschitz w.r.t. the state. We investigated an optimal control problem which amount to maximize biomass production under constraints on water available for irrigation. Using a comparison tool, we have first shown that the state constraint of this model is never satisfied for an optimal solution. Moreover, we have proved that optimal paths involve singular arcs such that the associated trajectory stays at the points at which the dynamics is non-differentiable. Next, we have shown that, under constraint on the available water, an optimal trajectory has to reach as fast as possible the domain for which the relative humidity is less than the optimal threshold for a maximal crop production, and then to maintain it in this domain until the harvesting time. This latter characteristic has led to the introduction of three control strategies:

- SMS strategy: this strategy consists in irrigating with possible multiple singular arcs.
- SOS strategy: this strategy is an SMS strategy with at most one singular arc at the maximal crop transpiration threshold.

- OS strategy: this strategy is a bang-bang strategy and consists in irrigating once with a single shot.

Finally, we have compared the three strategies and we have shown numerically that the SMS strategy has the greatest cost between the three possible strategies.

In Chapter 3, we have applied the minimal time of crisis problem to the crop model introduced in Chapter 2 to find an optimal strategy that deals with the constraint of the total water available for irrigation. Thanks to numerical simulations, we observe that an optimal solution consists in reaching as fast as possible the crisis threshold and staying on it as long as the water is available. This control policy is a part of the SOS strategy introduced in Chapter 2 which consists in maximizing the biomass while taking into account the constraint on the water.

In Chapter 4, we have introduced a new regularization procedure for the minimal time of crisis problem using an additional control that takes only two values $\{\pm 1\}$. This reformulation replaces the original problem that involves a discontinuous integrand w.r.t. the state by a smooth optimal control problem involving a mixed state-control constraint. We have then used a penalty method to avoid to deal with with this constraint. Despite the absence of convexity of the augmented velocity set, we have proved the convergence of the value function and optimal trajectories of the penalized problem to an optimal solution to the original problem. This regularization scheme has been tested successfully on several examples.

In Chapter 5, we investigate necessary optimality conditions under a less restrictive hypothesis than the usual one encountered in the literature. To do so, we have introduced a general regularization of the minimal time crisis problem. We have investigated the boundedness of the sequence of covectors in $L^\infty([0,T];\mathbb{R}^n)$ through a sequence of integrals $(I_n)_n$, that resulted from the regularization scheme. We have had the following results:

- If the sequence $(I_n)_n$ is bounded then the sequence of covectors is also bounded, moreover we have the necessary optimality condition of the minimal time of crisis problem.
- If all the crossing times are transverse, then $(I_n)_n$ is bounded and therefore we have the necessary optimality conditions of the minimal time of crisis problem.

- If the sequence $(I_n)_n$ is bounded, then all the crossing times are one-sided transverse.
- Suppose every crossing time is one-sided transverse, then under a condition on the regularizing sequence that does not involve geometric properties of optimal trajectories at the boundary of the constraint set. In that case, the sequence $(I_n)_n$ can be bounded, and we obtain the necessary optimality conditions of the minimal time of crisis problem.
- If the optimal path has at least a purely one purely tangent crossing time, then the sequence $(I_n)_n$ is not bounded.

These results extend the existing results in the literature that require an optimal path to have transverse crossing times for stating necessary optimal conditions.

Perspectives. There are several perspectives related to the first part. The model considered in the first part of the thesis is better associated with greenhouse-grown crops. One of the points that could be developed concerns water drainage and rainfall inputs in the soil humidity evolution. The plant needs water and nitrogen in variable proportions during its development; another perspective to consider is adding a nitrogen variable into the crop model.

We have seen in Chapter 5 that derivation of the necessary optimality conditions is strongly related to the sequence $(I_n)_n$, which in turn is related to the chosen regularization. We have proved that if $(I_n)_n$ is bounded, then the crossing times are at least one-sided transverse; however, we did not manage to show that it is reciprocal. A natural perspective is to investigate this reciprocal propriety. Another perspective is to consider a non-autonomous regularization that depends on the behavior of the optimal path on the boundary of the set K.

To the best of our knowledge, optimality conditions have not been considered when the trajectories hit the boundary of the constrained sets tangentially in a general hybrid optimal control problem. A perspective to investigate could be to use the procedure developed in Chapter 5 to obtain an extension of necessary optimality conditions for more general hybrid problems when an optimal trajectory one-sided transverse crossing time.

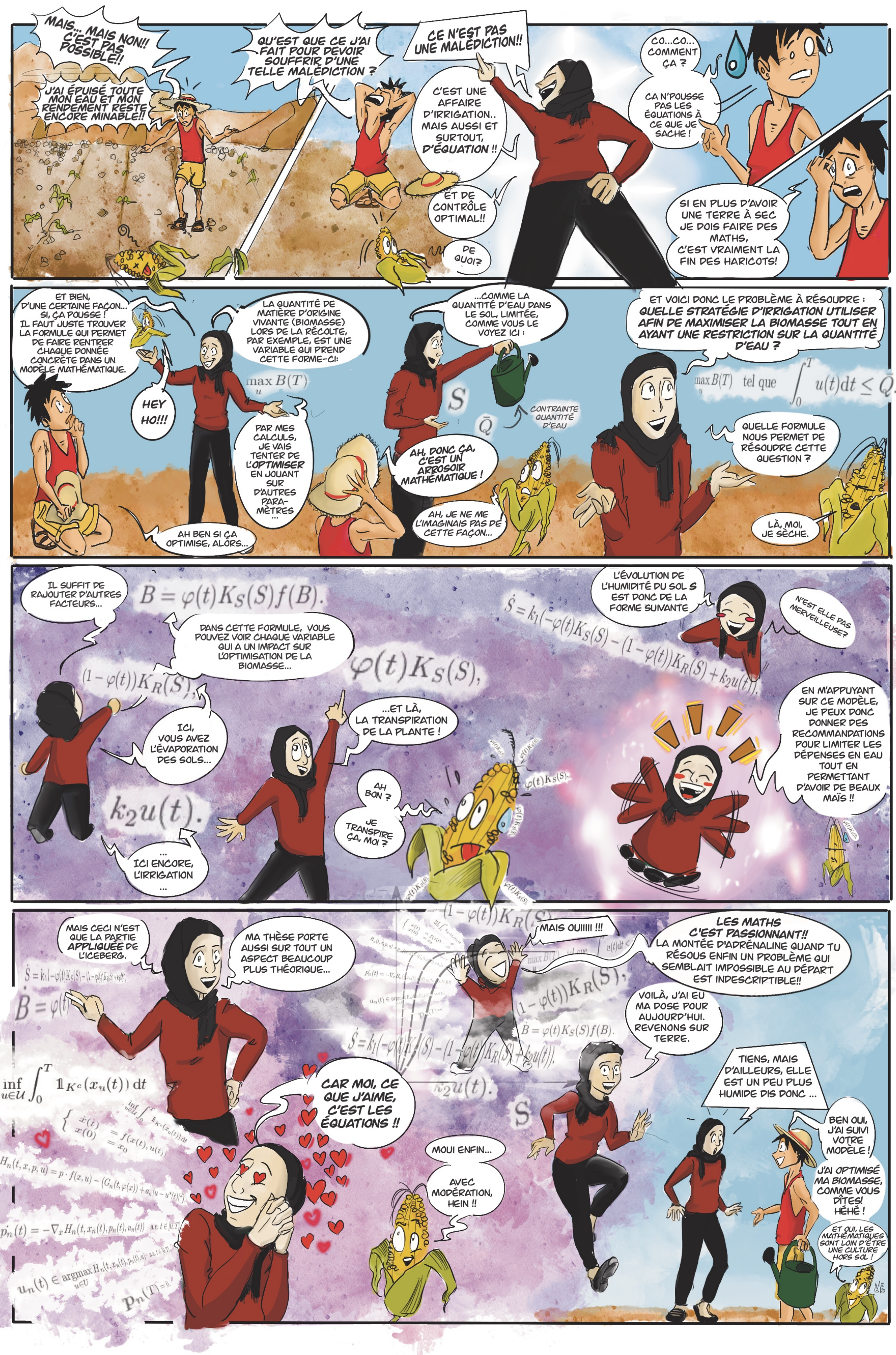


FIGURE 6.1 – The comic board was realized by Morgane Arrieta and Guillaume Bagnolini within the framework of the activities of the association Cosciences for the doctoral college of the university of Montpellier in collaboration with Kenza Boumaza.

BIBLIOGRAPHIE

[1] R. Allen, L. Pereira, D. Raes and M. Smith , *Crop evapotranspiration - Guidelines for computing crop water requirements - FAO Irrigation and drainage paper 56*, Fao, Rome, 1998.

[2] S.M. Aseev and A.I. Smirnov, *Necessary first-order conditions for optimal crossing of a given region*, Comput. Math. Model., vol. 18, 4, pp. 397–419, 2007.

[3] J.-P. Aubin, *Viability Theory*, Systems & Control: Foundations & Applications. Birkhäuser Boston, 1991.

[4] J.-P. Aubin, A.M. Bayen and P. Saint-Pierre, *Viability Theory, New Directions*, Second Editions, Springer, Heidelberg, 2011.

[5] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuuser, Boston, 1997.

[6] G. Barles, A. Briani and E. Trélat, *Value function for regional problems via dynamic programming and Pontryagin maximum principle*, Math. Control Relat. Fields, vol. 8, 3&4, pp. 509–533, 2018.

[7] T. Bayen, K. Boumaza and A. Rapaport, *A new proof of optimality conditions for the time of crisis via regularization*, 2021. hal-03153881v1.

[8] T. Bayen and L. Pfeiffer, *Second-order analysis for the time crisis problem*, Journal of Convex Analysis, vol. 27, 1, pp. 139–163, 2020.

[9] T. Bayen and A. Rapaport, *About Moreau-Yosida regularization of the minimal time crisis problem*, J. Convex Anal. 23, 1, pp. 263–290, 2016.

[10] T. Bayen and A. Rapaport, *About the minimal time crisis problem*, ESAIM Proc. Surveys , EDP Sci., vol. 57, pp. 1–11, 2017.

[11] T. Bayen and A. Rapaport, *Minimal time crisis versus minimum time to reach a viability kernel : a case study in the prey-predator model*, Optim. Control Appl. Meth., vol. 40, 2, pp. 330–350, 2019.

[12] N. Bertrand, S. Roux, O. Forey, M. Guinet and J. Wery, , *Simulating plant water stress dynamics in a wide range of bi-specific agrosystems in a region using the BISWAT model*, European Journal of Agronomy, vol. 99, 116–128, 2018.

[13] P. D. Brown, T. A. Cochrane, T. D. Krom, *Optimal on-farm irrigation*, scheduling with a seasonal water limit using simulated annealing, Agricultural Water Management 97(6):892–900 (2010).

[14] P. Bettiol and H. Frankowska, *Hölder continuity of adjoint states and optimal controls for state constrained problems*, Appl. Math. Optim., 57(1):125–147, 2008.

[15] B. Bonnard, L. Faubourg, G. Launay, and E. Trélat, *Optimal control with state constraints and the space shuttle re-entry problem*, J. Dynam. Control Systems, 9(2):155–199, 2003.

[16] J. F. Bonnans, V. Gaudrat and C. Saguez, *A domain control approach to state constrained control problems*, Notes in Control and Information Sciences, (100):72–90, 1988.

[17] J. F. Bonnans, D. Giorgi, B. Heymann, P. Martinon and O. Tissot, *Bocop HJB 1.0.1 - User Guide.* Technical Report INRIA 0467, 2015.

[18] J. F. Bonnans, C. Moreno and C. Saguez, *Contrôle de domaines temporels*, INRIA research report, 1984.

[19] N. Bonneuil, K. Müllers *Viable populations in a prey-predator system.* J Math Biol. 35(3):261-293, 1997.

[20] K. Boumaza, T. Bayen and A. Rapaport, *Penalty function method for the minimal time crisis problem,* ESAIM: PROCEEDINGS AND SURVEYS, Vol. 71, p. 21-32, August 2021.

[21] K. Boumaza, N. Kalboussi, A. Rapaport, S. Roux, C. Sinfort,*Optimal control of a crop irrigation model under water scarcity*, Optim Control Appl Meth. 1– 20, 2021.

[22] K. Brumbelow and A. Georgakakos, *Determining crop-water production functions using yield-irrigation gradient algorithms.* Agricultural Water Management, vol. 87, 151–161, 2007.

[23] P. E. CAINE, F. H. CLARKE, X. LIU AND R. B. VINTER, *A Maximum Principle for Hybrid Optimal Control Problems with Pathwise State Constraints*, Proceedings of the 45th IEEE Conference on Decision and Control, 4821–4825, 2006.

[24] P. E. CAINES AND M. S. SHAIKH, *Optimality Zone Algorithms for Hybrid Systems Computation and Control: From Exponential to Linear Complexity*, Proceedings of the 2005 IEEE International Symposium on, Mediterrean Conference on Control and Automation Intelligent Control, 1292–1297,2005.

[25] B. CHEVIRON, R. W. VERVOORT, R. ALBASHA, R. DAIRON, C. LE PRIOL AND J. MAILHOL , *A framework to use crop models for multi-objective constrained optimization of irrigation strategies*, Environmental Modelling & Software, 86, 145–157, 2016.

[26] F. H. CLARKE, *Functional Analysis, Calculus of Variation, Optimal control*, Graduate Texts in Mathematics, 264, Springer, London, 2013.

[27] F. H. CLARKE AND R. B. VINTER, *Optimal Multiprocesses*, SIAM Journal on Control and Optimization, 27(5): 1072–1091, 1989.

[28] F. H. CLARKE, YU.S. LEDYAEV, R.J. STERN, P.R. WOLENSKI, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics, Springer, 1998.

[29] R. COURANT, *Variational methods for the solution of problems of equilibrium and vibrations*, Bull. Amer. Math. Soc., 49(1):1–23,01 1943.

[30] A.V. DMITRUK, *The hybrid maximum principle is a consequence of Pontryagin maximum principle*, Systems Control Lett. 57, 11, pp. 964–970, 2008.

[31] A.V. DMITRUK AND A. M. KAGANOVICH, *Maximum principle for optimal control problems with intermediate constraints*, Comput. Math. Model., vol. 22, 2, pp. 180-215, 2011.

[32] A.V. DMITRUK AND A. M. KAGANOVICH, *Quadratic order conditions for an extended weak minimum in optimal control problems with intermediate and mixed constraints*, Discrete Contin. Dyn. Syst. 29, 2, pp. 523–545, 2011.

[33] A. DOMINGUEZ, J. DE JUAN, J. TARJUELO, R. MARTINEZ, A. MARTINEZ-ROMERO, *Determination of optimal regulated deficit irrigation strategies for maize in a semi-arid environment*, Agricultural Water Management 110,67–77, (2012).

[34] L. DOYEN, P. SAINT-PIERRE, *Scale of viability and minimal time of crisis*, Set-Valued Anal. 5, pp.227–246, 1997.

[35] J. DOORENBOS AND A. KASSAM, *Yield response to water*, FAO Irrigation and drainage paper (33), 257, 1979.

[36] T. FAULWASSER, M. KORDA, C. JONES, AND BONVIN, *On turnpike and disspativity properties of continuous-time optimal control problems*, Automatica, 81, 297–304, 2017.

[37] A. F. FILIPPOV,*Differential Equations with Discontinuous Righthand Sides*. Kluwer, Dordrecht, 1988.

[38] M. GARAVELLO, B. PICCOLI, *Hybrid necessary principle*, SIAM J. Control Optim., vol. 43, 5, pp. 1867–1887, 2005.

[39] B. S. GOH, G. LEITMANN, TL. VINCENT, *Optimal control of a prey-predator system*, Math Biosci, 19(3-4):263-286, 1974.

[40] B. S. GOH, *Management and Analysis of Biological Populations*, Amsterdam, The Netherlands: Elsevier, 1980.

[41] T. HABERKORN, E. TRÉLAT, *Convergence results for smooth regularizations of hybrid nonlinear optimal control problems*, SIAM . J. Control Optim., vol. 49, 4, pp. 1498–1522, 2011.

[42] R.F. HARTL AND G. FEICHTINGER , *A new sufficient condition for most rapid approach paths*, Journal of Optimization Theory and Applications, Vol; 54 (2), pp. 402–411, 1987.

[43] R. F. HARTL, S. P. SETHI, AND R. G. VICKSON, *A survey of the maximum principles for optimal control problems with state constraints.* SIAM Rev., 37(2):181–218, 1995.

[44] H. HERMES AND J.P. LASALLE, *Functional analysis and time optimal control*, Mathematics in Science and Engineering, Vol. 56, Elsevier, 1969.

[45] J.B. HIRIART-URRUTY AND C. LEMARÉCHAL, *Fundamentals of Convex Analysis*, Springer 2001.

[46] M. E. JENSEN, *Water consumption by agricultural plants*, (chapter 1), 1968.

[47] N. KALBOUSSI, S. ROUX, B. CHEVIRON, J. HARMAND, A. RAPAPORT, C. SINFORT *Apport de la modélisation pour l'aide à la décision en vue de la réutilisation agricole des eaux usées traitées*, Journal International Sciences et Techniques de l'Eau et de l'Environnement, Association Scientifique et Technique pour l'Eau et l'Environnement en Tunisie, 3(1):102-107, 2018.

[48] N. KALBOUSSI , S. ROUX, K. BOUMAZA, C. SINFORT, AND A. RAPAPORT , *About modeling and control strategies for scheduling crop irrigation.* IFAC Workshop on Control Methods for Water Resource Systems CMWRS, Delft (Netherlands), IFAC-PapersOnLine, Vol. 52(23), pp. 43–48, 2019.

[49] S. KLOSS, R. PUSHPALATHA, K. J. KAMOYO, N. SCHÜTZE, *Evaluation of crop models for simulating and optimizing deficit irrigation systems in arid and semi-arid countries under climate variability* Water resources management 26(4), pp. 997–1014, 2012.

[50] Z. LI, P.H. COURNÈDE AND J.C. MAILHOL , *Irrigation Optimization by Modeling of Plant-Soil Interaction*, 2011 Proceedings of the Conference on Applied Simulation and Modelling, 2011.

[51] D. LIBERZON, *Calculus of Variations and Optimal Control Theory: A Concise Introduction*, Princeton University Press, USA. 2011.

[52] S. O. LOPES, F. A. C. C. FONTES, A. C. D. CALDEIRA, AND R. M. S. PEREIRA, *Optimal control of irrigation with field capacity modes: Characterizing the minimal water consumption solution*, In 2018 13th APCA International Conference on Automatic Control and Soft Computing (CONTROLO), pages 300–305, 2018.

[53] S. O. LOPES , F. A. C. C. FONTES, R. M. S. PEREIRA, MDR. DE PINHO, AND A. M. GONÇALVES , *Optimal Control Applied to an Irrigation Planning Problem*, Mathematical Problems in Engineering, 2016.

[54] A. MIELE , *Extremization of Linear Integrals by Green's Theorem*, Optimization Techniques, Edited by G. Leitmann, Academic Press, New York, New York, pp. 69-98, 1962.

[55] K. OKAMURA, *Some Mathematical Theory of the Penalty Method for Solving Optimum Control Problems*, SIAM J. on Control, Series A, 2(3):317-331, 1965.

[56] OPTIRRIG, logiciel pour la génération, l'analyse et l'optimisation des scénarios d'irrigation des cultures : http://www.irstea.fr/innovations/optirrig-logiciel-generation-analyse-optimisation-scenarios-irrigation

[57] N. Pelak, R. Revellia and A. Porporato , *A dynamical systems framework for crop models: Toward optimal fertilization and irrigation strategies under climatic variability*, Ecological Modelling, Vol. 365, pp. 80-92, 2017.

[58] L.S. Pontryagin, V.G. Boltyanskiy, R.V. Gamkrelidze, E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*, The Macmillan Co., New York 1964.

[59] R. Rapaport and P. Cartigny , *Turnpike theorems by a value function approach*, ESAIM Control, Optimizationa and Calculus of variations, 10, 123-141, 2004.

[60] D. L. Russell , *Penalty Functions and Bounded Phase Coordinate Control*, SIAM J. on Control, Ser. A, 2(3):409-422, 1964.

[61] M. S. Shaikh and P. E. Caines, *On the Hybrid Optimal Control Problem: Theory and Algorithms*, IEEE Transactions on Automatic Control, 52(9):1587–1603, 2007.

[62] N. Schütze, G. H. Schmitz , *New planning tool for optimal climate change adaption strategies in irrigation*, Journal of Irrigation and Drainage Engineering, 136(12), pp. 836-846, 2010.

[63] N. Schütze, M. de Paly and U. Shamir , *Novel simulation-based algorithms for optimal open-loop and closed-loop scheduling of deficit irrigation systems*, Journal of Hydroinformatics, Vol. 14(1), 136-151, 2011.

[64] U. Shani, Y. Tsur and A. Zemel, *Optimal dynamics irrigation schemes*, Optimal Control, Applications and Methods, 25, 91–106, 2004.

[65] C. Silva, E. Trélat, *Asymptotic approach on conjugate points for minimal time bang–bang controls*, Systems Control Lett., vol. 59, 11, pp. 720–733, 2010.

[66] A. Smirnov *Necessary optimality conditions for a class of optimal control problems with discontinuous integrand*, Proceedings of the Steklov Institute of Mathematics, vol. 262, 1, pp. 213–230, 2008.

[67] M. SPENCE AND D. STARRETT , *Most Rapid Approach Paths in Accumulation Problems*, International Economic Review, Vol. 16, pp. 388–403, 1975.

[68] P. STEDUTO, T. C. HSIAO, E. FERERES, D. RAES, *Fao irrigation and drainage paper 66*, Crop yield response to water, Food and Agriculture Organization of the United Nations, (2012).

[69] P. STEDUTO, H. HSIAO, D. RAES AND E. FERERES , *AquaCrop, The FAO crop model to simulate yield response to water: I. Concepts and underlying principles*, Agronomy Journal, 101(3), 426–437, 2009.

[70] H. J. SUSSMANN, *A maximum principle for hybrid optimal control problems*, Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No.99CH36304), 425–430 vol.1, 1999.

[71] D. RAES, P. STEDUTO, T. C. HSIAO, E. FERERES, *Aquacrop—the fao crop model to simulate yield response to water: Ii. main algorithms and software description*, Agronomy Journal 101(3):438–447, (2009).

[72] Team Commands, Inria Saclay, BOCOP: an open source toolbox for optimal control. http://bocop.org

[73] E. TRÉLAT AND E. ZUAZUA , *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations, 258, 81–114, 2015.

[74] G. VICO AND A. PORPORATO , *Probabilistic description of crop development and irrigation water requirements with stochastic rainfall*, Water Resources Research, 49, 3, 1466–1482, 2013.

[75] R. VINTER, *Optimal Control*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 2000.

[76] W. WALTER, *Ordinary differential equations*, Springer, 1998.

[77] Y. WEN, S. SHANG, J. YANG, *Optimization of irrigation scheduling for spring wheat with mulching and limited irrigation water in an arid climate*, Agri- cultural Water Management, 192:33–44, (2017).

[78] A.-Q. XING AND C.-L. WANG, *Applications of the exterior penalty method in constrained optimal control problems*, Optimal Control Applications and Methods, 10(4):333–345, 1989.

[79] S. YOSIDA , *An optimal control problem of the prey-predator system*, Funkcialaj Ekvacioj, 25:283-293, 1982.

[80] LI. ZHONGPING *Modélisation de la croissance des plantes en interaction avec la ressource en eau et contrôle optimal de l'irrigation.* Autre. Ecole Centrale Paris, 2013. Français.

[81] M. I. ZELIKIN, V. F. BORISOV, *Theory of Chattering Control*, Systems & Control: Foundations & Applications, Birkhäuser, 1994.

[82] A. ZEMEL, *On turnpike solutions of some optimal growth problems*, IFAC Proceedings, 42(2):82–84 (2009).