Mathematical modelling and analysis of spatio-temporal models dedicated to tree-grass dynamics in humid savannas

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Mathematical modelling and analysis of spatio-temporal models dedicated to tree-grass dynamics in humid savannas

by :

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Dedication

To my wife and my children.

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Contents

Abstract

In wet ecosystems vegetation biome encompasses self organized physiognomies that describe complex dynamical processes ranging from homogeneous distributions of forests, grasslands and savannas, to heterogeneous distributions of trees and grasses. Spatio-temporal patterns of vegetation are characteristic features of wetland ecosystems and are observed on all continents. The development of a better understanding of their spatial dynamics is an issue of significant ecological and socio-economical importance for global climate regulation and the supply of necessary materials for human. Mathematical modelling is a useful tool to describe dynamics of complex systems, and several mathematical models have been devoted to the study of tree-grass dynamics in savanna ecosystems, but with a scarcely attention, of spatial mechanisms of tree and grass interactions. This work, dedicated to the modelling and analysis via partial differential equations of tree-grass dynamics in humid savannas, is divided in two main parts. In the first part, we propose and analyse a spatio-temporal model of tree-grass interactions in humid savannas. This first model is based on two integro-differential reaction-diffusion equations with, the reaction part of the model including intra and inter-specific competition kernels as well as a kernel acting indirectly as facilitation term that describes the reduction in tree mortality related to fires. The diffusion part is modelled via the Laplace operators. A qualitative analysis of this model reveals several ecological thresholds that shape the overall dynamics of the model. Thanks to linear stability analysis, the model accounts for the occurrence of space inhomogeneous solutions. All of these lead us to conclude that, the interplay between nonlocal competition and nonlocal facilitation can explain the spatial periodic physiognomy of vegetations observed in humid savannas. In the second part of this work, we consider nonlocal seed dispersal in order to describe the spatial propagation of both tree and grass biomasses. We therefore replace, the Laplace operator by integral operators and focus on the existence of travelling wave connecting the grassland homogeneous steady state to the forest homogeneous steady state . A qualitative analysis of this reaction-dispersion model leads to the characterisation by a mathematical expression depending on several model parameters of the minimal wave speed that controls the forest encroachment into the grassland. We therefore find that, the length of tree seed dispersal and the fire frequency can control the wave propagation.

Keywords : Savanna – Fire – Partial Differential Equation – Qualitative Analysis – Travelling Wave – Nonlocal Competition – Nonlocal Facilitation.

Résumé

DAns les écosystèmes humides le biome de végétation englobe des physionomies auto-organisées qui décrivent des processus dynamiques complexes allant des distributions homogènes de forêts, de prairies et de savanes, à des distributions hétérogènes d'arbres et d'herbes. Ces physionomies de végétations sont des traits caractéristiques des écosystèmes en zones humides et sont observées sur tous les continents. Le développement d'une meilleure compréhension de leur evolution spatio-temporelle est un enjeu d'importance écologique et socio-économique considérable, pour la régulation du climat mondial et de l'approvisionnement en matériaux nécessaires à l'homme. La modélisation mathématique est un outil utile pour décrire la dynamique de systèmes complexes, et plusieurs modèles mathématiques ont été consacrés à l'étude des dynamiques "arbres-herbes"dans les écosystèmes de savanes, mais ceci, avec une attention limitée sur les mécanismes spatiaux entre arbres et herbes. Ce travail, qui consiste en la modélisation et l'analyse via des équations aux dérivées partielles des dynamiques arbres/herbes dans les savannes humides, est divisée en deux parties principales. Dans la première partie, nous analysons un modèle spatio-temporel d'interactions arbres-herbes en zones de savanes humides. Ce premier modèle est basé sur deux équations de réaction-diffusion intégro-différentielles avec, dans la partie réaction du modèle des noyaux de compétition intra et inter spécifiques et un noyau agissant indirectement comme terme de facilitation en décrivant la réduction de la mortalité des arbres liée aux feux. La partie diffusion est modélisée via l'opérateur de Laplace. Une analyse qualitative de ce modèle révèle plusieurs seuils écologiques qui régulent la dynamique globale du modèle. Grâce à l'analyse de stabilité linéaire, le modèle rend compte de l'existence de solutions inhomogènes en espace. Ceci nous conduit à conclure que la présence conjointe de compétition non-locale et facilitation non-locale conduit à une structuration spatiale périodique dans les savanes humides. Dans la deuxième partie de ce travail, nous considérons la dispersion non locale des graines pour décrire la propagation spatiale de la biomasse des arbres et des graminées. Nous remplaçons donc les opérateurs de Laplace par des opérateurs intégraux et nous nous focalisons sur l'existence d'une onde progressive reliant l'état stationnaire homogène de prairie à l'état stationnaire homogène de forêt. Une analyse qualitative de ce modèle de réaction-dispersion nous permet de caractériser, par une expression mathématique dépendant de plusieurs paramètres du modèle, la vitesse minimale de l'onde progressive qui contrôle l'avancée de la forêt dans la prairie. Ainsi nous parvenons à la conclusion selon laquelle la longueur de dispersion des graines d'arbres et la fréquence des feux peuvent contrôler la vitesse de l'onde progressive.

Mots Clés : Savane – Feu– Equation aux Dérivées Partielles– Analyse qualitative – Onde Progressive – Compétition non-locale – Facilitation non-locale.

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General introduction

THe landscape structure in tropical ecosystems is principally characterized by three forms of physiognomies: (1) grassland described by a total dominance of grass cover, (2) forest characterised by a predominance of tree cover and (3) savanna defined as a mixture of tree and grass without one species excluding the other. All of these physiognomies are economically and ecologically important. For example, forests play a crucial role in regulating the global climate by storing carbon. Savannas provides for human materials needs by producing a range of resources including : (i) fodder for commercial agriculture and substance livestock farming; (ii) habitat for wildlife; (iii) a wide range of products such a wild foods, medicinal plants, fuelwood, construction materials. Grasslands are important because they provide the food base for grazing livestock. Moreover, such livestock also provides products such as fertilizer, transport, fibre and leather. Therefore, all of these ecosystems are typically important and then, the maintenance of each specificity is challenging due to anthropogenic global changes.

Occurring in many regions around the world, savannas are observed in a large range of mean-annual precipitation (MAP). In Africa, they particularly occur between 100mm and 1500mm (and sometimes more) of total MAP and occupy around 50% of the land area. However, within specific stretches of the rainfall, vegetation may sometimes exhibit, plausibly self-organized physiognomies also termed as vegetation mosaics. The self-organization of plants can result in patterns of different shapes including gap patterns, labyrinths, and spot patterns. Indeed as pointed out by Yatat Djeumen et al. [123], there are several empirical evidence that highlight the existence of vegetation pattern in wet ecosystems. These mosaics of vegetations display dense clusters of shrubs, grasses or trees that can be interpreted as regular spot structures or localized structures. Many researches are therefore focus on explaining how spatial patterns in savannas appear. They involved either desert versus herbaceous or woody vegetation in arid and semi arid regions (Lefever and Lejeune [62], Couteron and Lejeune [19], Pueyo et al. [79]) or grassland/savanna versus forest in humid regions (Yatat Djeumen et al. [122], Tega II et al. [103]). Consequently, the different class of vegetation mosaics, which can be characterized by how much rainfall they typically received should be affected by different set of processes. Humid savannas for example are characterized by frequent and intense fires. Fire in humid savannas, are know to prevent or at least delay the development on woody vegetation. Hence, they prevent trees and shrubs to depress grass production though competition for light and nutriments. The grass-fire feedback is widely acknowledged in literature as a force able to counteract the asymmetric competition of tree on grass, at least for climatic conditions within the savannas biomes that enables grass production. According to Accatino et al. [3], fire is crucial for preventing canopy closure in site where abundant mean annual rainfall would sustain forest. Additionally, fire can induce facilitation mechanisms due to protection effect, more precisely where vulnerable juvenile trees placed near to adults trees are protected from fire.

The Landscape structure is also correlated with tree-grass intra and inter-specific competition. It is now acknowledged that, the interplay between positive and negative feedbacks, are the main process for self-organisation (process leading to the appearance of spatial structures), principally in stressed environmental areas like arid and semi-arid savannas. In humid ecosystems, the physiognomies of vegetations should be jointly depend on facilitation mechanisms due to frequent/recurrent fires and on intra and inter-specific competition between trees and grasses (Tega II et al. [103])

Different research teams have reached useful and meaningful results by modelling vegetation dynamics via ordinary differential equation (ODE) and partial differential equation (PDE). Indeed, tree-grass interactions in savanna ecosystem have been mostly modelled though framework that implicitly acknowledge space (Tchuinte Tamen et al. [102], Yatat Djeumen et al. [125], Touboul et al. [107], Yatat Djeumen et al. [124]). Nevertheless, most of works using PDE formalism, have been carried out in relation to arid/semi-arid vegetation (where water is the most limited factor and fire are infrequent), in order to explain emergence of spatially periodic vegetation pattern. Based on ecological facts, and due to the scarcity of works taking into account spatial mechanisms for tree and grass dynamics in humid savannas, **this dissertation aims to build, study and discuss spatio-temporal models of the tree-grass dynamics in wet savannas**. We will specifically focus on :

(i) Self-organisation of vegetation, principally on the construction and analyse of a mathematical PDE-like model, allowing to illustrate the spatial structuring of vegetation in wet savannas zones where regular spot pattern (trees groves) have been causally reported in the presence of regular and intense fires.

(ii) Boundary dynamics in humid savannas, notably by the phenomena of forest encroachment into grassland area.

The works is organized follows:

- In the first chapter, we describe the context and motivation of our study. We therefore provide the ecological mechanisms recognized in literature to explain the tree-grass coexistence observed in humid savannas. In this vein, we present a non exhaustive literature review of mathematical works, that highlight the tree-grass dynamics in savannas, in order to show what has already been done and what may remain to be done in this area of research.

- The second chapter deals with mathematical tools for the analysis of systems of two reaction-diffusion equations. We put a particular emphasize on the monotone method (upper and lower solution method) when in a first step the reaction term is quasimonotone and in the second step when the reaction term does not satisfies any quasi-monotone property.

- In the third chapter, we propose and analyse a spatial explicit model, allowing to reach spatial pattern (namely regular spots and localized structures) which have been reported in fire prone savannas. This model relies on two non-local reaction diffusion equations with kernels of intra and interspecific interactions for woody and grassy biomasses. A qualitative analysis of the model reveals that monostability of the forest, grassland and savanna space-homogeneous steady states, and multistabilities are possible depending on some ecological thresholds. Thanks to nonlocal biomasses interactions, the model account for the occurrence of space inhomogeneous solutions including a possible spatial periodic structuring. Specifically, we find a range of nonlocal interactions (depending on model parameters) for the appearance of space inhomogeneous solutions. We therefore characterized the wavelength, when the inhomogeneous solution is periodic. We finally present numerical simulations to illustrate our theoretical results and we verify that the computed spatial wavelengths are in good agreement with the predictions from the theoretical analysis.

- In chapter four, we focus on the process of forest encroachment in savanna/grassland due to long seed dispersal. We therefore construct a reaction dispersion model, by considering nonlocal dispersal terms in both dynamics of tree and grass biomasses. We focus, on the existence of travelling wave and the minimal wave speed, connecting the forest and the grassland homogeneous steady state of the model. We mathematically establish the existence of the travelling wave through the construction of a truncate problem combining the upper and lower solutions method with the Schauder fixed point theorem. The characterisation of critical (minimal) wave speed obtained, gives some information on model parameters that are able to control the propagation of the wave. Our results imply that the increase of the length of tree-seed dispersal can accelerate the wave speed and the increase of fire frequency can slow down the process of forest encroachment.

CHAPTER ONE

ECOLOGICAL BACKGROUND AND LITERATURE REVIEW FOR MATHEMATICAL MODELLING OF TREE-GRASS DYNAMICS IN HUMID SAVANNAS

In this chapter, we provide a non-exhaustive literature review of savanna models, theories, and field-based observations and studies. Particular attention is paid to modeling methods and underlying assumptions, and how these have evolved over time.

1.1 Physiognomies of vegetations in wet ecosystem

Savannas are widely defined as ecological vegetations systems characterized by the long term coexistence of continuous grass cover and scattered or clustered trees. They have been also identified by biogeographers, as the biomes corresponding to warm mean annual temperatures $(> 20°C)$ and a broad range of intermediate mean annual rainfall $(100 - 2000\ \ mm)$ (Sarmiento [86], Youta Happi [126], Yatat Djeumen et al. [123], Abbadie et al. [1]). Covering a high proportion of the global terrestrial land surface and thus have a significant role in earth-atmosphere feedback processes (Woodward et al. [118], Bond [13]), savannas occupy $ca.$ 12% of the global land surface (February and Higgins [41]) and $ca.$ 50% of the land area in Africa. Therefore, different classes of savannas, can be characterized by how much rainfall they typically receive. Savannas receiving less than $650\ \ mm$ mean annual rainfall (MAR) has been classified as arid or semi arid savannas. Savannas areas receiving between $650 - 1000\ \ mm$ MAR are considered as mesic savannas and finally savannas receiving more than $1000\ \ mm$ MAR has been identified as humid savannas (Staver et al. [98]). All of these classes of savannas should be affected by different modalities of tree-grass coexistence and these modalities have been the subject of several researches in ecology known as savanna problem: "how trees and grasses coexists over a wide climatic, edaphic and historical condition? (see Sankaran et al. [84], Higgins et al. [51], Yatat Djeumen et al. [125],Yatat Djeumen et al. [121]).

In humid tropical ecosystems, the structure of landscape is in majority composed by these three types of physiognomies: **forest** characterized by a predominance of tree layer, **grassland** characterized by a grass layer total dominance and finally **savanna defined as the mixture (or coexistence) of both trees and grasses species**. However, due to external factors of disturbances like climatic changes, fires , herbivory, human actions, the uniform distribution of these three different types of physiognomies can appear in a non uniform way (like we can see in figure 1.1). In fact, in many regions around the world, vegetation may sometimes exhibits spectaculars organized spatial features that can either be periodic (see panel (a) of figure 1.1) or random (see panel (b) of figure 1.1) termed as patchy vegetations or vegetation mosaics (patterns). Then, empirical evidences suggest that vegetation mosaics in humid regions barely feature periodic patterns and display dense cluster of shrubs, grasses or trees that can be interpreted as regular spot structures or localized structures (Tlidi et al. [106],Tega II et al. [103]). Most often, they are aperiodic but, with quite sharp boundaries like isolated groves or savanna patches encircled by forests (Tega II et al. [103]). The study of vegetation patterns, is motivated by their widespread occurrence in many humid landscapes cover around the world and by the possibility to infer from their presence features, informations on the underlying process, including the susceptibility of the system to abrupt shifts to savanna and/or grassland to forest.

(a) Pattern of spots of forest vegetation within a grassland matrix as observable in Zambia (image from 01/05/2014 accessed on Google Earth®).

(b) Pattern of spots of forest vegetation within a savanna matrix as observable in Cameroon (Mpem-Djim National Park) from an UAV-borne photograph taken on 16/12/2019, P. Couteron).

Figure 1.1: Some vegetation mosaics of trees and grasses in Zambia and in Cameroon.

Also interesting phenomenon observed in wet ecosystem is woody encroachment. For example Jeffery et al. [55] reported in ‶La Lopé National Park ″in Gabon that without fire event, forest invade the grassland mosaics of grassland-forest (see figure 1.2). This changing balance of trees relatively to grasses has been classed as a form of land-degradation (Devine et al. [25]). In fact, the suppression of grasses by encroaching tree species, which are often unpalatable to domestic livestock can have negative impact on livelihoods. Large scale vegetations changes has also consequences for energy, carbon and water budgets (Woodward et al. [118]). Moreover, changes in the composition of savannas are particularly important in Africa, which hosts a large and rapidly growing proportion of the world human population where, many of whom are pastoralists (Scholes and Archer [91]). Precisely woody encroachment for example, has profound implications for biodiversity: it decreases landscape heterogeneity, reducing the diversity of invertebrates, birds and large mammals (Devine et al. [25]).

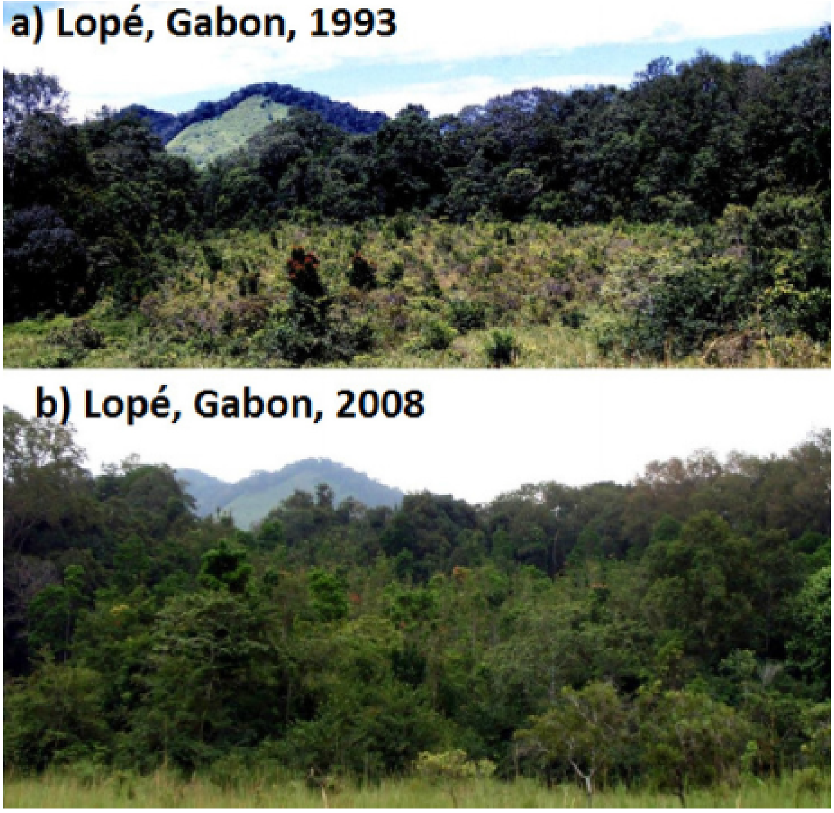

Figure 1.2: Forest encroachment in savanna patch in Lope National park

Vegetation transitions in tropical ecosystems are classically described as resulting from an interplay of process depending on: rainfall, soil moisture, herbivory and fire. In fact, below 650 mm MAR, fire and rainfall are rare, there is not enough moisture in most year to produce continuous fuelbeds. Then in this stressed environmental context, the vegetation physiognomies oscillates between desert to tree or grass spot of vegetation physiognomies. In contrast, from above 1000 mm MAR, fire are regularly frequent and intense. Then vegetation landscape principally shift from savanna/or grassland to forest. According to Archibald et al. [6], the dominant factors to control vegetation mosaics, changes from soil moisture in arid savannas to fire and herbivory in mesic and humid savannas. It is therefore acknowledged that, fire is one of the key factors that shape the physiognomy of savannas vegetations, in general, and particularly, in humid savannas where rainfall is sufficient to promote very high grass biomass production which in turn constitutes the principal fuel for fires. But, as a response to the negative impact of fires, trees have developed 'defence' or resilience mechanisms in order to limit or to reduce the fire-induced tree mortality. Indeed the process of tree-tree facilitation or cooperation by promoting the germination of tree's seeds, the recruitment of new trees by improving the conditions under canopy (shading, litter and nutriments, enhanced water infiltration) play a crucial role for reducing or control tree fire mortality.

1.1.1 Fires in humid savannas

Fire is an important process in savanna, acting like a facilitator for the co-existence of trees and grasses. It is generally regarded as the dominant factor of disturbances, preventing canopy closure in humid savannas where rainfalls are sufficiently abundant. In fact in wet savannas, rainfall promote large accumulation of grass biomass, which constitute fuel load for fire in the dry season. According to Goel et al. [45], fire experiments have repeatedly shown that, frequents fires can maintain savanna in region where closed canopy is climatically possible. The overall effect of fire on woody vegetation depends upon on the interaction elements defining the fire regimes including intensity, frequency and season. Then, fire can influence woody and herbaceous vegetation biomass composition, structures and dynamics. More precisely, the transition from a shrub to an adult tree is possible if the young tree undergoes a long interval without fires, to allow it to reach a threshold beyond which he is no longer likely to be killed. Note that trees in humid savannas are fire resistant. Thus once a rod crosses the threshold of fire resistance, it probability of mortality will decline as the stem grows. However, the trees that do not survive to fire have a high probability of regrowth.

The role of fire in the maintenance of structure and function in African savannas, is probably the oldest issue in savanna ecology, but certain aspects remain contentious. One point of view, noting the ubiquitous occurrence of fires in savannas and the tendency for woody plant density in savannas to increase when fires are excluded, concludes that savannas are fire sub-climaxes to woodland or forest. Another view stresses the long history of fire in Africa, with the numerous plant adaptations to surviving fires observed in all the continents in spite of very different floras, and presents fire as a modifier of savanna structure rather than a primary determinant of savanna distribution (Scholes and Archer [91], Bond et al. [15]). According to Abbadie et al. [1], savanna fires are set by man for various purposes (clearing, protection against uncontrolled fires, hunting, grazing management). As a result, fires are frequent, usually occurring every $1-5$ years in wet savannas. They can be considered as relatively mild compared to forest fires. Fires burn the grass layer and the young trees included in it, leaving adult trees alive, affecting tree recruitment but not significantly adult survival. Fuel load typically ranges between 2 and 10 $Mgha^{-1}$. Consequently, understanding the impact of fire on the demography of savanna trees and shrubs is necessary for understanding human impacts in tropical savannas (Hoffmann and Solbrig [54]). At low fuel load there is no fires while above a sufficient grass biomass, fires intensity and potential impact on woody individuals increases rapidly before reaching a saturation (Van Wilgen et al. [112], Scheiter and Higgins [90], Staver et al. [98], Yu and D'odorico [127]).

Approaches to fire management have evolved over the past century and some protected areas have been undergone multiple change of fire management. African savannas are dynamic ecosystem and fire interacts with climatic cycles varying levels of herbivory and increasing pressures brought about by rising human population. Fire characteristics such as the frequency, intensity, seasonality, extend of burn and type of fuel describe the fire regime. In savannas, the intensity and the frequency of fire are both generally what managers try to manipulate in order to maintain particular vegetation structure. Fires of higher intensity (flame length >2 m) caused greater mortality and topkill than fires of lower intensity (flame length <2 m). In African savanna ecosystems, active fire management has been practiced for many decades. Policies and practices have changed as new evidence on the role of fire has emerged (Van Wilgen et al. [113]). Nevertheless, the implementation of a chosen fire regime is difficult, even when agreement on its nature is reached (Van Wilgen et al. [113], Diouf et al. [27]). Unplanned fires (often resulting from factors beyond the control of managers) burn large areas, upsetting agreed-upon fire targets or policy decisions (Van Wilgen et al. [113]). Based on field experiments conducted in the Kruger National Park, South Africa, Govender et al. [46] concluded that the mean fire intensities showed no significant differences between annual burns and burns in the biennial, triennial and quadrennial categories, despite lower fuel loads in annual burns, suggesting that seasonal fuel moisture effects linked to season overrode those of fuel load. Mean fire intensity in sexennial burns was less than half that of other burns. Therefore, managers of African savannas can manipulate fire intensity by choosing the season of fire, and further by burning in years with higher or lower fuel loads in concert to achieve specific vegetation structural objectives (Abbadie et al. [1],Govender et al. [46], Smit et al. [93], Jeffery et al. [55]).

1.1.2 Herbivory in savannas

According to Staver and Bond [96] herbivory is one of the key process structuring vegetations in savanna, especially in Africa where large mammal herbivores communities remain intact. It have been generally proposed as a driver for woody encroachment. Specifically increase grazing reduces grass cover, which will later decrease the fire intensity. Then it is now recognized that grazing may be not detrimental, and even favorable for plants (i.e., the herbivory optimization hypothesis (HOH), McNaughton (McNaughton [69], Abbadie et al. [1]). In particular, herbivory can promote grassland soil nitrogen cycling which strongly influences plant responses to grazing. Furthermore, herbivores can largely influence the temporal changes in tree/grass balance, directly through the reduction in competition intensity or indirectly through the reduction in fire frequency and intensity (Scholes and Archer [91], Van Langevelde et al. [110], Sankaran et al. [84], Sankaran et al. [85], Abbadie et al. [1]). Indeed, browsing herbivores reduce woody vegetation, such that trees are either killed or reduced in size, the effect of elephants being most profound and well known. The combined effect of fire and browsing led to find that fire causes the decline in woody vegetation whereas browsers inhibit recovery (Van Langevelde et al. [110]). On the other hand, grass biomass removal through grazing leads to reduced fuel load, which makes fire less intense and, thus, less damaging to trees; consequently, it may result in an increase in woody vegetation.

1.1.3 Self-organization of vegetation in humid savannas induced by the interplay between competition and facilitation

The self-organization of plant systems in humid savannas relates to a set of processes or interactions leading to the appearance of stable spatial structures of vegetation. These interactions are the result of multiples interactions between trees and grasses, constituent elements of savannas. It is typically a loop of positives and negatives interactions existing simultaneously between trees and/or grasses but presenting different spatial ranges.

1.1.3.1 The concept of competition in humid savannas

The term competition is used in the sense of negative interferences of impacts of one plant over another. In vegetation dynamic, it refers to the rivalry between plants for the use or consumption of resources. The classical competition theory subdivides the process of competition between to main angles: the intra and inter-specific competition, and tends to predicts that, the intra-specific competition should be greater than the interspecific competition because plants of same species are share same requirements of resources. In humid savannas, competition generally takes place for access to light on the one hand, and for access to nutritive resources in the soil on the other hand.

Above the ground surface, the light resource is essential for the growth of plants, and becomes a source of competition when it is limiting. This resource is essential to the growth of plants because it allows them, through the process of photosynthesis, to produce carbon assimilates which are constituents part of various aerial and underground organs. Light is a unidirectional resource. Although the direction of light rays varies during a day, the plant which will be taller than the others, will capture the light while it will cause shading and therefore, a weaker resource availability for lower plants. The plant position in the layers of the canopy, is therefore decisive for access to light resource and gives them, a disproportionate advantage. Then, competition for light is said to be "asymmetrical" by relation to the size of the plants.

Below the surface of the ground, there is a large of resources useful for the plant growth. It contains macro and micro nutrients (such as nitrogen and phosphorus) and water resources. These resources, are not present and accessible as a homogeneous way in the ground . Unlike the light plants have a competitive advantage proportional to their occupation, of the soil by roots: competition for soil resources is said to be "symmetrical" with respect to plant size. The acquisition and use of soil resources, are based on four main processes: the demand of the plant, the transport of resources in the plant, ground exploration in space and ground exploration in time. These processes depend essentially on the structure of the roots system of plants, and their abilities to grow in depth, in breadth and to extract the resource. The request of the plant is dependent on the aerial biomass produced and the acquisition of water by the roots (wich depend on the leaf area deployed,and the evapotranspiration caused by these leaves). Moreover, soil resources can vary in quantity, availability and they are not located in the same way in a soil profile .

1.1.3.2 The concept of facilitation in humid savannas

In the field of ecology, and more precisely in biological interactions, we speak of facilitation to describe the situations where, the presence of a species improves the installation, the life or the survival of others without the relationship of interdependence don't be very strong. The term facilitation encompasses a wide range of mechanisms modifying the availability of resources independent of their direct uptake or influencing environmental conditions. A plant or a group of plants can modify the availability of resources either locally via the improvement of the porosity of the soil or by interaction with the climate.

Traditionally, ecologist have emphasized the role of competition between trees and/or grasses as being the key determinants of savannas structures (Dohn et al. [29] and references therein). While the importance of competition in structuring ecological communities is widely recognized, there is also a growing appreciation of the role of facilitation in structuring social communities. Indeed facilitation is a process that needs to me more integrate in vegetation dynamics, this in relation to the type on environmental context. According to Dohn et al. [29], the higher mean annual temperature (i.e increased environment stress) may increase the importance of facilitative mechanism. However due to the abundance of precipitation that limit tress for plant in humid context, and according to Martinez-Garcia et al. [68], who suggest that fire may be frequent and intense in humid savanna, the mechanism of facilitation in wet area, may played in the process on controlling or reducing the fire effect on tree mortality. Tega II et al. [103] have suggested a procedure based on the increase of tree biomass and the reduction of grass biomass in the area, by the interspecific competition for light induced by tree on grasses. These two process consequently reduce the fire effect on tree mortality.

To highlight the increased of tree biomass, Tega II et al. [103] referred to Li. et al. [66] (and references therein), who suggested, facilitation of trees by other trees via hydrolic lift (movement of water from wet to dry soil layers through tree roots). In fact, tree canopy shade, reduce soil water, and plant water loss, by reducing understorey temperature and evapotranspiration. To highlight the process to tree-grass interspecific competition, Tega II et al. [103] have suggested that, tree canopy shade reduce light availability for grasses. In fact, as water availability increase with increased rainfall, light may begin to replace as the limiting factor in grass photosynthetic reactions, thus shading by trees may begin to inhibit subcanopy primary production in wetter environment systems.

1.2 Mathematical modelling of Savanna dynamics

Different teams have reach useful results, by modelling vegetation dynamic via ordinary differential equation (ODE), and partial differential equations (PDE). A line of modelling has addressed grass-tree dynamics in presence of fire mediated interactions. A minimalistic two variables model (Yatat Djeumen et al. [124]) ODE, proved able to provide reasonable predictions of vegetations physiognomies (savannas, grasslands, forests) along the entire rainfall gradient (from equatorial forest to desert).

There has been little use of PDE models, for the topic of spatial vegetation pattern in humid context (but see Yatat Djeumen et al. [122]). Most of the works done using PDE formalism, has been carried out in relation to arid/semiarid vegetation, where patterns of vegetations spot (groves) in flooding plain are observed and has been modelled with an integro-differential equation (Lejeune et al. [65], Martinez-Garcia et al. [68]).

One may remark that, other works with impulsive differentials equations (Tchuinte et al. [100], Yatat Djeumen et al. [121]), and stochastic equations (D'odorico et al. [28], Hanan et al. [48], De Michele et al. [23], Beckage et al. [11]) have addressed the question of tree grass coexistence but some of them are beyond the scope of our works.

1.2.1 Mathematical modelling of savanna dynamics with ordinary differential equations

The question raised by observed or putative dynamics within savanna biome have triggered and increased interest in term of modelling. Pioneering works Walker et al. [116] and Walker and Noy-Meir [115], first used system of Ordinary Differential equations (ODE) to address the particular case of arid fire immune savannas in which, excessive grazing fosters bush encroachment. This line of modelling featured grass and woody biomasses as state variable and aimed at explicitly depicting their interactions in relation to soil moisture dynamics. A such it became a paradigm for interaction model's involving a limited resource. Another line of ODE-based modelling build on the application to savannas of the initial concept of asymmetric competition (Tilman [105]) through a simple framework that allows considering both direct and disturbance-mediated plant interactions. Tilman's framework reinterpretation used two states variables, namely cover fractions of grass (G) and tree (T) assumed exclusive and summing between zero and one was done by Accatino et al. [2].

1.2.1.1 The Accatino et al. (2010) model [2]

Accatino et al. (2010) [2] focused on the domain of stability of tree-grass coexistence with respect to influencing "biophysical" variables (climate, herbivory). Based on Tilman (1994) [105] works, Accatino et al. [2] added additional terms in equations proposed by Tilman (1994) [105]. In fact, Tilman [105] considered two species (P_1 and P_2) model, in which the competition is for habitat sites and he assumed that:

1. The superior competitor P_1 (tree) colonized area where there is not yet a superior competitor.
2. The inferior competitor P_2 (grass) colonized only place not yet occupied by either superior or inferior competitor.
3. The competition between a superior and an inferior competitor has a depressive effect on inferior competitor.

Thus, Tilman (1994) [105] obtained the following system

$$\begin{cases} \dfrac{dP_1}{dt} &= c_1P_1(1-P_1) - m_1P_1, \\ \\ \dfrac{dP_2}{dt} &= c_2P_2(1-P_1-P_2) - m_2P_2 - c_1P_1P_2. \end{cases} \tag{1.1}$$

Therefore, the terms added by Accatino et al. [2] reflect the mortality of grasses and trees due to fires fG and $\delta_f fGT$ respectively where δ_f represents tree fire sensitivity. Moreover, to take into account explicitly rainfall on savanna dynamics Accatino et al. (2010) [2] added a third equation which reflects the dynamics of soil moisture and they assumed that the colonization rates of both trees and grasses are linearly dependent on soil moisture. They proposed

$$\begin{cases} \dfrac{dS}{dt} &= \dfrac{p}{w_1}(1-S) - \varepsilon S(1-T-G) - \tau_T ST - \tau_G SG, \\ \\ \dfrac{dT}{dt} &= \gamma_T ST(1-T) - \delta_T T - \delta_F fGT, \\ \\ \dfrac{dG}{dt} &= \gamma_G SG(1-T-G) - \gamma_T STG - \delta_{G0}G - fG. \end{cases} \tag{1.2}$$

From their model, Accatino et al. [2] obtained stability situations involving grassland, forest and savanna equilibria. They also obtained bistability situations involving also grassland and forest equilibria, and involving savanna and forest equilibria which depend fires and rainfall parameters. Accatino et al. [2] assumed that, vegetation component are mutually-exclusive which is not totally satisfactory. Moreover, considering a linear function between grass and fire is not also in agreement with observations (Scheiter and Higgins [90], Staver et al. [98]). Another critic that can be done for the model proposed by Accatino et al. [2] is that the direct depressive effect of grass on shrubs and seedlings as evidenced in Scholes and Archer [91] is not taken into account.

1.2.1.2 The De Michele et al. (2011) model [22]

Extending the model of Accatino et al. [2] and taking into account considerations made by Van Langevelde et al. [110], De Michele et al. [22] introduced two additional terms in the model proposed by Accatino et al. [2]. In fact, according to Van Langevelde et al. [110], in savannas ecosystems, coexistence between tress and grasses is controlled by fires, browsers and grazers. Then the terms added by De Michele et al. [22] in the Accatino et al. [2] model, reflect the activities of grazers (gG) and browsers (bT). Therefore, they proposed the following model

$$\begin{cases} \dfrac{dS}{dt} = \dfrac{p}{w_1}(1-S) - \varepsilon S(1-T-G) - \tau_T ST - \tau_G SG, \\ \\ \dfrac{dT}{dt} = \gamma_T ST(1-T) - \delta_T T - bT - \delta_F fGT, \\ \\ \dfrac{dG}{dt} = \gamma_G SG(1-T-G) - \gamma_T STG - \delta_{G0} G - gG - fG. \end{cases} \tag{1.3}$$

Using the linear stability analysis, De Michele et al. [22] found the same situations of stability and bistability as identified by Accatino et al. [2] and these (multy) stabilities are regulated by fires, rainfall and herbivory (browsing and grazing) parameters. Nevertheless, since the model proposed and studied by De Michele et al. [22] is and extension of the model studied by Accatino et al. [2], then the same critics done for Accatino et al. [2] model are still valid (see section 1.2.1.1).

1.2.1.3 The Tchuinte et al. (2014) model [102]

Tchuinte Tamen et al. [102] in this paper, questioned the assumption according to which the fire frequency f could be constant like in Accatino et al. [2] and De Michele et al. [22] works. In fact, if most fires start from human ignition (Govender et al. [46]), fire are strongly constrained by available grass fuel and its distribution across space. Then in Tchuinte Tamen et al. [102], they keep the fire frequency f constant and they modulated it by $\omega(G)$, which will stay its low branch as long grass biomass is not sufficient quantity. They proposed and analysed the following simple model of tree-grass dynamics by taking into account fire as continuous events:

$$\begin{cases} \dfrac{dG}{dt} = (\gamma_G - \delta_{G0})G - \mu_G G - \gamma_{TG} TG - \lambda_{fG} fG, \\ \\ \dfrac{dT}{dt} = (\gamma_T - \delta_T)T - \mu_T T^2 - \lambda_{fT} f\omega(G)T, \end{cases} \tag{1.4}$$

with $G(0) = G_0, \;\; T(0) = T_0$, positive initial conditions and $\omega(G) = \dfrac{G^\beta}{G^\beta + \alpha^\beta}$, with α controlling the location of the point where ω is half of its maximum and β controlling the rate of increase of ω. ω is a function of grass biomass which express the causality between grass biomass and fire intensity. Moreover ω is see as an indirect proxy of ignitable dry grass biomass available at the middle of the dry season. In the mathematical analyses of their model, Tchuinte Tamen et al. [102] take $\omega(G)$ as a generic monotonously increasing function of grass biomass and the case were it is on the form of Holling type II or III function is also discuss on their paper.

Their model deals with tree-grass pattern in arid, semi arid and mesic ecosystem and is able to predict several equilibria among which pure cover type i.e bare soil, grassland, forest along with several levels of tree-grass mixtures. Their model featured various bistability situations: between forest and grassland, between forest and one of the tree-grass equilibrium with low tree biomass.

However, Tchuinte Tamen et al. [102] have considered fire as a forcing factor, whose effects are independent of tree size development. In fact, they treated woody cover as a single factor with no distinction between seedlings/sapling which are highly fire sensitive, and mature trees which are largely immune to fire damage. This modelling choice on fire effect done by Tchuinte Tamen et al. [102] is questionable.

1.2.1.4 The Yatat et al. (2014) model [125]

Yatat Djeumen et al. [125] constructed and analysed a mathematical model in order to study the interaction of tree and grass that explicitly makes fire intensity dependant of the grass biomass. Contrary to Tchuinte Tamen et al. [102], and in order to take into account the role of fire in savanna dynamics, Yatat Djeumen et al. [125] considered a tree-grass compartmental mood with one compartment for grasses and two for trees, namely fire-sensitive individuals (like seedlings, sapling, shrubs) and non-sensitive individuals mature trees. To build up their model, Yatat Djeumen et al. [125] considered the following assumptions:

1. The grass vs. sensitive-tree competition has a negative feedback on sensitive tree dynamics.
2. The grass vs. non sensitive-tree has a negative feedback on grass dynamics.
3. After an average time expressed in years, the sensitive tree biomass becomes non sensitive to fire.

Then, the Yatat Djeumen et al. [125] model is given by the following three coupled of nonlinear ordinary differential equations where T_{NS} denotes the class of non-sensitive tree biomass, T_S, the class of sensitive tree and, the class of grass biomass (G):

$$\left\{\begin{aligned} \frac{dT_S}{dt} &= (\gamma_S T_S + \gamma_{NS} T_{NS})\left(1 - \frac{T_S + T_{NS}}{K_T}\right) - (\mu_S + \omega_S + \sigma_G G + f\eta_S \omega(G))T_S, \\ \frac{dT_{NS}}{dt} &= \omega_S T_S - \mu_{NS} T_{NS}, \\ \frac{dG}{dt} &= \gamma_G \left(1 - \frac{G}{K_G}\right) G - (\sigma_{NS} T_{NS} + f\eta_G + \mu_G)\, G, \end{aligned}\right. \tag{1.5}$$

with, $T_S(0) = T_{S_0}$, $T_{NS}(0) = T_{NS_0}$, $G(0) = G_0$, and $\omega(G) = \dfrac{G^2}{G^2 + g_0^2}$, where g_0^2 is the value of grass biomass at which fire intensity reach to half saturation.

Their size structured model reveal three possible equilibria excluding tree-grass coexistence (desert, grassland, forest) along with equilibria for which woody and grassy biomass show durable coexistence (i.e savanna vegetation). They also identified ecological meaningful thresholds that defined in parameter space regions of monostability, bistability and tristability. However it is possible to make this model of Yatat Djeumen et al. [125] more realistic by taking into account, on the one hand and in explicit way, the role of precipitation in the dynamics of trees and grasses biomasses. On the other hand, the factors of cooperation which can exist between trees to reduce the effect of fire.

1.2.1.5 The Yu and D'Odoricco (2014) model [127]

Yu and D'odorico [127] developed an ecohydrological framework to explain the mechanisms underlying grass displacement by woody plant encroachment. They investigated the interaction of woody plants and grasses with soil water and light through a coupled energy and water balance model accounting for the competitive advantage of woody plants over grasses. Yu and D'odorico [127] model assumed that:

1. woody plants have preferential access to soil water and that in the absence of disturbances (e.g., fires) they would outcompete grasses.
2. bottleneck effects associated with the higher susceptibility of woody plant seedlings and saplings to fires and drought affects vegetation dynamics only by slowing down the rate of woody plant establishment and growth(Bond [13]);
3. fire frequency depends only on grass cover and all woody plants have the same susceptibility to fire killings.

Yu and D'odorico [127] model couples the surface energy and soil water balance with vegetation dynamics for two plant functional types: woody plant cover (f_c) and grass cover (f_g).

- The vegetation dynamics:

  Yu and D'odorico [127] proposed the following ordinary differential equations according to tree-grass interactions:

$$\left\{\begin{array}{rcl} \dfrac{df_c}{dt} & = & bf_c\left(1-\dfrac{f_c}{f_{c\max}}\right)-df_c g(f_g), \\ \\ \dfrac{df_g}{dt} & = & \beta f_g\left(1-\dfrac{f_c}{f_{g\max}(f_c)}\right), \end{array}\right. \tag{1.6}$$

  where b and β (per year) are growth coefficients of tree and grass respectively; $f_{c\max}$ and $f_{g\max}(f_c)$ are carrying capacities of tree and grass respectively; d (per year) is a parameter determining the death rate of woody plants by fires; and $g(f_g)$ is the term defined by:

$$g(f_g)=\frac{\eta f_g^2}{1+(\eta-1)f_g^2}. \tag{1.7}$$

  In system (1.6) the carrying capacity $f_{g\max}(f_c)$ is defined by the following expression:

$$f_{g\max}(f_c) = (1 - f_c)d_{bgm} + f_c d_{cgm}, \tag{1.8}$$

where d_{cgm} and d_{bgm} are maximum density of grasses under and between canopies respectively.

- The soil moisture dynamics:

  Rodriguez-Iturbe et al. [82] and Laio et al. [61], Yu and D'odorico [127] considered that the soil moisture dynamics under and between canopies are expressed through soil water balance equations:

$$\begin{cases} nZ_R\dfrac{dS_c}{dt} &= I_c - E_c - L_c, \\ \\ nZ_R\dfrac{dS_b}{dt} &= I_b - E_b - L_b, \end{cases} \tag{1.9}$$

  where n (dimensionless) is the soil porosity, Z_R (mm) is the effective rooting depth, S_c and S_b are the relative soil moistures $0 < S_c, S_b < 1$, I_c and I_b are the infiltration rates $(mm.yr^{-1})$, E_c and E_b are the sum of soil evaporation and transpiration, L_c and L_b are the drainage rates $(mm.yr^{-1})$. The subscripts c and b refer to sites under and between canopies, respectively. The mean landscape soil moisture is then calculated as

$$S = (1 - f_c)S_b + f_c S_c. \tag{1.10}$$

Systems (1.6) and (1.9) are analyzed using linear stability methods. Yu and D'odorico [127] showed that the reduction in grass cover is likely an effect of woody plant encroachment. They found that in arid environments grass cover is limited mainly by water availability with no major shading effects because the tree cover is relatively low. Yu and D'odorico [127] also showed that the grass-fire feedback can lead to the emergence of bistable dynamics both at the dry and wet sites. Moreover they showed that bistable dynamics associated with grasslands and woodlands occur with the mean annual rainfall (MAR) in the 450–1050 mm range. In their model, Yu and D'odorico [127] assumed that woody plants are better competitors for soil water resources and therefore can outcompete grasses in the absence of disturbances which is not undisputable considering empirical sources. Indeed, in some ecosystems grasses are better competitors where they share the same belowground rooting space with woody seedlings and saplings (e.g., Scholes and Archer [91], Bond [13]).

1.2.1.6 The Touboul et al. (2018) model [107]

Touboul et al. [107] have thoroughly explored the dynamic of a simple model for landscape dynamics of grass, savanna trees and forest tree in interaction with savanna. Their model also assume fire spread has a threshold response to tree cover and savanna tree can vary in the degree to which they excludes fire and that fire, in turn, impact tree population dynamics by preventing savanna sapling

from maturing into adult tree by killing forest trees regardless of their size.
Touboul et al. [107] base their model on units of aerial cover where, G, S and T represent the fractional cover of grass, sapling and trees respectively. They incorporate the threshold response of fire to grass cover G into the sapling-to-tree recruitment term ω. At hight G, fire spread and sapling to tree recruitment ω is low; at low G fire cannot spread and ω is hight. All of these yields to the following ODE system:

$$\begin{cases} \dot{G} &= \mu S + \nu T - \beta G T, \\ \\ \dot{S} &= \beta G T - (\omega(G) + \mu)\, S, \\ \\ \dot{T} &= \omega(G) S - \nu T, \end{cases} \tag{1.11}$$

where, μ and ν are the mortality rates of savanna saplings and adult trees respectively. β is the birth rate of savanna saplings and the sapling-to-tree recruitment coefficient ω is a decreasing function of grass. Touboul et al. [107] considers that, because G, S and T represent fraction of landscape $G + S + T = 1$, and this assumptions allowing them, by expressing one of variables as a function of the two others two reduce the previous system in (1.11) .

The analysis of system (1.11) reveals that depending on parameter value, the system can have multiple equilibria resulting from general sigmoidal response of ω to G but largely independent of its precise functional form. Touboul et al. [107] also showed that depending on the birth and mortality rate of tree, the landscape can develops into grassland or alternatively into a completely wooded landscape and both state can coexist an be stable in a wide region of parameters delineated by codimension-one saddle node bifurcations.

Touboul et al. [107] also consider an extension of system (1.11) by adding an additionnal forest tree functional F. They allowed savanna tree (S and T) to vary in the degree to which they excluded grasses, thereby preventing fire to spread. Therefore they denoted this with $\gamma \in [0, 1]$ with the convention that $\gamma = 1$ corresponds to savanna tree allowing fire to spread as well grass alone and that, $\gamma = 0$ corresponds to perfectly shady savanna trees:

$$\begin{cases} \dot{G} &= \mu S + \nu T - \beta G T + \varphi(G + \gamma(S + T))F - \alpha G F, \\ \\ \dot{S} &= \beta G T - (\omega(G + \gamma(S + T)) + \mu)\, S - \alpha S F, \\ \\ \dot{T} &= \omega(G + \gamma(S + T))S - \nu T - \alpha T F, \\ \\ \dot{F} &= (\alpha(1 - F) - \varphi(G + \gamma(S + T)))\, F, \end{cases} \tag{1.12}$$

where, α is the forest tree birth rate and φ is the forest mortality rate. Touboul et al. [107] analysed the dynamics of system (1.12) by investigating how the degree to which savanna trees and sapling prevent fire spread qualitatively. Their analysis reveals that :

(i) diverse cyclic behaviours, including limit and homo-and heteroclinic cycle can occur for broad range of parameters space.

(ii) Large shifts in landscape structure can result from endogenous dynamics, not just from external drivers.

Touboul et al. [107] like Accatino et al. [2] and De Michele et al. [22] assumed that vegetation component are mutually exclusive but this modelling choice is questionable become sometimes grass vegetation for example, develops under tree crow and thus it is therefore more adapted to model with vegetation dynamics as biomasses.

1.2.1.7 The Yatat et al. (2021) model [124]

In Yatat Djeumen et al. [124] it is analysed a model aiming to recovering as dynamical outcomes of tree-grass interactions, the wide range of vegetation physiognomies observables in the savanna biomes along the rainfall gradients at regimal scales. This model is based on two ordinary differential equations for woody and herbaceous vegetations, explicitly express as functions of mean annual precipitation with the aim to study model prediction in direct relation to rainfall and fire frequency gradients.
In their model the following assumptions where done:

1. Limits puts by rainfall on woody and grass biomass development.
2. Asymmetric interactions between woody and herbaceous plant life forms.
3. Positive feedback between grass biomass and fire intensity and decreased fire impact with tree height.

Precisely, they explicitly express the growth of both woody and herbaceous vegetation as a function of the mean annual rainfall with the aim to study model predictions in direct relation to rainfall and fire frequency gradients. They further assume that the carrying capacity of grass and tree to be increasing and bounded functions of water availability. Similarly, the assumed that the effect of tree biomass on grass biomass is modelled by a non-linear function of mean annual rainfall that take either negative values (meaning facilitation) or positives values (for competition). Finally, contrary to Tchuinte Tamen et al. [102], they split fire frequency from final impact fire on woody biomass in a multiplicative way $(f \times \omega(G) \times \vartheta(T))$ to highlight the fact that grass biomass control both fire spread and local fire intensity which impacts differently small and large woody individuals.
Based on this assumptions, Yatat Djeumen et al. [124] proposed and analysed the following minimal model, given by the set of two nonlinear ODE:

$$\left\{\begin{array}{lcl} \dfrac{dG}{dt} & = & \dfrac{\gamma_G \mathbf{W}}{b_G + \mathbf{W}} G \left(1 - \dfrac{G}{K_G(\mathbf{W})}\right) - \delta_G G - \eta_{TG}(\mathbf{W}) T G - \lambda_{fG}(\mathbf{W}) f G, \\ \\ \dfrac{dT}{dt} & = & \dfrac{\gamma_T \mathbf{W}}{b_T + \mathbf{W}} T \left(1 - \dfrac{T}{K_T(\mathbf{W})}\right) - \delta_T T - f \vartheta(T) \omega(G) T, \end{array}\right. \tag{1.13}$$

where:

(i) G and T (in $t.ha^{-1}$) stand for grass and tree biomasses respectively.

(ii) $\mathbf{W}$ describes the mean annual precipitation.

(iii) $\eta_{TG}(\mathbf{W})$ is the non-linear function described the tree effects on grass biomass.

(iv) The function $\omega(G)$ describe the fire intensity, and it is in on Holling type III form.

(v) $\vartheta(T)$ express a function of fire induced woody biomass mortality and it is a decreasing function of tree-biomass.

(vi) $\dfrac{\gamma_G \mathbf{w}}{b_G + \mathbf{w}}$ and $\dfrac{\gamma_T \mathbf{w}}{b_T + \mathbf{w}}$ are the annual production of grass and tress respectively, where γ_G and γ_T (in yr^-1) express the maximal growth of grass and tree biomass respectively.

(vii) The half saturation b_G and b_T express how quickly growth increase with water availability.

The Yatat Djeumen et al. [124] is fully tractable and sufficient to produce a realistic bifurcation diagram rendering the picture of vegetation physiognomies in the savannas biomes. In fact, their model delimited domains of monostability (forest, savanna), bistability (e.g forest-grassland or forest-savanna) and even tristability. This model differs fundamentally from existing tree-grass models in that MAP is explicitly in the parameters of biomass logistic growth. However, Yatat Djeumen et al. [124] did not take particular emphasis on the cooperation factors that may exist between trees depending on whether one is in arid or humid environmental context in order to reduce fire effects.

1.2.2 Mathematical modelling of savanna dynamics with partial differential equations

In wet savanna areas, vegetation may sometimes exhibit plausibly self-organized physiognomies also termed as patchy vegetation or vegetation mosaics. Indeed, as pointed out by Yatat Djeumen et al. [123], there are several empirical evidences that highlight the existence of vegetation mosaics. Patches of vegetation display dense clusters of shrubs, grasses or trees and can be interpreted as regular spot structures or localized structures (Tlidi et al. [106]). These mosaics involve either bare soil ("desert") versus vegetation (herbaceous or woody) in arid, semi-arid regions (Lefever and Lejeune [62]; Lefever et al. [64]; Lefever and Turner [63]; Couteron and Lejeune [19]; Couteron et al. [20]; HilleRisLambers et al. [52]; Rietkerk et al. [81]; Gilad et al. [43]; Pueyo et al. [79, 80]; Deblauwe et al. [24]), or grasslands/savannas versus forests in temperate as well as humid tropical regions (Youta Happi [126]; Hirota et al. [53]; Jeffery et al. [55]; Xu et al. [120]; Stall et al. [95] and references therein; see also figure 1.1). Empirical evidences suggest that vegetation mosaics in humid regions barely feature periodic patterns. Most often, they are aperiodic but, with quite sharp boundaries like isolated groves or savanna patches encircled by forests. According Borgogno et al. [16], the study of vegetation patterns is motivated by their widespread occurrence in dryland landscapes and by the possibility to infer from their presence and features useful information on the underlying processes, including the susceptibility of the system to abrupt shifts as a result of climate change or anthropogenic disturbances.

To explicitly acknowledge spatial mechanisms within savanna-like ecosystems ecosystems of tree-grass interactions such as seeds production and dispersion, vegetation propagation, life forms competition, several authors proposed and studied savanna models that rely on partial differential equations(PDE). Spatial mechanisms in savannas can further be classified into two different parts: local mechanisms such as local vegetation propagation, local competition for ressources and non-local mechanisms such as non-local seeds dispersal, non-local competition/facilitation for light and nutrients. These two kinds of mechanisms induced two kinds of savanna PDE models, namely local models (acknowledging only local mechanisms) and non-local models (acknowledging both local and non-local mechanisms). We point out that non-local spatial mechanisms are expressed through kernel functions.

Two main modelling directions was followed by the different research teams who addressed the questions of predictions of spatio temporal dynamic in savanna area:

(i) Models featured coupled of reaction-diffusion type.

(ii) Models based on a single integro-differential equation depicting vegetation dynamics with non-local kernel modelling of plant-plant interactions.

In fact, in order to built a mathematical model for savanna ecosystems, it is necessary to think about the number of sate variable, the type of state variable and the resulting model.

1.2.2.1 The Martinez Garcia et al. (2013) model [68]

Martinez-Garcia et al. [68] proposed a model of tree density in Mesic savanna by considering the long range competition among tree and the effect of fire indirectly acting as local facilitation mechanism. Then, they developed a minimalistic model of savannas, that consider two factors as already mentioned, thought to be important to structure mesic savannas: tree-tree competition and fire, with a focus on spatially nonlocal competition. They explored the conditions under which, their model can produce non-homogeneous spatial distributions. Martinez-Garcia et al. [68], therefore proposed and evolution equation in 1.14, for the space dependent density of tree cover $\rho(\mathbf{x}, t)$:

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} = \frac{b\sigma \exp\left(-\delta \int_{\mathbb{R}} G(\mathbf{x} - r)\rho(\mathbf{r}, t) dr\right)}{\sigma + 1 - \rho(\mathbf{x}, t)} \rho(\mathbf{x}, t) \left(1 - \rho(\mathbf{x}, t)\right) - \alpha\rho(\mathbf{x}, t). \tag{1.14}$$

The corresponding grass cover can be deduced by assuming the constant unity.
The blue term in (1.14), described the probability of establishment of a tree and this latter, is related to two independent phenomena: the probability of surviving the competition (which depend on tree crowding in a local neighborhood, decaying exponentially with the density of surrounding tree) and the probability of surviving a fire event, where δ is the a parameter that modulates the strength of competition and σ governs the resistance to fire.

Their central result is that, the nonlocal competition promote the clustering of trees. Whether or not this occur depends entirely on the shape of the competition kernel. Then, they showed the different spatial distributions of trees that occur as competition becomes more intense.

However, Martinez-Garcia et al. [68] in this paper, have neglected the nonlocal mechanisms, firstly for tree seed dispersion, secondly for protection induce by mature trees to limit fire effect on sensitive tree (this later, play a crucial role in the process of tree growth and tree mortality). Another critic is that, Martinez-Garcia et al. [68] proposed an evolution equation of density cover, but it is acknowledged that, vegetation are not mutually exclusive.

1.2.2.2 The Yatat Djeumen et al. (2018) model [122]

In their paper, Yatat Djeumen et al. [122] have extended the previous temporal models, developed in Yatat Djeumen et al. [125] and Tchuinte Tamen et al. [102], using diffusion operators into spatio temporal model in order to study, the long term dynamics of mosaics of forest and grassland in a humid context of central Africa. They developed, a space explicit tree-grass interaction model, taking into account spatial aspects of tree-grass interactions by constructing, a system of partial differential equations that belongs to the family of reaction-diffusion equation. Their resulting system of PDE reads as:

$$\begin{cases} \dfrac{\partial T_S}{\partial t} & = & D_S \dfrac{\partial^2 T_S}{\partial x^2} + (\gamma_S T_S + \gamma_{NS} T_{NS}) \left(1 - \dfrac{T_S + T_{NS}}{K_T}\right) - (\mu_S + \omega_S + \gamma_G G + f\eta_S \omega(G)), \\ \dfrac{\partial T_{NS}}{\partial t} & = & D_{NS} \dfrac{\partial^2 T_{NS}}{\partial x^2} + \omega_S T_S - \mu_{NS} T_{NS}, \\ \dfrac{\partial G}{\partial t} & = & D_G \dfrac{\partial^2 G}{\partial x^2} + \gamma_G \left(1 - \dfrac{G}{K_G}\right) G - (\gamma_{NS} T_{NS} + \mu_G + f\eta_G) G \end{cases} \tag{1.15}$$

with $\omega(G) = \dfrac{G^2}{G^2 + g_0^2}$, x belong to the one spatial domain $\Omega = (-d; d)$, $0 \leq T_S(0,x) = T_{S_0}(x),\ 0 \leq T_{NS}(0,x) = T_{NS_0}(x),\ 0 \leq G(0,x) = G_0(x)$. In addition they also consider homogeneous Newman boundary condition for three state variable, i.e:

$$\frac{\partial T_{NS}(x,t)}{\partial x} = \frac{\partial T_S(x,t)}{\partial x} = \frac{\partial G(x,t)}{\partial x} = 0, \text{ at } x = -d \text{ and } x = d.$$

The main motivation of their model, is to depict or characterize the impact of fire events on possible forest invasion, as bistable travelling wave solution for a context of humid savanna in central Africa. Due to the lack of monotonicity of the previous system in (1.15), and for a mathematical simplification, Yatat Djeumen et al. [122] proposed a reformulation of (1.15) by considering one compartment for tree biomass. Therefore the new system follow the Tchuinte Tamen et al. [102] model:

$$\left\{\begin{array}{rcl}\dfrac{\partial T}{\partial t} & = & D_T\dfrac{\partial^2 T}{\partial x^2}+\gamma_T\left(1-\dfrac{T}{K_T}\right)T-\left(\mu_T+\sigma_G G+f\eta_T\omega(G)\right)T,\\[2ex] \dfrac{\partial G}{\partial t} & = & D_G\dfrac{\partial^2 G}{\partial x^2}+\gamma_G\left(1-\dfrac{G}{K_G}\right)G-\left(\mu_G+\sigma_T T+f\eta_G\right)G\end{array}\right. \tag{1.16}$$

The analytical treatment of system (1.16) highlight several stability and multi-stability results involving spatially homogeneous forest, grassland and savanna solutions. Yatat Djeumen et al. [122] have also discussed for this system, the existence of monostable and bistable grassland forest travelling wave solution. They have founded that depending of fire return time as well as difference in diffusion potential of woody and herbaceous vegetation, fire events are able to greatly slow down or even stop the progression of forest in humid regions.

Yatat Djeumen et al. [122] in this work, didn't considered nonlocal interactions which are empirically evidence as mechanisms of plant dispersal, plant competition for light and nutriment as well plant protection from fires.

1.2.2.3 The Wuyst et al. (2019) model [119]

Wuyts et al. [119] have constructed, a reaction diffusion model of Amazonian tree cover in the purpose to reproduce, the observed spatial distribution of forest versus savanna when, forced by heterogeneous environmental and anthropogenic variables, even though bistability was underestimated. They performed analytical analysis of their model, by deriving the Maxwel Point (MP) of the homogeneous reaction-diffusion equation without savanna trees, as a function of rainfall and the showed that, the front between forest and non-forest settles as this point as long as savanna tree cover near the front remains sufficiently low. In fact, the MP is a well-understood concept in phase transition theory used for example in physics and mathematical biology. In such application, it is the point of external conditions (e.g pressure or temperature) where two separate equilibrium phases of the considering system have the same free energy. In fact, when there is a gradient of external conditions, the front between the stable steady state pins (i.e settles) at the MP and that is what Wuyts et al. [119] found their model.

Their full system of PDE representing cover type as a function af space and time in the given space give:

$$\left\{\begin{array}{rcl}\partial_t S & = & R_S\left(1-S-T-T\right)T-Q_0\left(1-h\Phi(T,F)\right)S-M_S S-R_F SF+D_S\nabla^2_S,\\[2ex] \partial_t T & = & Q_0\left(1-h\Phi(T,F)\right)S-M_T T-R_F TF,\\[2ex] \partial_t F & = & R_F\left(1-F-\right)F-b\Phi(F,T)F-M_F F-cF+D_F\nabla^2_F,\end{array}\right. \tag{1.17}$$

where $\Phi(T,F)=\dfrac{\tau^{-1}Y_c^4}{Y_c^4(T+F)}$, and S is savanna sapling cover, T the savanna adult tree cover, F

forest tree cover and Φ the fraction of area burnt, $Y \in \{S, T, F\}$. and Y_c is the critical value below when fire spread occur and T the maximum return time. $Q_0(1-h\Phi)$ is the recruitment rate of savanna saplings into adult savanna trees. b is the sensitivity of front tree cover.
From the homogeneous system without savanna tree ($S = T = 0$) and via numerical continuation, a MP is derived.

However, the Wuyts et al. [119] model has a lack of mathematical tractability, due to the fact that, authors deals principally with numerical simulations to have some information about their models. Therefore it is not easy to use mathematical analysis to infer or either to understand the behaviours and the properties of their models.

1.2.2.4 The Goel et al. (2020) model [45]

In Goel et al. [45], it is examined the contributions of dispersal to determine biome distribution, by using a two dimensional (2D) reaction diffusion model . In fact, theoretical work on reaction diffusion models show that, coupling diffusion with a mean-field bistable model in a one dimensional (1D) landscape with a prediction gradient can yields spatially aggregated biome distribution. However, one obvious limitation of that type of model is that they treat dispersal as a one dimensional process, evenght though it is more realistic to model dispersal using two dimensional (2D) reaction diffusion model. Goel et al. [45] assumed in their paper that, the landscape consists of fractional groups of plants: forest trees and grasses. They designed by $T(\mathbf{r}, t)$ the density of forest in a spatial point $\mathbf{r}$ of the domain at the time t and $G(\mathbf{r}, t)$ the density of grasses. Without loss of generality, they considered the constant unity, i.e $T(\mathbf{r}, t) + G(\mathbf{r}, t) = 1$. Therefore, the instantaneous dynamic of biomes can be represented in terms of single dynamical variable T, with grass represented as $G = 1 - T$. Then, the model studied in Goel et al. [45] is formulated by the following equation:

$$\frac{\partial T}{\partial t} = P(1-T)T - \Phi T + D\nabla^2 T. \tag{1.18}$$

The study of equation (1.18) reveals that, the dynamic in the 2D reaction diffusion model with an underlying precipitation gradient can in a first step reproduce the both overlap in the precipitation ranges over which savanna and forest biomes occur and that is missing from the 1D diffusion model. In a second step the Goel et al. [45] model reproduce the spatial aggregation properties of biome. Moreover, the 2D diffusion model suggest that this precipitation overlap might not be maintained by hysteresis. Dynamically, their model predicts that dispersal (via diffusion in 2D), may increase the resilience of tropical biome in response to global change.

The same critics done for Wuyts et al. [119] model are still valid because the authors also deals with numerical simulation to highlight spatial aggregation of biome in savanna-forest mosacis.

1.3 Conclusion

Due to scarcely attention on the problematic that concern the prediction of the spatial dynamic of vegetation in the humid savannah zone, two lines of research deserve to be addressed: (i) the prediction, using a spatially explicit mathematical model, depicting possibly spatial periodic structures observed in the humid savannah zone with particular emphasis on the period of the patterns observed; (ii) The construction of a mathematical model allowing to predict the possible transitions of mosaics observed with a particular emphasis on the speed of transitions between the mosaics observed. In fact, a deep literature reviews allows us to observed that, spatial interactions mechanism traduced in spaced have been scarcely taken into account in the study of tree grass dynamic. Indeed, tree-grass interactions in savanna ecosystem have been mostly modelled though framework that implicitly acknowledge space (Accatino et al. [2], Tchuinte Tamen et al. [102],Yatat Djeumen et al. [125], Yu and D'odorico [127],Touboul et al. [107],Yatat Djeumen et al. [124]). Most of works done using PDE formalism, where carried out in the arid and semi-arid environmental context (Lefever and Lejeune [62], HilleRisLambers et al. [52], Couteron and Lejeune [19], Lefever et al. [64], Lefever and Turner [63]). Only a few mathematical tractable and space explicit tree-grass interaction model have been designated for wet savanna (Yatat Djeumen et al. [122], Wuyts et al. [119], Goel et al. [45]). Except Yatat Djeumen et al. [122], most of PDE models relied on numerical treatment to render some spatial structures and relate them to process. Then, mathematical PDE models that allows mathematical tractability are thus more desirable in the context of wet savanna.

CHAPTER TWO

MONOTONE ITERATIVE METHOD FOR REACTION-DIFFUSION EQUATIONS

In this chapter, we provide some mathematical basic knowledge in the prospect to help the readability of our contributions in doctoral dissertation. We principally refer to the book of Pao [77]:“ Nonlinear parabolic and elliptic equations".

2.1 Derivation of reaction-diffusion equations

One of the basic theories in the formulation of governing equations for physical problems is the principle of conservation. When the problem under consideration involves a reaction process accompanied by diffusion, this principle leads to a set of partial differential equation for the unknown quantities of system. If we consider a single quantity $u(x,t)$, called density function at time t and position x in a diffusion medium Ω in $\mathbb{R}^n$, the principle of conservation states that: “for any subdomain R of Ω with boundary surface S, the rate of change of mass density is equal to the rate of flux across S plus the rate of generation within R.” This statement is the balance relation in which the flux denoted by the vector J, is the density flow per unit surface area per unit time. Let ν be the outward normal vector on S and q_0 the rate of generation per unit volume per unit time in R. Assume that u, J and q_0 are continuous in x, J has a continuous partial derivatives with respect to the components of x and u has a continuous derivative in t. Then the balance relation may be expressed as:

$$\frac{d}{dt}\int_R a_0 u dx = -\int_S J.\nu ds + \int_R q_0 dx, \tag{2.1}$$

where a_0 is a constant. The negative sign in the surface integral in (2.1) represents the density flow into the region R through the boundary surface S. Since the divergence theorem:

$$\int_S J.\nu ds = \int_R \nabla.J dx \tag{2.2}$$

where ∇ is the gradient operator in x. Equation (2.1) is reduced to:

$$\int_R \left(a_0 u_t + \nabla.J - q_0\right) dx = 0, \tag{2.3}$$

where $u_t = \frac{\partial u}{\partial t}$. The continuity assumption on u_t and $\nabla.J$ and the arbitrariness of the subdomain R imply that

$$a_0 u_t + \nabla.J - q_0 \text{ in } \Omega. \tag{2.4}$$

This equation (2.4) is often referred to as the equation of the principle of conservation variation.
To relate the diffusion flux J to the density function u it is assumed that in absence of convection, the flux is proportional to the negative gradient of density:

$$J = -D^* \nabla u, \tag{2.5}$$

where D^* is a strictly positive function in Ω. Substituting (2.5) in (2.4) yields to the following equation:

$$u_t = \nabla.(D\nabla u) + q, \tag{2.6}$$

where $D = D^*/a_0$, $q = q_0/a_0$. The function D is called the diffusion coefficient. The term $\nabla.\,(D\nabla u)$ represents the rate of change due to the diffusion, and q is the rate of change due to reaction. In many reaction-diffusion-type problems, q depends on the density function u and possibly on (x, t) explicitly. Writing $q = f(x, t, u)$, the equation in (2.6) leads to the reaction diffusion equation:

$$u_t - \nabla.\,(D\nabla u) = f(x, t, u). \tag{2.7}$$

2.2 Boundary Conditions

When the diffusion medium Ω is a bounded domain in $\mathbb{R}^n$, the reaction diffusion equations are supplemented by suitable boundary on the boundary surface $\partial\Omega$. The appropriate condition on the boundary depends on the physical mechanism surrounding the diffusion medium. Then, the condition on the boundary depends on the material properties both inside and outside the diffusion medium. If the outside environment is know there are the following three basic types of boundary conditions:

1. **Dirichlet boundary condition**:
   When the density function is specified on the boundary surface, the boundary condition is given in the form:

   $$u(x, t) = h(x, t) \ \ (x \in \partial\Omega, \ \ t > 0), \tag{2.8}$$

   where h is a density function with which the diffusion medium is in contact.

2. **Neumann boundary condition**:
   When the flux across the boundary surface is presented, the boundary condition becomes:

   $$\partial u/\partial\nu = h(x, t) \ \ (x \in \partial\Omega, \ \ t > 0), \tag{2.9}$$

   where h represents the rate of flow of the density and $\partial u/\partial\nu$ is the directional derivative of u

in the direction ν. Precisely $\partial/\partial\nu$ is considered as the outward normal derivative on $\partial\Omega$. One special interest is the homogeneous Neumann boundary condition

$$\partial u/\partial\nu = 0 \ \ (x \in \partial\Omega, \ \ t > 0), \tag{2.10}$$

which plays a particular role in the qualitative behaviour of the solution. The physical meaning of (2.10) is that the boundary surface is completely insulated so that there is no flow across the boundary.

3. **Robin boundary condition**:
   By considering the outward normal derivative on u on $\partial\Omega$ and denote the surrounding density by $h_0(t,x)$, then :

   $$\partial u/\partial\nu = \beta(h_0 - u), \tag{2.11}$$

   where β is a proportionality constant, which can vary from point to point on $\partial\Omega$. Since h_0 is known, the boundary condition can be written as :

   $$\partial u/\partial\nu + \beta\nu = h(x,t) \ \ (x \in \partial\Omega, \ \ t > 0), \tag{2.12}$$

   where $h = \beta h_0$. This condition in (2.12) is called Robin boundary condition.

All of three types of boundary conditions can be cast into the general form :

$$\mathbf{B}u = h(x,t) \ \ (x \in \partial\Omega, \ \ t > 0), \tag{2.13}$$

and $\mathbf{B}$ is referred to the boundary operator defined by:

$$\mathbf{B}u = \alpha_0(x)\partial u/\partial\nu + \beta_0(x)u. \tag{2.14}$$

2.3 Linear reaction-diffusion Equations

In this section we introduce some basic definitions and collect some facts for linear parabolic boundary-value problems.
Let Ω be either a bounded or an unbounded open domain in $\mathbb{R}^n$, and let $\partial\Omega$ be the boundary of Ω. For each $T > 0$, let $D_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$. Denote by:

- $C^m(\Omega)$, the set of all continuous functions, whose partial derivatives up to the mth order are continuous in Ω.
- $C^{m,l}(D_T)$ the set of functions whose l-times derivatives in t and m-times derivatives in x are continuous in D_T.
- $C^\alpha(\Omega)$ the set of all Hölder continuous functions in Ω.

Similar notations are used for $C^m(\overline{\Omega})$ and $C^{l,m}(\overline{D}_T)$, where $\overline{\Omega}$, $\overline{D}_T$ are the respective closure of Ω and D_T. When $m = 0$ we denote by $C(\Omega)$, $C(\overline{\Omega})$, $C(D_T)$, $C(\overline{D}_T)$ the set of continuous functions in Ω, $\overline{\Omega}$, D_T and $\overline{D}_T$ respectively. The norms in $C(\Omega)$ and $C(D_T)$ are defined by:

$$|u|_0^{\Omega} = \sup_{x\in\Omega} |u(x)| \text{ and } |u|_0^{D_T} = \sup_{(x,t)\in D_T} |u(x,t)|. \tag{2.15}$$

Similar norms with respect to $\overline{\Omega}$, $\overline{D}_T$ are defined for $C(\overline{\Omega})$ and $C(\overline{D}_T)$.
We next collect some facts concerning the linear time-dependant boundary value-problem:

$$\begin{cases} u_t - D\nabla^2 u + c_0 u = q(x,t) & \text{in} \quad D_T, \\ \mathbf{B}u = \partial u/\partial \nu = h(x,t) & \text{on} \quad S_T, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.16}$$

where D is a positive constant, c_0 is a continuous function, and q, h, and u_0 are given internal and boundary data.

Definition 2.1. (Fundamental solution)(Pao [77])
A function $\Gamma \equiv \Gamma(x,t)$ is called a fundamental solution of the parabolic operator $\partial/\partial_t - D\nabla^2 + c_0$ in $\mathbb{R}^n \times (0;T]$ if for any fixed $(y,s) \in \mathbb{R}^n \times (0;T]$, Γ satisfies the equation:

$$\Gamma_t - D\nabla^2\Gamma + c_0 = \delta(x-y)\delta(t-s), \tag{2.17}$$

where δ is the Dirac δ-function.
If c_0 is a constant function Γ is given by:

$$\Gamma(x,t) = (4\pi Dt)^{-n/2} \exp\left[-\left(c_0 t + \frac{|x|^2}{4Dt}\right)\right]. \tag{2.18}$$

Proposition 2.1. (see also Pao [77]) Let $q \equiv q(x,t)$ be a measurable function in D_T and $\Phi \equiv \Phi(x,t)$ a bounded continuous function on S_T. Then,

(i) the “volume potential"

$$V_0(x,t) = \int_0^t \int_\Omega \Gamma(x-y, t-s) q(y,s) dyds, \tag{2.19}$$

is a continuous function in $\overline{D}_T$ if q is bounded in D_T.

(ii) V_0 is Hölder continuous in $\overline{D}_T$ for every $\alpha \in (0,1)$ if q is continuous in $\overline{D}_T$.

(iii) the “single layer potential"

$$V_1(x,t) = \int_0^t \int_{\partial\Omega} \Gamma(x-y, t-s)\Phi(y,s) dyds, \tag{2.20}$$

is Hölder continuous on $\overline{D}_T$ for any $\alpha \in (0,1)$.

(iv) there exists $\alpha^* \in (0,1)$ such that the "double layer potential"

$$V_2(x,t) = \int_0^t \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu_y}(x-y,t-s)\Phi(y,s)dyds, \tag{2.21}$$

is Hölder continuous on S_T for any $\alpha \leq \alpha^*$.

Theorem 2.1. (**Integral representation**)(see Pao [77])
Let q be Hölder continuous in x, uniformly in $\overline{D}_T$, then for any continuous function h on S_T and u_0 in $\overline{\Omega}$ the linear time-dependant boundary value-problem (2.16) has a unique solution u which is Hölder continuous in x, uniformly in $\overline{D}_T$. Moreover, u can be represented by the formula:

$$\begin{aligned} u(x,t) &= \int_\Omega \Gamma(x-y,t)u_0(y)dy + \int_0^t \int_\Omega \Gamma(x-y,t-s)q(y,s)dyds \\ &\quad + \int_0^t \int_{\partial\Omega} \Gamma(x-y,t-s)\Phi(y,s)dyds, \end{aligned} \tag{2.22}$$

where Γ is the fundamental solution of the operator $\partial/\partial_t - D\nabla^2 + c_0$, Φ is governed by the integral equation:

$$\Phi(x,t) = 2\int_0^t \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu_y}(x-y,t-s)\Phi(y,s)dyds - 2H(x,t), \tag{2.23}$$

and $H(x,t)$ is given by:

$$H(x,t) = \int_\Omega \frac{\partial \Gamma}{\partial \nu_x}(x-y,t)u_0(y)dy + h(x,t) + \int_0^t \int_\Omega \frac{\partial \Gamma}{\partial \nu_x}(x-y,t-s)q(y,s)dyds. \tag{2.24}$$

A consequence of linear theory of parabolic equations concern the density Φ of the integral equation (2.22). By considering a sequence of functions $\left\{H^{(k)}\right\}$ which converges pointwise on S_T to a function H and if $\Phi^{(k)}$ is the solution of the integral equation:

$$\Phi^{(k)}(x,t) = 2\int_0^t \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu_y}(x-y,t-s)\Phi(y,s)dyds - 2H^{(k)}(x,t), \tag{2.25}$$

and Φ is verified equation (2.23), then we have the following result:

Proposition 2.2. (see Pao [77]) Let $\left\{H^{(k)}\right\}$ be a sequence of continuous functions that converges pointwise on S_T to H, and let $\Phi^{(k)}$, Φ be the respective solutions of (2.25) and (2.23). Then $\left\{\Phi^{(k)}\right\}$ converge pointwise on S_T to Φ. Moreover if H is continuous on S_T so is Φ.

A basic and important tool in the development of monotone method is the maximum principle for parabolic operators.

Theorem 2.2. (see Pao [77]) Let $w \in C^{2,1}(D_T)$ such that

$$w_t - \nabla^2 w + cw \geq 0, \ \ (x,t) \in D_T,$$

where $c \equiv c(x,t)$ is a bounded and positive function in D_T. If w attains a minimum value $m_0 \leq 0$ at some point in D_T then $w(x,t) = m_0$ throughout D_T. If $\partial\Omega$ has the inside sphere property and w attains a minimum at some point (x_0, t_0) on S_T, then $\partial w/\partial\nu < 0$ at (x_0, t_0) whenever w is not a constant.

Proposition 2.3. If $w \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ and satisfies the relation:

$$\begin{cases} w_t - D\nabla^2 w + cw \geq 0 & \text{in} \quad D_T, \\ \mathbf{B}w \geq 0 & \text{on} \quad S_T, \\ w(x,0) \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{2.26}$$

where $c \equiv c(x,t)$ is any bounded and positive function in D_T, then $w \geq 0$ in $\overline{D}_T$.

Proof. Assume by contradiction that there exists a point $(x_0, t_0) \in \overline{D}_T$ such that $w(x_0, t_0)$ is a negative minimum. Because $w(x_0, 0) \geq 0$, (x_0, t_0) must be either in D_T or on S_T. Since w can not be a negative constant, the maximum principle in Theorem (2.2) implies that $(x_0, t_0) \in S_T$. Now by the boundary inequality $\partial w/\partial\nu \geq 0$ at (x_0, t_0), which contradicts the maximum principle in Theorem (2.2). Then $w(x,t) \geq 0$ in $\overline{D}_T$. □

We address now the method to establish an existence theorem for reaction-diffusion equation under Neumann boundary condition.

2.4 Monotone Iterative method for reaction-diffusion equation

The basic idea of this method is that by using an upper solution or an lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iterations is monotone and converges to a solution of the problem.
To illustrate the method, let us consider the time-dependent problem:

$$\begin{cases} u_t - D\nabla^2 u = f(x,t,u) & \text{in} \quad D_T, \\ \mathbf{B}u = \partial u/\partial\nu = h(x,t) & \text{on} \quad S_T, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.27}$$

where the functions f, h and u_0 are assumed Hölder continuous in their respective domains. We therefore have the following definitions.

Definition 2.2. A function $\tilde{u} \in C(\overline{D}_T) \cap C^{2,1}(D_T)$ is called an upper solution of (2.27) if it satisfies the relation

$$\begin{cases} \tilde{u}_t - D\nabla^2 \tilde{u} \geq f(x,t,\tilde{u}) & \text{in} \quad D_T, \\ \mathbf{B}\tilde{u} = \partial\tilde{u}/\partial\nu = h(x,t) & \text{on} \quad S_T, \\ \tilde{u}(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.28}$$

Similarly, $\hat{u} \in C(\overline{D}_T) \cap C^{2,1}(D_T)$ is called a lower solution if it satisfies all the reversed inequalities in (2.28).

The functions $\tilde{u}$, $\hat{u}$ are called ordered upper and lower solutions if $\tilde{u} \geq \hat{u}$ in $\overline{D}_T$. It is therefore defined the sector $\langle \hat{u}, \tilde{u} \rangle = \left\{ u \in C(\overline{D}_T);\ \ \hat{u} \leq u \leq \tilde{u} \right\}$.

Remark 2.1. Every solution of (2.27) is an upper solution as well lower solution. However upper and lower solution exist unless the problem has no solution in D_T.

To ensure the existence of a solution it is necessary to impose some condition on the reaction function. A basic assumption is that for some bounded functions $\underline{c} \equiv \underline{c}(x,t)$ and $\overline{c} \equiv \overline{c}(x,t)$, f satisfy the condition:

$$-\underline{c}(u_1 - u_2) \leq f(x,t,u_1) - f(x,t,u_2) \leq -\overline{c}(u_1 - u_2) \ \text{ for } \ \hat{u} \leq u_2 \leq u_1 \leq \tilde{u}. \tag{2.29}$$

Without loss of generality, we may assume that $\underline{c}(x,t)$ is Hölder continuous in $\overline{D}_T$. In view of (2.29) the function

$$F(x,t,u) = \underline{c}u + f(x,t,u), \tag{2.30}$$

is monotone nondecreasing in u for $u \in \langle \hat{u}, \tilde{u} \rangle$ and Hölder continuous in $\overline{D}_T \times \langle \hat{u}, \tilde{u} \rangle$. Furthermore, F satisfy the Lipschitz condition

$$|F(x,t,u_1) - F(x,t,u_2)| \leq K|u_1 - u_2| \ \text{ for } \ u_1, u_2 \in \langle \hat{u}, \tilde{u} \rangle, \tag{2.31}$$

where K may be taken as an upper bound of $|\underline{c}(x,t)| + |\overline{c}(x,t)|$ in $\overline{D}_T$. Clearly, (2.29) holds when f is Lipschitz continuous in u. However, in the construction of monotone sequence only the left-hand side Lipschitz condition in (2.29) is needed; the right-hand side Lipschitz condition is used to ensure the uniqueness of the solution.
By adding $\underline{c}u$ on both sides of the differential equation in (2.27) and choosing a suitable initial iteration $u^{(0)}$, it is constructed a sequence $\left\{u^{(k)}\right\}$. successively from the iteration process:

$$\begin{cases} u_t^{(k)} - D\nabla^2 u^{(k)} + \underline{c}u^{(k)} = \underline{c}u^{(k-1)} + f(x,t,u^{(k-1)}) & \text{in} \quad D_T, \\ \mathbf{B}u^{(k)} = \partial u / \partial \nu = h(x,t) & \text{on} \quad S_T, \\ u^{(k)}(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.32}$$

Since for each k the right-hand side of (2.32) is know, the existence theorem for linear parabolic boundary-value problems implies that the sequence $\left\{u^{(k)}\right\}$ is well defined. Of particular interest is the sequence obtained from (2.32) with an upper solution or a lower solution as the initial iteration. Denote the sequence with the initial iteration $u^{(0)} = \tilde{u}$ by $\left\{\overline{u}^{(k)}\right\}$ and the sequence with $u^{(0)} = \hat{u}$ by $\left\{\underline{u}^{(k)}\right\}$, and refer to them as upper and lower sequences, respectively. The following lemmas shows that in a first step that the two sequences are well defined and in a second step gives the monotone property of these two ones.

Lemma 2.1. Let condition (2.29) be satisfied. Then the two sequences $\left\{\overline{u}^{(k)}\right\}$ and $\left\{\underline{u}^{(k)}\right\}$ are well defined, and $\overline{u}^{(k)}$, $\underline{u}^{(k)}$ are in $C^{\alpha}(D_T)$ for each k.

Proof. Let $\left\{u^{(k)}\right\}$ be either the upper sequence or the lower sequence and let:

$$q^{(k)}(x,t) = F(x,t,u^{(k)}(x,t)).$$

$u^{(k)}$ is uniquely determined if the function $q^{(k-1)}$ is continuous in D_T and is Hölder continuous in x. Consider the case $k = 1$. The Hölder continuity of $u^{(0)}$ and F imply for (x,t) and (y,s) in D_T that:

$$\begin{aligned} |q^{(0)}(x,t) - q^{(0)}(y,s)| &\le |F(x,t,u^{(0)}(x,t)) - F(y,s,u^{(0)}(x,t))| + |F(y,s,u^{(0)}(x,t)) - F(y,s,u^{(0)}(y,s))| \\ &\le H_\alpha\left(|x-y|^\alpha + |t-s|^\alpha\right) + K|u^{(0)}(x,t) - u^{(0)}(y,s)|, \\ &\le \left(H_\alpha + KH_\alpha^{(0)}\right)\left(|x-y|^\alpha + |t-s|^\alpha\right) \end{aligned}$$

where H_α and $H_\alpha^{(0)}$ are the Hölder constants (with exponent α) of F and $u^{(0)}$ respectively. This shows that $q^{(0)}$ is Hölder continuous in D_T with exponent α. In view of Theorem 2.1 page 29, a unique solution $u^{(1)}$ to (2.32) exists and is in $C^{\alpha}(D_T)$. Replacing $u^{(0)}$ by $u^{(1)}$ the same argument shows that $q^{(1)}(x,t)$ is Hölder continuous in D_T with the same exponent α. Hence the solution $u^{(2)}$ to (2.32) exists and is in $C^{\alpha}(D_T)$. An induction argument leads to the conclusion of the lemma. ☐

Lemma 2.2. Let $\tilde{u}$, $\hat{u}$ ne ordered upper and lower solutions of system (2.27), and let f satisfy (2.29). Then the sequences $\left\{\overline{u}^{(k)}\right\}$, $\left\{\underline{u}^{(k)}\right\}$ possess the monotone property:

$$\hat{u} \le \underline{u}^{(k)} \le \underline{u}^{(k+1)} \le \overline{u}^{(k+1)} \le \overline{u}^{(k)} \le \tilde{u} \ \text{ in } \ D_T \tag{2.33}$$

for every $k = 1, 2, ...$

Proof. Let $w = \overline{u}^{(0)} - \overline{u}^{(1)} = \tilde{u} - -\overline{u}^{(1)}$. By (2.28), (2.32), and $\overline{u}^{(0)} = \tilde{u}$,

$$\begin{aligned} w_t - D\nabla^2 w + \underline{c}w &= \left(\tilde{u}_t - D\nabla^2\tilde{u} + \underline{c}\tilde{u}\right) - \left(\underline{c}\overline{u}^{(0)} + f(x,t,\overline{u}^{(0)})\right), \\ &= \tilde{u}_t - D\nabla^2\tilde{u} - f(x,t,\tilde{u}) \ge 0, \end{aligned}$$

$$\begin{aligned} &\mathbf{B}w = \mathbf{B}\tilde{u} - h(x,t) \ge 0, \\ &w(x,0) = \tilde{u}(x,0) - u_0(x) \ge 0. \end{aligned}$$

In view of proposition 2.3, $w \ge 0$ in $\overline{D}_T$, which leads to $\overline{u}^{(1)} \le \overline{u}^{(0)}$. A similar argument using the property of a lower solution gives $\underline{u}^{(1)} \ge \underline{u}^{(0)}$. Next, let $w^{(1)} = \overline{u}^{(1)} - \underline{u}^{(1)}$. By systems (2.29) and (2.32), $w^{(1)}$ satisfies the relation:

$$\begin{aligned} w_t^{(1)} - D\nabla^2 w^{(1)} + \underline{c}w^{(1)} &= \left(\underline{c}\overline{u}^{(0)} + f(x,t,\overline{u}^{(0)})\right) - \left(\underline{c}\underline{u}^{(0)} + f(x,t,\underline{u}^{(0)})\right), \\ &= \underline{c}\left(\tilde{u} - \hat{u}\right) + f(x,\tilde{u}) - f(x,t,\hat{u}) \ge 0, \end{aligned}$$

$$\begin{aligned} &\mathbf{B}w^{(1)} = h(x,t) - h(x,t) = 0, \\ &w^{(1)}(x,0) = u_0(x) - u_0(x) = 0. \end{aligned}$$

Again, by proposition 2.3, $w^{(1)} \geq 0$. The above conclusion shows that $\underline{u}^{(0)} \leq \underline{u}^{(1)} \leq \overline{u}^{(1)} \leq \overline{u}^{(0)}$. Assume, by induction:

$$\underline{u}^{(k-1)} \leq \underline{u}^{(k)} \leq \overline{u}^{(k)} \leq \overline{u}^{(k-1)} \text{ in } \overline{D}_T. \tag{2.34}$$

Then by systems (2.29) and (2.32), the function $w^{(k)} = \overline{u}^{(k)} - \overline{u}^{(k+1)}$ satisfies the relation:

$$w_t^{(k)} - D\nabla^2 w^{(k)} + \underline{c}w^{(k)} = \left(\underline{c}\overline{u}^{(k-1)} + f(x, \overline{u}^{(k-1)})\right) - \left(\underline{c}\overline{u}^{(k)} + f(x, \overline{u}^{(k)})\right).$$

Since $\mathbf{B}w^{(k)} = 0$ on S_T and $w^{(k)}(x,0) = 0$ in Ω, proposition 2.3 implies that $w^{(k)} \geq 0$, that is $\overline{u}^{(k+1)} \leq \overline{u}^{(k)}$. Similar reasoning gives $\underline{u}^{(k+1)} \leq \underline{u}^{(k)}$ and $\overline{u}^{(k+1)}$ and $\overline{u}^{(k+1)} \leq \overline{u}^{(k)}$. The monotone method property in 2.33 follows by the principle of induction. ☐

Lemma 2.3. The pointwise limits

$$\lim_{k\to+\infty} \overline{u}^{(k)}(x,t) = \overline{u}(x,t) \text{ and } \lim_{k\to+\infty} \underline{u}^{(k)}(x,t) = \underline{u}(x,t) \tag{2.35}$$

exists and satisfy the relation:

$$\hat{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq \underline{u} \leq \overline{u} \leq \overline{u}^{(k+1)} \leq \overline{u}^{(k)} \leq \tilde{u} \text{ in } \overline{D}_T \tag{2.36}$$

where $k = 1, 2, ...$

Proof. Since by the Lemma 2.2, the sequence $\left\{\overline{u}^{(k)}\right\}$ is monotone nonincreasing and is bounded from below and the sequence $\left\{\underline{u}^{(k)}\right\}$ is monotone nondecreasing and is bounded from above, the pointwise limits of these sequences exist and their limits are denoted by $\overline{u}$ and $\underline{u}$. Moreover by (2.32) the limits $\overline{u}$ and $\underline{u}$ satisfy the relation (2.36). ☐

Lemma 2.4. If the limits $\overline{u}$ and $\underline{u}$ in (2.35) are solutions of (2.27), then $\overline{u} = \underline{u}$ and is the unique solution in the sector $\langle \hat{u}, \tilde{u} \rangle$.

Proof. Let $w = \underline{u} - \overline{u} \leq 0$. Then w satisfies the relation:

$$w_t - D\nabla^2 w = f(x,t,\underline{u}) - f(x,t,\overline{u}) \geq -\overline{c}(\overline{u} - \underline{u}) = \overline{c}w,$$

and the boundary and initial condition $\mathbf{B}w = 0$ on S_T, $w(x,0) = 0$ in Ω. By the proposition 2.3, $w \geq 0$ in $\overline{D}_T$ which ensure that $\overline{u} = \underline{u}$. Now if u^* is any solution in the sector $\langle \hat{u}, \tilde{u} \rangle$ then by considering u^*, $\tilde{u}$ and $\hat{u}$, u^* as ordered upper and lower solutions, since the sequence $\left\{\overline{u}^{(k)}\right\}$ with $\overline{u}^{(0)} = u^*$ consists of the same function u^* for every k, the above conclusion implies that $u^* \geq \underline{u}$. Similarly, $u^* \leq \overline{u}$. This implies that $\overline{u} = u^* = \underline{u}$ and u^* is the unique solution of (2.27). ☐

Now we show that $\overline{u}^{(k)}$ and $\underline{u}^{(k)}$ are ordered upper and lower solutions of (2.27).

Lemma 2.5. For each k, $\underline{u}^{(k)}$ and $\overline{u}^{(k)}$ are respectively lower and upper solution of (2.27) and $\underline{u}^{(k)} \leq \overline{u}^{(k)}$ in $\overline{D}_T$.

Proof. By the iteration process (2.32) and the conditions (2.29) and (2.36), $\overline{u}^{(k)}$ satisfies the relation:

$$\begin{aligned}
\underline{u}_t^{(k)} - D\nabla^2\underline{u}^{(k)} &= -\underline{c}\left(\underline{u}^{(k)} - \underline{u}^{(k-1)}\right) + f(x,t,\underline{u}^{(k-1)}),\\
&= -\left[\underline{c}\left(\underline{u}^{(k)} - \underline{u}^{(k-1)}\right) + f(x,t,\underline{u}^{(k)}) - f(x,t,\underline{u}^{(k-1)})\right] + f(x,t,\underline{u}^{(k)}),\\
&\leq f(x,t,\underline{u}^{(k)}),
\end{aligned}$$

and $\overline{u}^{(k)}$ satisfies the relation:

$$\begin{aligned}
\overline{u}_t^{(k)} - D\nabla^2\overline{u}^{(k)} &= \underline{c}\left(\overline{u}^{(k-1)} - \overline{u}^{(k)}\right) + f(x,t,\overline{u}^{(k-1)}),\\
&= \underline{c}\left(\overline{u}^{(k-1)} - \overline{u}^{(k)}\right) + \left(f(x,t,\overline{u}^{(k-1)}) - f(x,t,\overline{u}^{(k)})\right) + f(x,t,\overline{u}^{(k)}),\\
&\geq f(x,t,\overline{u}^{(k)}).
\end{aligned}$$

It follows from the boundary and initial condition that $\underline{u}^{(k)}$ and $\overline{u}^{(k)}$ are respectively lower and upper solution of (2.27). The relation $\underline{u}^{(k)} \leq \overline{u}^{(k)}$ follows from Lemma 2.2. ☐

Lemma 2.6. Let $\tilde{u}$, $\hat{u}$ be upper and lower solution of (2.27) and f be a C^1-function in u. Then, $\tilde{u} \leq \hat{u}$. In particular, if $\tilde{u}$ is an upper solution (resp., $\hat{u}$ is a lower solution) and u^* is the solution of (2.27), then $\tilde{u} \leq u^*$ (resp., $\hat{u} \geq u^*$).

Proof. Let $w = \tilde{u} - \hat{u}$. Then by the definition of $\tilde{u}$, $\hat{u}$ and the mean value theorem,

$$\begin{aligned}
&w_t - D\nabla^2 w \geq f(x,t,\tilde{u}) - f(x,t,\hat{u}) = f_u(x,t,\hat{\eta})w,\\
&\mathbf{B}w \geq h - h = 0,\\
&w(w,0) \geq u_0 - u_0 = 0,
\end{aligned}$$

where $\hat{\eta}$ is an intermediate value between $\hat{u}$ and $\tilde{u}$. By proposition 2.3, $\tilde{u} \geq \hat{u}$. Since every solution u^* may be considered as a lower solution or an upper solution the relation $\tilde{u} \geq u^*$ and $\hat{u} \leq u^*$ follow immediately. ☐

Now we justify that the limits $\overline{u}$, $\underline{u}$ of upper and lower sequences coincide and yield a unique solution of (2.27) by showing that this later satisfy the integral equation:

$$\begin{aligned}
u(x,t) &= \int_\Omega \Gamma(x-y,t)u_0(y)dy + \int_0^t \int_\Omega \Gamma(x-y,t-s)(F(u))(y,s)dyds\\
&\quad + \int_0^t \int_{\partial\Omega} \Gamma(x-y,t-s)\Phi(y,s)dyds
\end{aligned} \tag{2.37}$$

where Φ is the density given by (2.23).

Lemma 2.7. $\overline{u}$, $\underline{u}$ defined in (2.35) satisfy the integral equation 2.37.

Proof. Let $\left\{u^{(k)}\right\}$ be either the upper sequence or the lower sequence and let u be the limit of $\left\{u^{(k)}\right\}$

as $k \longrightarrow +\infty$. By the integral representation in Theorem 2.1, the solution $u^{(k)}$ of (2.32) is given by:

$$\begin{aligned} u^{(k)}(x,t) &= \int_\Omega \Gamma(x-y,t)u_0(y)dy + \int_0^t \int_\Omega \Gamma(x-y,t-s)(F(u^{(k-1)}))(y,s)dyds \\ &+ \int_0^t \int_{\partial\Omega} \Gamma(x-y,t-s)\Phi^{(k-1)}(y,s)dyds. \end{aligned} \tag{2.38}$$

The density $\Phi^{(k)}$ which depends on k is given by (2.23) where $H(t,x)$ replaced by $H^{(k)}(x,t)$ and:

$$H^{(k)}(x,t) = \int_\Omega \frac{\partial\Gamma}{\partial\nu_x}(x-y,t)u_{i,0}(y)dy + h(x,t) + \int_0^t \int_\Omega \frac{\partial\Gamma}{\partial\nu_x}(x-y,t-s)\left(F(u^{(k)})\right)(y,s)dyds, \tag{2.39}$$

Since $F(u^{(k)}) \longrightarrow F(u)$ as $k \longrightarrow +\infty$, the dominated convergence theorem implies that $\left\{H^{(k)}\right\}$ converges to the function H in (2.23). The boundedness of $F(u)$ and the continuity of h and u_0 ensure that H is continuous on S_T. Moreover, the sequence $\left\{\Phi^{(k)}\right\}$ converge to a continuous function Φ which satisfy the integral equation (2.23). It follows by letting $k \longrightarrow +\infty$ in (2.38) that the limit u of $\left\{u^{(k)}\right\}$ satisfies the integral equation (2.37). □

Theorem 2.3. Let $\tilde{u}$, $\hat{u}$ be ordered upper and lower solutions of (2.27) and let f satisfy (2.29). Then the sequences $\left\{\overline{u}^{(k)}\right\}$, $\left\{\underline{u}^{(k)}\right\}$ converge monotonically to a unique solution u of (2.27) and :

$$\hat{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1} \leq u \leq \overline{u}^{(k+1)} \leq \overline{u}^{(k)} \leq \tilde{u} \quad \text{in } \overline{D}_T \tag{2.40}$$

Proof. By lemma 2.3 the sequences $\left\{\overline{u}^{(k)}\right\}$, $\left\{\underline{u}^{(k)}\right\}$ converge to their respective limits $\overline{u}$, $\underline{u}$, which satisfy the relation (2.36). In view of lemma 2.7, theses limits are solutions of the integral equation (2.37). Following the Theorem (2.1), the solution of the integral equation (2.37) is a solution of (2.27) if the function $q(x,t) \equiv (F(u))(x,t)$ is continuous in D_T and is locally Hölder continuous in x, uniformly in t. We have $\displaystyle\int_\Omega \Gamma(x-y,t)u_0(y)dy$ is in $C^{2+\alpha}(D_T)$ and the volume potential at the right-hand side of (2.37) is continuous in $\overline{D}_T$. The density Φ and the function H are continuous on S_T then proposition (2.1), implies that the single layer potential :

$$V_1 = \int_0^t \int_{\partial\Omega} \Gamma(x-y,t-s)\Phi(y,s)dyds,$$

is Hölder continuous in $\overline{D}_T$. By the integral representation (2.37), u is continuous in $\overline{D}_T$ and so is the function q. The continuity of q ensures that the volume potential in (2.37) is Hölder continuous in $\overline{D}_T$, which leads to the Hölder continuity of u. It follows from the Hölder continuity of $f(x,t,u)$ that q is Hölder continuous in $\overline{D}_T$. An application of Theorem 2.1 shows that u is the solution of Neumann boundary-value problem. Finally since $\overline{u}$, $\underline{u}$ are solutions of (2.27) lemma 2.6 guarantees that $\overline{u} = \underline{u}$ and is the unique solution of 2.27. This completes the the proof of the theorem. □

2.5 Monotone method for Coupled systems of reaction diffusion equations

The monotone method and its associated upper and lower solution leads to two monotone sequences each of which converges to a unique solution of an integral equation. For coupled systems of equations the definition of upper and lower solutions and the construction of monotone sequences depend on the quasi monotone property of the reaction function system. To illustrate the basic idea of this method, we consider a coupled system of two reaction-diffusion equations in the form

$$\begin{aligned}
&(u_i)_t - D_i\nabla^2 u_i = f_i(x,t,u_1,u_2) && \text{in} \quad D_T, \\
&B_i u_i = h_i(x,t) && \text{on} \quad S_T \quad (i=1,2), \\
&u_i(x,0) = u_{i,o}(x) && \text{in} \quad \Omega,
\end{aligned} \tag{2.41}$$

where B_i is the operator in the form

$$B_i \equiv \alpha_i(x,t)\frac{\partial}{\partial \nu}. \tag{2.42}$$

It is assumed that for each $i = 1,2$ that initial functions $h_i, u_{i,0}$ all satisfies the same conditions as the corresponding functions in the scalar problem (2.27). The functions f_i are assumed Hölder continuous in $D_T \times J_1 \times J_2$ for some bounded subset $J_1 \times J_2 \in \mathbb{R}^2$.

Given any nonnegative constant γ and any regions Q (open or closed, such as D_T or $\overline{D}_T$) it is denoted by $\mathcal{C}^\gamma(Q)$ the product function space of $C^\gamma(Q)$ taken N times. This mean that for any vector function $\mathbf{u} \equiv (u_1, ..., u_N)$ in $\mathcal{C}^\gamma(Q)$, the components $u_1, ..., u_N$ of $\mathbf{u}$ are all $C^\gamma(Q)$. The norm of $\mathbf{u}$ is defined by:

$$||\mathbf{u}||_\gamma = |u_1|_\gamma + ... + |u_N|_\gamma, \tag{2.43}$$

where $|u|_\gamma$ is in the norm $C^\gamma(Q)$.

Definition 2.3. A function $f_i \equiv f_i(u_1, ..., u_N)$ is said to be quasimonotone nondecreasing (resp. nonincreasing) if for fixed u_i, f_i is nondecreasing (resp., nonincreasing) in u_j for $j \neq i$.

Hence in the case of a vector function $\mathbf{f} \equiv (f_1, f_2)$ of two components there are three basic types of quasimonotone functions. These are given by the following:

Definition 2.4. A function $\mathbf{f} \equiv (f_1, f_2)$ is called quasimonotone nondecreasing (resp., nonincreasing) in $J_1 \times J_2$ if both f_1 and f_2 are quasimonotone nondecreasing (resp., nonincreasing) for $(u_1, u_2) \in J_1 \times J_2$. When f_1 is quasimonotone nonincreasing and f_2 is quasimonotone nondecreasing (or vice versa), then $\mathbf{f}$ is called mixed quasimonotone.

Remark 2.2. If $\mathbf{f}$ is a C^1-function or quasi C^1-function, then the three types of quasimonotone function in Definition (2.3) are reduced to the form

- $\dfrac{\partial f_1}{\partial u_2} \geq 0, \quad \dfrac{\partial f_2}{\partial u_1} \geq 0$ (nondecreasing quasimonotonicity).

- $\dfrac{\partial f_1}{\partial u_2} \leq 0, \quad \dfrac{\partial f_2}{\partial u_1} \leq 0$ (nonincreasing quasimonotonicity).
- $\dfrac{\partial f_1}{\partial u_2} \leq 0, \quad \dfrac{\partial f_2}{\partial u_1} \geq 0$ (mixed quasimonotonicity)

for $(u_1, u_2) \in J_1 \times J_2$.

Definition **2.5.** A pair of functions $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ in $\mathcal{C}(\overline{D}_T) \cap \mathcal{C}^{1,2}(\overline{D})$ are called ordered upper and lower solutions of (2.41) if they satisfy the relations:

(a) $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ and

(b) $B_i \tilde{u}_i \geq h_i(t,x) \geq B_i \hat{u}_i$ on S_T, $\tilde{u}_i(0,x) \geq u_{i,0}(x) \geq \hat{u}_i(0,x)$ in Ω, and

(b_1)

$$\begin{aligned}
(\tilde{u}_1)_t - D_1 \nabla^2 \tilde{u}_1 - f_1(x,t,\tilde{u}_1,\tilde{u}_2) \geq \;& 0 \; \geq (\hat{u}_1)_t - D_1 \nabla^2 \hat{u}_1 - f_1(x,t,\hat{u}_1,\hat{u}_2), \\
(\tilde{u}_2)_t - D_2 \nabla^2 \tilde{u}_2 - f_2(x,t,\tilde{u}_1,\tilde{u}_2) \geq \;& 0 \; \geq (\hat{u}_2)_t - D_2 \nabla^2 \hat{u}_2 - f_1(x,t,\hat{u}_1,\hat{u}_2),
\end{aligned} \tag{2.44}$$

when (f_1, f_2) is quasimonotone nondecreasing,

(b_2)

$$\begin{aligned}
(\tilde{u}_1)_t - D_1 \nabla^2 \tilde{u}_1 - f_1(x,t,\tilde{u}_1,\hat{u}_2) \geq \;& 0 \; \geq (\hat{u}_1)_t - D_1 \nabla^2 \hat{u}_1 - f_1(x,t,\hat{u}_1,\tilde{u}_2), \\
(\tilde{u}_2)_t - D_2 \nabla^2 \tilde{u}_2 - f_2(x,t,\hat{u}_1,\tilde{u}_2) \geq \;& 0 \; \geq (\hat{u}_2)_t - D_2 \nabla^2 \hat{u}_2 - f_1(x,t,\tilde{u}_1,\hat{u}_2),
\end{aligned} \tag{2.45}$$

when (f_1, f_2) is quasimonotone nonincreasing, and

(b_3)

$$\begin{aligned}
(\tilde{u}_1)_t - D_1 \nabla^2 \tilde{u}_1 - f_1(x,t,\tilde{u}_1,\hat{u}_2) \geq \;& 0 \; \geq (\hat{u}_1)_t - D_1 \nabla^2 \hat{u}_1 - f_1(x,t,\hat{u}_1,\tilde{u}_2), \\
(\tilde{u}_2)_t - D_2 \nabla^2 \tilde{u}_2 - f_2(x,t,\tilde{u}_1,\tilde{u}_2) \geq \;& 0 \; \geq (\hat{u}_2)_t - D_2 \nabla^2 \hat{u}_2 - f_1(x,t,\hat{u}_1,\hat{u}_2),
\end{aligned} \tag{2.46}$$

when (f_1, f_2) is mixed quasimonotone.

Remark **2.3.**

- In the above definition $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ are required to be in $\mathcal{C}(\overline{D}_T) \cap \mathcal{C}^{1,2}(D_T)$.
- The ordering relation $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ is meant in the usual component-wise sense, that is, $\tilde{u}_i \geq \hat{u}_i$ in $\overline{D}_T$ for each i.

2.5.1 Monotone sequences for coupled parabolic equations

Suppose for a given type of quasimonotone reaction function, there exist a pair of ordered upper and lower solutions $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$. Define the sector

$$\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \equiv \left\{ (u_1, u_2) \in \mathcal{C}(\overline{D}_T); \;\; (\hat{u}_1, \hat{u}_2) \leq (u_1, u_2) \leq (\hat{u}_1, \hat{u}_2) \right\}.$$

If $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ is contained in $J_1 \times J_2$, then in the definition of quasimonotone function it suffices to take $J_1 \times J_2 = \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. In the following discussion we consider each types of reaction function in the sector $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. In addition, it is assumed that there exist bounded functions $\underline{c}_i \equiv \underline{c}_i(t,x)$ such that for every (u_1, u_2), (v_1, v_2) in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, (f_1, f_2) satisfies the one-sided Lipschitz condition

$$\begin{aligned} f_1(x,t,u_1,u_2) - f_1(x,t,v_1,v_2) &\geq -\underline{c}_1(u_1 - v_1), \quad \text{when} \quad u_1 \geq v_1, \\ f_2(x,t,u_1,u_2) - f_2(x,t,v_1,v_2) &\geq -\underline{c}_2(u_2 - v_2), \quad \text{when} \quad u_2 \geq v_2. \end{aligned} \tag{2.47}$$

To ensure the uniqueness of the solution, it is also assumed that there exist bounded functions $\overline{c}_i \equiv \overline{c}_i(t,x)$ such that for every (u_1, u_2), (v_1, v_2) in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$,

$$\begin{aligned} f_1(x,t,u_1,u_2) - f_1(x,t,v_1,v_2) &\leq -\overline{c}_1(u_1 - v_1), \quad \text{when} \quad u_1 \geq v_1, \\ f_2(x,t,u_1,u_2) - f_2(x,t,v_1,v_2) &\leq -\overline{c}_2(u_2 - v_2), \quad \text{when} \quad u_2 \geq v_2. \end{aligned} \tag{2.48}$$

It is clear that, if there exist bounded functions $K_i \equiv K_i(x,t)$ such that (f_1, f_2) satisfies the Lipschitz condition

$$|f_i(x,t,u_1,u_2) - f_i(x,t,v_1,v_2)| \leq K_i \left(|u_1 - v_1| + |u_2 - v_2|\right) \;\; (i = 1, 2), \tag{2.49}$$

for $(u_1, u_2), \;\; (v_1, v_2) \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$,
then both conditions (2.47) and (2.48) are satisfied. However, in the hypotheses (2.47) and (2.48) the functions $\underline{c}_i$, $\overline{c}_i$ are not required to be positive. This weakened condition plays important role in the study of the qualitative behaviour of the solution.
Without loss of generality, it as assumed that the functions $\underline{c}_i$ in (2.47) are Hölder continuous in $\overline{D}_T$. Then,

Proposition 2.4. The functions F_1 and F_2 given by

$$F_i(x,t,u_1,u_2) = \underline{c}_i(x,t) u_i + f_i(x,t,u_1,u_2), \quad (i = 1, 2) \tag{2.50}$$

are Hölder continuous in $D_T \times \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and are monotone nondecreasing in u_i.

Define operators $\mathbb{L}_i$ by:

$$\mathbb{L}_i u_i = (u_i)_t - D_i \nabla^2 u_i + \underline{c}_i u_i, \quad (i = 1, 2) \tag{2.51}$$

then the differential equations in (2.41) are equivalent to

$$\mathbb{L}_i u_i = F_i(t, x, u_1, u_2) \quad \text{in} \;\; D_T \quad (i = 1, 2). \tag{2.52}$$

Starting from a suitable initial iteration $(u_1^{(0)}, u_2^{(0)})$, it is constructed a sequence $\left\{\mathbf{u}^{(k)}\right\} = \left\{u_1^{(k)}, u_2^{(k)}\right\}$ from the iteration process:

$$\begin{array}{rcl} \mathbb{L}_i u_i^{(k)} & = & F_i\left(u_1^{(k-1)}, u_2^{(k-1)}\right), \\ B_i u_i^{(k)} & = & h_i(x,t), \quad (i=1,2) \\ u_i^{(k)}(0,x) & = & u_{i,0}(x). \end{array} \tag{2.53}$$

The existence of $\left\{u_1^{(k)}, u_2^{(k)}\right\}$ is guaranteed by Theorem 2.1. To ensure that this sequence is monotone and converges to a solution of (2.41), it is necessary to choose a suitable initial iteration. The choice of this function depends on the type of quasimonotone property of (f_1, f_2).

(i) *Quasimonotone nondecreasing function*: For this type of quasimonotone function it suffices to take either $(\tilde{u}_1, \tilde{u}_2)$ or $(\hat{u}_1, \hat{u}_2)$ as the initial iteration $(u_1^{(0)}, u_2^{(0)})$. Denotes these two sequences by $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ and $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ respectively where $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ and $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$. The following lemma give the monotone property of these two sequences

Lemma 2.8. For quasimonotone nondecreasing (f_1, f_2) the two sequences $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ and $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ possess the monotone property

$$\underline{u}_i^{(k)} \le \underline{u}_i^{(k+1)} \le \overline{u}_i^{(k+1)} \le \overline{u}_i^{(k)} \quad \text{in } \overline{D}_T \quad (i=1,2) \tag{2.54}$$

where $k = 0, 1, 2, ...$

Proof. Let $w_i^{(0)} = \overline{u}_i^{(0)} - \overline{u}_i^{(1)} = \tilde{u}_i - \overline{u}_i^{(1)}$, $i = 1, 2$. By the definition 2.5, we have :

$$\begin{array}{rcl} \mathbb{L}_i w_i^{(0)} & = & (\tilde{u}_i)_t - D_i \nabla^2 \tilde{u}_i + \underline{c}_i \tilde{u}_i - F_i(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) \\ & = & (\tilde{u}_i)_t - D_i \nabla^2 \tilde{u}_i - f_i(\tilde{u}_1, \tilde{u}_2) \ge 0, \\ B_i w_i^{(0)} & = & B_i \tilde{u}_i - h_i \ge 0, \\ w_i^{(0)}(x,0) & = & \tilde{u}_i(x,0) - u_i(x,0) \ge 0. \end{array} \tag{2.55}$$

In view of proposition 2.3, $w_i^{(0)} \ge 0$, which shows that $\overline{u}_i^{(1)} \le \overline{u}_i^{(0)}$. Using the property of a lower solution the same reasoning gives $\underline{u}_i^{(1)} \ge \underline{u}_i^{(0)}$. Let $w_i^{(1)} = \overline{u}_i^{(1)} - \underline{u}_i^{(1)}$, $i = 1, 2$. Then by (2.53) and the monotone property of F_i ,

$$\mathbb{L}_i w_i^{(1)} = F_i(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) - F_i(\underline{u}_1^{0}, \underline{u}_2^{(0)}) \ge 0. \tag{2.56}$$

Since:

$$B_i w_{(i)}^1 = h_i - h_i = 0, \quad w_i(x,0) = u_{i,0} - u_{i,0} = 0, \tag{2.57}$$

it follows again from proposition 2.3 that $w_i^{(1)} \ge 0$. The above conclusions yield the relation

$\underline{u}_i^{(0)} \leq \underline{u}_i^{(1)} \leq \overline{u}_i^{(1)} \leq \overline{u}_i^{(0)}$ for $i = 1, 2$. Assume, by induction that :

$$\underline{u}_i^{(k-1)} \leq \underline{u}_i^{(k)} \leq \overline{u}_i^{(k)} \leq \overline{u}_i^{(k-1)} \quad (i = 1, 2). \tag{2.58}$$

Then by (2.53) and the monotone property of F_i the function $w_i^{(k)} = \overline{u}_i^{(k)} - \overline{u}_i^{(k+1)}$ satisfies the relation:

$$\mathbb{L}_i w_i^{(k)} = F_i(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}) - F_i(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}) \geq 0, \tag{2.59}$$

and the boundary and initial conditions in (2.57). This lead to the result $\overline{u}_i^{(k+1)} \leq \overline{u}_i^{(k)}$. A similar argument gives $\underline{u}_i^{(k+1)} \geq \underline{u}_i^{(k)}$ and $\underline{u}_i^{(k+1)} \leq \overline{u}_i^{(k+1)}$ for result $i = 1, 2$. Result (2.54) follows by induction. □

(ii) *Quasimonotone nonincreasing function*: When the reaction function (f_1, f_2) is quasimonotone nonincreasing, it is choose $(\tilde{u}_1, \hat{u}_2)$ or $(\hat{u}_1, \tilde{u}_2)$ as the initial iteration in (2.53). The corresponding sequences is denoted by $\left\{\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$, respectively, where $(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\tilde{u}_1, \hat{u}_2)$ and $(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\hat{u}_1, \tilde{u}_2)$. The following lemma gives the monotone property of these sequences:

Lemma 2.9. For quasimonotone nonincreasing (f_1, f_2) the two sequences $\left\{\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ possess the monotone property in relation (2.54).

Proof. Let $w_1^{(0)} = \overline{u}_1^{(0)} - \overline{u}_1^{(1)} = \tilde{u}_1 - \overline{u}_1^{(1)}$, $w_2^{(0)} = \underline{u}_2^{(1)} - \underline{u}_2^{(0)} = \underline{u}_2^{(1)} - \hat{u}_2$. By (2.45) and (2.53),

$$\begin{aligned}
\mathbb{L}_1 w_1^{(0)} &= (\tilde{u}_1)_t - D_1 \nabla^2 \tilde{u}_1 + \underline{c}\tilde{u}_1 - F_1\left(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}\right), \\
&= (\tilde{u}_1)_t - D_1 \nabla^2 \tilde{u}_1 - f_1(\tilde{u}_1, \hat{u}_2) \geq 0, \\
\mathbb{L}_2 w_2^{(0)} &= F_2\left(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}\right) - \left((\hat{u})_t - D_2 \nabla^2 \hat{u}_2 + \underline{c}_2 \hat{u}_2\right), \\
&= f_2(\tilde{u}_1, \hat{u}_2) - \left((\hat{u}_2)_t - D_2 \nabla^2 \hat{u}_2\right).
\end{aligned} \tag{2.60}$$

Since $w_i^{(0)}$, $i = 1, 2$ satisfy the boundary and initial inequalities in (2.55) proposition (2.3), implies that $w_i^{(0)} \geq 0$. This proves that $\overline{u}_1^{(1)} \leq \overline{u}_1^{(0)}$, $\underline{u}_2^{(1)} \geq \underline{u}_2^{(0)}$. A similar argument, using the property of $(\hat{u}_1, \tilde{u}_2)$, gives $\underline{u}_1^{(1)} \geq \underline{u}_1^{(0)}$, $\overline{u}_2^{(1)} \leq \overline{u}_2^{(0)}$.
Let $w_i^{(1)} = \overline{u}_i^{(1)} - \underline{u}_i^{(1)}$, $i = 1, 2$. By (2.53), $w_i^{(1)}$ satisfies the relation:

$$\begin{aligned}
\mathbb{L}_1 w_1^{(1)} &= F_1\left(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}\right) - F_1\left(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}\right), \\
&= \left[\underline{c}_1\left(\tilde{u}_1, \hat{u}_1\right) + f_1(\tilde{u}_1, \hat{u}_2) - f_1(\hat{u}_1, \hat{u}_2)\right] + \left[f_1(\hat{u}_1 - \hat{u}_2) - f_1(\hat{u}_1, \tilde{u}_2)\right],
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}_2 w_2^{(1)} &= F_2\left(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}\right) - F_2\left(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}\right), \\
&= \left[\underline{c}_2\left(\tilde{u}_2 - \hat{u}_2\right) + f_2(\hat{u}_1, \tilde{u}_2) - f_2(\hat{u}_1, \hat{u}_2)\right] + \left[f_2(\hat{u}_1, \hat{u}_2) - f_2(\tilde{u}_1, \hat{u}_2)\right].
\end{aligned}$$

Since by (2.47) and the quasimonotone nonincreasing property of (f_1, f_2), the right-hand side of the above two equations are both nonnegative. Proposition 2.3 ensures that $w_i^{(1)} \geq 0$. This leads to the relation $\underline{u}_i^{(0)} \leq \underline{u}_i^{(1)} \leq \overline{u}_i^{(1)} \leq \overline{u}_i^{(0)}$, $i = 1, 2$. The proof for the monotone property (2.54) follows by a similar induction argument for lemma (2.8). □

(iii) *Mixed quasimonotone function*: The construction of monotone sequences for mixed quasimonotone functions requires the use of both upper and lower solutions simultaneously. When f_1 is quasimonotone nonincreasing, and f_2 is quasimonotone nondecreasing, the monotone iteration process is given by:

$$\begin{array}{ll} \mathbb{L}_1\overline{u}_1^{(k)} = F_1(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}), & \mathbb{L}_2\overline{u}_2^{(k)} = F_2(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}) \\ \mathbb{L}_1\underline{u}_1^{(k)} = F_1(\underline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}), & \mathbb{L}_2\underline{u}_2^{(k)} = F_2(\underline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}) \end{array} \tag{2.61}$$

where $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$.

Lemma 2.10. For mixed quasimonotone (f_1, f_2) the sequences $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ and $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$ given by (2.61) possess the monotone property (2.54)

Proof. Let $w_i^{(0)} = \overline{u}_i^{(0)} - \overline{u}_i^{(1)} = \tilde{u}_i - \overline{u}_i^{(1)}$, $i = 1, 2$. By (2.4) and (2.46),

$$\begin{array}{rcl} \mathbb{L}w_1^{(0)} & = & (\tilde{u}_1)_t - D_1\nabla^2\tilde{u}_1 - f_1(\tilde{u}_1, \hat{u}_2) \geq 0, \\ \mathbb{L}w_2^{(0)} & = & (\tilde{u}_2)_t - D_2\nabla^2\tilde{u}_2 - f_2(\tilde{u}_1, \tilde{u}_2) \geq 0, \end{array}$$

Since $w_i^{(0)}$ satisfies the boundary and initial inequalities in (2.55), Proposition (2.3) implies that $\overline{u}_i^{(1)} \leq \overline{u}_i^{(0)}$, $i = 1, 2$. Using the equations for $(\underline{u}_1^{(1)}, \underline{u}_2^{(1)})$ in (2.46) and the property of a lower solution, a similar argument gives $\underline{u}_i^{(1)} \geq \underline{u}_i^{(0)}$. Let $w_i^{(1)} = \overline{u}_i^{(1)} - \underline{u}_i^{(1)}$, $i = 1, 2$. By (2.47) and the mixed quasimonotone property of (f_1, f_2), :

$$\begin{array}{rcl} \mathbb{L}_1 w_1^{(1)} & = & [\underline{c}_1(\tilde{u}_1 - \hat{u}_1) + f_1(\tilde{u}_1, \hat{u}_2) - f_1(\hat{u}_1, \hat{u}_2)] + f_1(\hat{u}_1, \hat{u}_2) - f_1(\hat{u}_1, \tilde{u}_2), \\ & \geq & 0, \\ \mathbb{L}_2 w_2^{(1)} & = & [\underline{c}_2(\tilde{u}_2 - \hat{u}_2) + f_2(\tilde{u}_1, \tilde{u}_2) - f_2(\tilde{u}_1, \hat{u}_2)] + f_2(\tilde{u}_1, \hat{u}_2) - f_2(\hat{u}_1, \hat{u}_2), \\ & \geq & 0. \end{array}$$

This implies that $w_i^{(1)} \geq 0$, which leads to the relation $\overline{u}_i^{(1)} \geq \underline{u}_i^{(1)}$. The above conclusions shows that $\underline{u}_i^{(0)} \leq \underline{u}_i^{(1)} \leq \overline{u}_i^{(1)} \leq \underline{u}_i^{(0)}$, $i = 1, 2$. Assume by induction that (2.58) holds. Then by (2.47) and the mixed quasimonotone property of (f_1, f_2) the functions $w_i^{(k)} = \overline{u}_i^{(k)} - \overline{u}_i^{(k+1)}$,

$i = 1, 2$, satisfy the condition (2.57) and the relation:

$$\begin{aligned}
\mathbb{L}_1 w_1^{(k)} &= \left[\underline{c}_1(\overline{u}_1^{(k-1)} - \overline{u}_1^{(k)}) + f_1\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}\right) - f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k-1)}\right)\right] \\
&\quad + f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k-1)}\right) - f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right), \\
&\geq 0, \\
\mathbb{L}_2 w_2^{(k)} &= \left[\underline{c}_2(\overline{u}_2^{(k-1)} - \overline{u}_2^{(k)}) + f_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}\right) - f_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k)}\right)\right] \\
&\quad + f_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k)}\right) - f_2\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right), \\
&\geq 0.
\end{aligned}$$

It follows again from Proposition (2.3) that $\overline{u}_i^{(k+1)} \leq \overline{u}_i^{(k)}$. Similar arguments leads to $\underline{u}_i^{(k+1)} \geq \underline{u}_i^{(k)}$ and $\overline{u}_i^{(k+1)} \geq \underline{u}_i^{(k)}$. The conclusion of the lemma follows by induction. □

The above construction of monotone sequences yields a sequence of ordered upper and lower solution for (2.41), which are given the following:

Lemma 2.11. Let $(\tilde{u}_1, \tilde{u}_2)$, $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (2.41) and let (f_1, f_2) be quasimonotone and satisfy condition (2.47). Then for each type of quasimonotone (f_1, f_2), the corresponding iterations $(\overline{u}_1^{(k)}, \overline{u}_2^{(k)})$ and $(\underline{u}_1^{(k)}, \underline{u}_2^{(k)})$ given by Lemmas (2.8) to (2.10) are ordered upper and lower solutions.

Proof. Consider the case where (f_1, f_2) is quasimonotone nondecreasing. Then by (2.47) and (2.53),

$$\begin{aligned}
(\overline{u}_i^{(k)})_t - D_i \nabla^2 \overline{u}_i^{(k)} &= \left[\underline{c}_i\left(\overline{u}_i^{(k-1)} - \overline{u}_i^{(k)}\right) + f_i\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}\right) - f_i\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k-1)}\right)\right] \\
&\quad + \left[f_i\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k-1)}\right) - f_i\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right)\right] + f_i\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right), \\
&\geq f_i\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right),
\end{aligned}$$

for $i = 1$, and a similar relation holds for $i = 2$. Since $\underline{u}_i^{(k)}$ satisfies the boundary and initial conditions in (2.53), the above inequality shows that $(\overline{u}_1^{(k)}, \overline{u}_2^{(k)})$ is an upper solution. The proof for the lower solution is the same. When (f_1, f_2) is quasimonotone nonincreasing it suffices to show that the pair $(\overline{u}_1^{(k)}, \underline{u}_2^{(k)})$ and $(\underline{u}_1^{(k)}, \overline{u}_2^{(k)})$ satisfy the inequalities in (2.48). Since by the construction of the sequences for quasimonotone nonincreasing functions,

$$\begin{aligned}
(\overline{u}_1^{(k)})_t - D_1 \nabla^2 \overline{u}_1^{(k)} &= \left[\underline{c}_1\left(\overline{u}_i^{(k-1)} - \overline{u}_1^{(k)}\right) + f_1\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}\right) - f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k-1)}\right)\right] \\
&\quad + \left[f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k-1)}\right) - f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right)\right] + f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right), \\
(\underline{u}_2^{(k)})_t - D_2 \nabla^2 \underline{u}_2^{(k)} &= -\left[\underline{c}_2\left(\underline{u}_2^{(k)} - \underline{u}_2^{(k-1)}\right) + f_2\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k)}\right) - f_2\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}\right)\right] \\
&\quad + \left[f_2\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k)}\right) - f_2\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right)\right] + f_2\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right),
\end{aligned}$$

condition (2.47) and the quasimonotone nonincreasing property of (f_1, f_2) imply that:

$$\begin{aligned}
(\overline{u}_1^{(k)})_t - D_1 \nabla^2 \overline{u}_1^{(k)} &\leq f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right), \\
(\underline{u}_2^{(k)})_t - D_2 \nabla^2 \underline{u}_2^{(k)} &\geq f_2\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right).
\end{aligned}$$

A similar argument gives:

$$\begin{aligned}
(\underline{u}_1^{(k)})_t - D_1\nabla^2\underline{u}_1^{(k)} &\leq f_1\left(\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right), \\
(\overline{u}_2^{(k)})_t - D_2\nabla^2\overline{u}_2^{(k)} &\geq f_2\left(\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right).
\end{aligned}$$

This shows that $(\overline{u}_1^{(k)}, \overline{u}_2^{(k)})$ and $(\underline{u}_1^{(k)}, \underline{u}_2^{(k)})$ are ordered upper and lower solutions for quasimonotone nonincreasing functions. Finaly for mixed quasimonotone (f_1, f_2), $(\overline{u}_1^{(k)}, \overline{u}_2^{(k)})$ and $(\underline{u}_1^{(k)}, \underline{u}_2^{(k)})$ are determined by (2.61). In view of (2.47) and the mixed quasimonotone property of (f_1, f_2),

$$\begin{aligned}
(\overline{u}_1^{(k)})_t - D_1\nabla^2\overline{u}_1^{(k)} &= \left[\underline{c}_1\left(\overline{u}_i^{(k-1)} - \overline{u}_1^{(k)}\right) + f_1\left(\overline{u}_1^{(k-1)}, \underline{u}_2^{(k-1)}\right) - f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k-1)}\right)\right] \\
&\quad + \left[f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k-1)}\right) - f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right)\right] + f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right), \\
&\geq f_1\left(\overline{u}_1^{(k)}, \underline{u}_2^{(k)}\right),
\end{aligned}$$

$$\begin{aligned}
(\overline{u}_2^{(k)})_t - D_2\nabla^2\overline{u}_2^{(k)} &= \left[\underline{c}_2\left(\overline{u}_2^{(k-1)} - \overline{u}_2^{(k)}\right) + f_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k-1)}\right) - f_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k)}\right)\right] \\
&\quad + \left[f_2\left(\overline{u}_1^{(k-1)}, \overline{u}_2^{(k)}\right) - f_2\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right)\right] + f_2\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right), \\
&\geq f_2\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right).
\end{aligned}$$

A similar argument gives:

$$\begin{aligned}
(\underline{u}_1^{(k)})_t - D_1\nabla^2\underline{u}_1^{(k)} &\leq f_1\left(\underline{u}_1^{(k)}, \overline{u}_2^{(k)}\right), \\
(\underline{u}_2^{(k)})_t - D_2\nabla^2\underline{u}_2^{(k)} &\leq f_2\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right).
\end{aligned}$$

This shows that $(\overline{u}_1^{(k)}, \overline{u}_2^{(k)})$ and $(\underline{u}_1^{(k)}, \underline{u}_2^{(k)})$ are upper and lower solutions for mixed quasimonotone functions.This completes the proof of the lemma. ☐

Lemmas (2.8) to (2.10) imply that each type of quasimonotone functions the corresponding sequence obtained from (2.54) and (2.61) converge monotonitically to some limit. Define:

$$\lim_{k\to\infty} \overline{u}_i^{(k)}(x,t) = \overline{u}_i(x,t), \quad \lim_{k\to\infty} \underline{u}_i^{(k)}(t,x) = \underline{u}_i(t,x). \tag{2.62}$$

We show that under conditions (2.47) and (2.48), $\overline{u}_i = \underline{u}_i = u_i$ and $\mathbf{u} = (u_1, u_2)$ is the unique solution of (2.41). The proof result is based of integral representation for reaction-diffusion problems. Since for each k problem (2.53) is the same as in (2.74), the solution $u_i^{(k)}$ is given by:

$$\begin{aligned}
u_i^{(k)}(x,t) &= \int_\Omega \Gamma_i(x-y,t)u_{i,0}(y)dy + \int_0^t\int_\Omega \Gamma_i(x-y,t-s)\left(F_i(\mathbf{u}^{(k-1)})\right)(y,s)dyds \\
&\quad + \int_0^t\int_{\partial\Omega} \Gamma_i(x-y,t-s)\Phi_i^{(k-1)}(y,s)dyds.
\end{aligned} \tag{2.63}$$

Here the density $\Phi_i^{(k)}$ depends on k and is governed by the integral equation:

$$\Phi_i^{(k)}(x,t) = 2\int_0^t \int_\Omega \frac{\partial \Gamma_i}{\nu_x}(x-y,t-s)\Phi^{(k)}(y,s)dyds - 2H_i^{(k)}(x,t), \tag{2.64}$$

where $H_i^{(k)}$ is given by:

$$H_i^{(k)}(x,t) = \int_\Omega \frac{\partial \Gamma_i}{\partial \nu_x}(x-y,t)u_{i,0}(y)dy + h_i(x,t) + \int_0^t \int_\Omega \frac{\partial \Gamma_i}{\partial \nu_x}(x-y,t-s)\left(F_i(\mathbf{u}^{(k)})\right)(y,s)dyds, \tag{2.65}$$

where $\mathbf{u}^{(k)} = (u_1^{(k)}, u_2^{(k)})$ and $(F_i(\mathbf{u}))(x,t) = F_i(x,t,u_1(x,t),u_2(x,t))$.
Following the same argument as in the proof of theorem 2.3 page 35, we have the following existence-comparison theorems for each of three types of reaction functions.

Theorem 2.4. Let $(\tilde{u}_1, \tilde{u}_2)$, $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (2.41), and let (f_1, f_2) be quasimonotone nondecreasing in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and satisfy the conditions (2.47) and (2.48). Then problem (2.41) has a unique solution $\mathbf{u} \equiv (u_1, u_2)$ in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Moreover, the sequences $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$, $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$, obtained from (2.53) with $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ converge monotonically to (u_1, u_2) and satisfies the relation:

$$(\hat{u}_1, \hat{u}_2) \leq (\underline{u}_1^{(k)}, \underline{u}_2^{(k)}) \leq (u_1, u_2) \leq (\overline{u}_1^{(k)}, \overline{u}_2^{(k)}) \leq (\tilde{u}_1, \tilde{u}_2) \quad \text{in } \overline{D}_T \tag{2.66}$$

for every $k = 1, 2, ...$

Proof. We consider the sequences $\left\{u_1^{(k)}, u_2^k\right\}$ represents either $\left\{\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right\}$ or $\left\{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right\}$. Since by Lemma 2.8, this sequences converge monotonically to some limits (u_1, u_2) as $k \longrightarrow +\infty$. The continuity and monotonicity property of F_i implies that $F_i(u_1^{(k)}, u_2^{(k)})$ converges monotonically to $F_i(u_1, u_2)$. Moreover, the function $H_i^{(k)}$ in (2.65) converge monotonically to the function

$$H_i(x,t) = \int_\Omega \frac{\partial \Gamma_i}{\partial \nu_x}(x-y,t)u_{i,0}(y)dy + h_i(x,t) + \int_0^t \int_\Omega \frac{\partial \Gamma_i}{\partial \nu_x}(x-y,t-s)\left(F_i(\mathbf{u})\right)(y,s)dyds. \tag{2.67}$$

The continuity of h_i, $u_{i,0}$ and the boundedness of $F_i(\mathbf{u})$ imply that H_i is continuous on S_T. By Proposition 2.2 page 29, the sequences $\left\{\Phi_i^{(k)}\right\}$ converges to a continuous function Φ_i which satisfy the integral equation (2.23). It follows by applying the dominated convergence to the integral representation (2.63) for $\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right)$ that the limit (u_1, u_2) satisfy the integral equation:

$$\begin{aligned} u_i(x,t) \;\; = \;\; & \int_\Omega \Gamma_i(x-y,t)u_{i,0}(y)dy + \int_0^t \int_\Omega \Gamma_i(x-y,t-s)\left(F_i(\mathbf{u})\right)(y,s)dyds \\ & + \int_0^t \int_{\partial\Omega} \Gamma_i(x-y,t-s)\Phi_i(y,s)dyds. \end{aligned} \tag{2.68}$$

Using the function $q \equiv q(x,t) = F_i(x,t,u_1(x,t),u_2(x,t))$ the same reasoning as in the proof of Theorem 2.3 page 35, shows that (u_1, u_2) is a solution of (2.41). To show the uniqueness of the solution we observe the function $w_i = \underline{u}_i - \overline{u}_i$ satisfies the relation:

$$\begin{aligned} &(w_i)_t - D_i \nabla^2 w_i - \overline{c} w_i = f_i(x, t, \underline{u}_i) - f_i(x, t, \overline{u}_i) + \overline{c}(\overline{u}_i - \underline{u}_i) \geq 0, \\ &B_i w_i = 0. \end{aligned} \tag{2.69}$$

In view of the proposition 2.3, $w \geq 0$ leading to the conclusion $\underline{u} = \overline{u}$. □

Theorem 2.5. Let $(\tilde{u}_1, \tilde{u}_2)$, $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (2.41), and let (f_1, f_2) be quasimonotone nonincreasing in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and satisfy the conditions (2.47) and (2.48). Then problem (2.41) has a unique solution $\mathbf{u} \equiv (u_1, u_2)$ in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Moreover, the sequences $\left\{ \overline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$, $\left\{ \underline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$, obtained from (2.53) with $(\overline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\tilde{u}_1, \hat{u}_2)$ and $(\underline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\hat{u}_1, \tilde{u}_2)$ converge monotonically to (u_1, u_2). The monotone property of the sequences is in the sense of (2.54).

Theorem 2.6. Let $(\tilde{u}_1, \tilde{u}_2)$, $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (2.41), and let (f_1, f_2) be mixed quasimonotone in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and satisfy the conditions (2.47) and (2.48). Then problem (2.41) has a unique solution $\mathbf{u} \equiv (u_1, u_2)$ in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Moreover, the sequences $\left\{ \overline{u}_1^{(k)}, \overline{u}_2^{(k)} \right\}$, $\left\{ \underline{u}_1^{(k)}, \underline{u}_2^{(k)} \right\}$, obtained from (2.61) with $(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}) = (\tilde{u}_1, \tilde{u}_2)$ and $(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}) = (\hat{u}_1, \hat{u}_2)$ converge monotonically to (u_1, u_2) and satisfy relation (2.66).

2.6 Coupled of reaction-diffusion equations with Nonquasimonotone functions

Let $w_i = e^{-\gamma t} u_i$, where γ is a positive constant to be chosen. Then problem (2.41) is transformed to the form:

$$\begin{aligned} &(w_i)_t - D_i \nabla^2 w_i + \gamma w_i = f_i^*(x, t, \mathbf{w}) && \text{in} \quad D_T, \\ &B_i w_i = h_i^*(x, t) && \text{on} \quad S_T \quad (i = 1, 2), \\ &w_i(x, 0) = u_{i,o}(x) && \text{in} \quad \Omega, \end{aligned} \tag{2.70}$$

where $\mathbf{w} = (w_1, w_2)$, $h_i^* = e^{-\gamma t} h_i$ and

$$f_i^*(x, t, \mathbf{w}) = e^{-\gamma t} f_i(x, t, e^{\gamma t} \mathbf{w}). \tag{2.71}$$

It is assumed that there exist some bounded functions $K_i \equiv K_i(t, x) \in D_T$ such that

$$|f_i(x, t, \mathbf{u}) - f_i(x, t, \mathbf{v})| \leq K_i |\mathbf{u} - \mathbf{v}| \quad \text{for } \ \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \tag{2.72}$$

where $|\mathbf{u} - \mathbf{v}| \equiv |u_1 - v_1| + |u_2 - v_2|$, then

Proposition 2.5. f_i^* satisfies the global Lipschitz condition:

$$|f_i^*(x, t, \mathbf{w}) - f_i^*(x, t, \mathbf{v})| \leq K_i |\mathbf{w} - \mathbf{v}|. \tag{2.73}$$

Given any function $\mathbf{w}^{(0)} \equiv (w_1^0, w_2^0)$ in $\mathcal{C}^\alpha(\overline{D}_T)$, it is constructed a sequence $\left\{\mathbf{w}^{(k)}\right\} \equiv \left\{w_1^{(k)}, w_2^{(k)}\right\}$ successively from the iteration process:

$$\begin{aligned}
&\left(w_i^{(k)}\right)_t - D_i\nabla^2 w_i^{(k)} + \gamma w_i^{(k)} = f_i^*(x,t,\mathbf{w}^{(k-1)}),\\
&B_i w_i^{(k)} = h_i^*(x,t),\\
&w_i^{(k)}(x,0) = u_{i,0}(x).
\end{aligned} \tag{2.74}$$

It is show that under global Lipschitz condition in (2.73) this sequence converges to a unique solution of (2.70) for any $\mathbf{w}^{(0)} \in \mathcal{C}^\alpha(\overline{D}_T)$. The convergence proof is based on contraction mapping theorem in the banach space $\mathcal{C}(\overline{D}_T)$ equipped with the norm:

$$||\mathbf{w}||_0 = \sup_{(x,t)\in D_T} \max\left\{|w_1(x,t)|, |w_2(x,t)|\right\}, \quad (\mathbf{w} = (w_1, w_2)).$$

Let:

- $X = C(\overline{D}_T) \cap C^\alpha(D_T)$,
- $\mathcal{X} = \mathcal{C}(\overline{D}_T) \cap \mathcal{C}^\alpha(D_T)$,

and defines operators:

$$A_i : \ D(A_i) \longrightarrow C(D_T) \ \text{ and } \ f_i^* : \mathcal{X} \longrightarrow X$$

by:

$$\begin{aligned}
&A_i w_i = (w_i)_t - D_i\nabla^2 w_i + \gamma w_i, \quad (w_i \in D(A_i)),\\
&f_i^*(\mathbf{w}) = f_i^*(x,t,\mathbf{w}), \quad (\mathbf{w} \in \mathcal{X}),
\end{aligned} \tag{2.75}$$

where $D(A_i)$ is the domain of A_i given by

$$D(A_i) = \left\{w_i \in C(\overline{D}_T) \cap C^{1,2}(D_T);\ Bw_i = h_i^*,\ w_i(x,0) = u_{i,0}(x)\right\}. \tag{2.76}$$

Clearly A_i is a sublinear operator with domain $D(A_i)$ contained in X and range $R(A_i)$ in $C(D_T)$. Next it is defined the operators $\mathcal{A} :\ D(\mathcal{A}) \longrightarrow \mathcal{C}(D_T)$ and $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{X}$ by:

$$\mathcal{A}\mathbf{w} = (A_1 w_1, A_2 w_2), \tag{2.77}$$

and

$$\mathcal{F}(\mathbf{w}) = (f_1^*(\mathbf{w}), f_2^*(\mathbf{w})), \tag{2.78}$$

where $D(\mathcal{A}) = D(A_1) \times D(A_2)$.

With the definition problem (2.70) is equivalent to the equation

$$\mathcal{A}\mathbf{w} = \mathcal{F}(\mathbf{w}), \quad (\mathbf{w} \in D(\mathcal{A})), \tag{2.79}$$

and the sequences of iterations in (2.74) becomes

$$\mathcal{A}\mathbf{w}^{(k)} = \mathcal{F}(\mathbf{w}^{(k-1)}), \quad (\mathbf{w}^{(k)} \in D(\mathbf{A})). \tag{2.80}$$

The following lemma gives some properties of the operators A_i and $\mathcal{A}$.

Lemma 2.12. For any $\gamma > 0$ the inverse operators A_i^{-1}, $\mathcal{A}^{-1}$ exits and

$$\begin{aligned} &|A_i^{-1}w_i - A_i^{-1}w_i'|_0 \leq \gamma^{-1}|w_i - w_i'|_0, \\ &||\mathcal{A}^{-1}\mathbf{w}_i - \mathcal{A}^{-1}\mathbf{w'}_i||_0 \leq \gamma^{-1}||\mathbf{w}_i - \mathbf{w'}_i||_0. \end{aligned} \tag{2.81}$$

Proof. Let $v = w_i - w_i'$ and let (x_i, t_i) be a point in $\overline{D}_T$ such that $|v|_0 = |v(x_i, t_i)|$, where w_i, w_i' are any two functions in $D(A_i)$. We first show that

$$v(x_i, t_i)(A_iw_i - A_iw_i') \geq |v|_0^2 \quad (w_i, w_i' \in D(A_i)).$$

This inequality is trivially satisfied when $v(x_i, t_i) = 0$. If $v(x_i, t_i) \neq 0$, then $v(x_i, t_i)$ is either a positive maximum or a negative minimum in $\overline{D}_T$. It is obvious from $v(x_i, 0) = 0$ that $t_i > 0$. This implies that $v_t(x_i, t_i) = 0$ when $t_i < T$ and

$$v(x_i, t_i)v_t(x_i, t_i) \geq 0 \quad \text{when} \;\; t_i = T. \tag{2.82}$$

Moreover, $\dfrac{\partial v}{\partial \nu}(x_i, t_i) = 0$, then $x_i \in \Omega$. Moreover, $v_{x_j}(x_i, t_i) = 0$ for every $j = 1, 2, ..., n$ and

$$v(x_i, t_i)\left(D_i \sum_{j=1}^{n} \frac{\partial^2 v(x_i, t_i)}{\partial x_j^2}\right) \leq 0. \tag{2.83}$$

This imply from (2.82) and (2.83) that:

$$v(x_i, t_i)\left(v_t(x_i, t_i) - D_i\nabla^2 v(x_i, t_i)\right) \geq 0.$$

It follows from the definition of A_i that:

$$v(x_i, t_i)\left(A_iw_i - A_iw_i'\right)(x_i, t_i) \geq \gamma|v(x_i, t_i)|^2 = \gamma|v|_0^2.$$

By the relation:

$$|v|_0|A_iw_i - A_iw_i'|_0 \geq v(x_i, t_i)\left(A_iw_i - A_iw_i'\right)(x_i, t_i),$$

the operator A_i possesses the property:

$$|A_iw_i - A_iw_i'|_0 \geq \gamma|w_i - w_i'|_0. \tag{2.84}$$

Hence the inverse A_i^{-1} exists on $R(A_i)$ and satisfies the first inequality in (2.81). Since the addition over i of the relation (2.84) gives

$$||\mathcal{A}\mathbf{w}-\mathcal{A}\mathbf{w}'||_0 \geq \gamma||\mathbf{w}-\mathbf{w}'||_0, \quad (\mathbf{w},\mathbf{w}' \in R(\mathcal{A})), \tag{2.85}$$

it follows that $\mathcal{A}^{-1}$ exists on $R(\mathcal{A})$ and satisfy the second inequality in (2.81). □

In view of lemma (2.12) the equations (2.79) and (2.80) can be written as

$$\mathbf{w} = \mathcal{A}^{-1}\mathcal{F}(\mathbf{w}), \quad (\mathbf{w} \in D(\mathcal{A})), \tag{2.86}$$

and

$$\mathbf{w}^{(k)} = \mathcal{A}^{-1}\mathcal{F}(\mathbf{w}^{(k-1)}), \quad (\mathbf{w}^{(k)} \in D(\mathcal{A})), \tag{2.87}$$

respectively. Let $\overline{K} = \max(K_1;K_2)$. The following lemma shows that for $\gamma > \overline{K}$, the operator $\mathcal{A}^{-1}\mathcal{F}$ is a contraction mapping in $\mathcal{X}$ with respect to the norm in $\mathcal{C}(\overline{D}_T)$.

Lemma 2.13. Let $\gamma > \overline{K}$ and consider $\mathcal{A}^{-1}\mathcal{F}$ as mapping from $\mathcal{X}$ into $\mathcal{C}(\overline{D}_T)$. Then $\mathcal{A}^{-1}\mathcal{F}$ possesses the contraction property

$$||\mathcal{A}^{-1}\mathcal{F}(\mathbf{w})-\mathcal{A}^{-1}\mathcal{F}(\mathbf{w}')||_0 \leq \left(\frac{\overline{K}}{\gamma}\right)||\mathbf{w}-\mathbf{w}'||_0, \quad (\mathbf{w},\mathbf{w}' \in \mathcal{X}). \tag{2.88}$$

Proof. For each i and each $\mathbf{v} \in \mathcal{X}$, the function $f_i^*(\mathbf{v})$ is in $\mathcal{X}$. Then there exits a unique $u_i \in D(A_i)$ such that $A_i u_i = f_i^*(\mathbf{v})$. This implies that $f_i(\mathbf{v}) \in R(A_i)$ and thus $\mathcal{F}(\mathbf{v}) \in R(\mathcal{A})$. This implies that $f_i^*(\mathbf{v}) \in R(A_i)$ and thus $\mathcal{F}(\mathbf{v}) \in R(\mathcal{A})$. Hence by Lemma 2.12, $\mathcal{A}^{-1}\mathcal{F}$ is well defined on $\mathcal{X}$. Since by (2.73),

$$|f_i^*(\mathbf{w})-f_i^*(\mathbf{w}')| \leq K_i|\mathbf{w}-\mathbf{w}'| \leq K_i||\mathbf{w}-\mathbf{w}'||_0, \tag{2.89}$$

the function $\mathcal{F}$ satisfies the Lipschitz condition:

$$||\mathcal{F}(\mathbf{w})-\mathcal{F}(\mathbf{w}')||_0 \leq \overline{K}||\mathbf{w}-\mathbf{w}'||_0, \quad (\mathbf{w},\mathbf{w}' \in \mathcal{X}). \tag{2.90}$$

It follows from Lemma 2.12 that,

$$||\mathcal{A}^{-1}\mathcal{F}(\mathbf{w})-\mathcal{A}^{-1}\mathcal{F}(\mathbf{w}')||_0 \leq \gamma^{-1}||\mathcal{F}(\mathbf{w})-\mathcal{F}(\mathbf{w}')||_0 \leq \left(\overline{K}/\gamma\right)||\mathbf{w}-\mathbf{w}'||_0. \tag{2.91}$$

□

The contraction property of $\mathcal{A}^{-1}\mathcal{F}$ and the regularity argument for solutions of integral equations lead to the following existence-uniqueness result.

Theorem 2.7. Let $f_i(t,x,\mathbf{u})$ satisfy the gobal Lipschitz condition (2.73). Then the problem (2.41) has a unique solution. Moreover, $\mathbf{u}$ is the limit of the sequence $\left\{\mathbf{u}^{(k)}\right\}$ given by the equation:

$$\left(u_i^{(k)}\right)_t - D_i\nabla^2 w_i^{(k)} = f_i(x,t,\mathbf{u}^{(k-1)}) \quad \text{in} \quad D_T. \tag{2.92}$$

Proof. For the existence-uniqueness problem it suffices to show the same result for the transformed problem (2.70). By the contraction property of $\mathcal{A}^{-1}\mathcal{F}$, the sequence $\left\{w_1^{(k)}, w_2^{(k)}\right\}$ given by (2.87) converge in $\mathcal{C}(\overline{D}_T)$ to a unique solution $\mathbf{w} = (w_1^*, w_2^*)$ of (2.86), where $\mathbf{w}^{(0)}$ is any function in $\mathcal{X}$. Since (2.87) is equivalent to (2.74), the functions $w_i^{(k)}$ can be represented by the integral formula (2.63). In this integral representation $\underline{c}_i = \gamma$ and the functions u_i and $F_i(\mathbf{u})$ should be replaced, respectively by w_i and $F_i(\mathbf{w})$,$i = 1, 2$. By the uniform convergence of $\left\{w_1^{(k)}, w_2^{(k)}\right\}$, the limit $\mathbf{w}^* = (w_1, w_2)$ is the unique solution of the corresponding integral equation. Since $\mathbf{w}^*$ is continuous in $\overline{D}_T$ the same argument as in the proof of Theorem (2.4) shows that $\mathbf{w}^*$ is the unique solution of (2.70). This ensures the existence and uniqueness of a solution $\mathbf{u}$ for the problem (2.41). It is easily seen by letting $u_i^{(k)} = e^{\gamma t} w_i^{(k)}$ that the sequence given by (2.74) is reduced to (2.92). The convergence of $\left\{w_1^{(k)}, w_2^{(k)}\right\}$ to $\mathbf{w}^*$ in $\mathcal{C}(\overline{D}_T)$ implies that $\mathbf{u}^{(k)} \longrightarrow \mathbf{u} = e^{\gamma t}\mathbf{w}^*$ as $k \longrightarrow +\infty$. This proves the theorem. ☐

Let $\mathcal{D} = \Omega \times [0; \infty)$, $\overline{\mathcal{D}} = \overline{\Omega} \times [0; \infty)$.

Theorem 2.8. (See Pao [77]) Suppose for each $r > 0$ there are constants $K_i = K_i(r)$ such that :

$$|f_i(x, t, \mathbf{u}) - f_i(x, t, \mathbf{v})| \leq K_i |\mathbf{u} - \mathbf{v}|, \quad \text{when } |\mathbf{u}| \leq r, \ |\mathbf{v}| \leq r, \ (x, t) \in \mathcal{D}. \tag{2.93}$$

Then there exists $T_0 \leq \infty$ such that a unique solution u to (2.41) exists in $\overline{D_{T_0}} = \overline{\Omega} \times [0; T_0)$. Moreover, either $\mathbf{u}$ exists globally or it blows-up at some finite T_0.

We next extend the defintion of coupled upper and lower solution for nonquasimonotone functions.

Definition 2.6. A pair of functions $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ in $\mathcal{C}^{2,1}(D_T) \cap \mathcal{C}(\overline{D}_T)$ are called generalized coupled upper and lower solutions of (2.41) if $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ and if they satisfy the boundary-initial inequalities in (2.41) and the differential inequalities:

$$\begin{array}{ll} (\tilde{u}_i)_t - D_i \tilde{u}_i \geq f_i(x, t, \mathbf{v}), & \text{for all } \mathbf{v} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle, \ \text{with } v_i = \tilde{u}_i, \\ (\hat{u}_i)_t - D_i \hat{u}_i \leq f_i(x, t, \mathbf{v}), & \text{for all } \mathbf{v} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \ \text{with } v_i = \tilde{u}_i, \end{array} \quad i = 1, 2. \tag{2.94}$$

In the following theorem, we show that the existence of generalized coupled upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ ensures the existence of a unique solution in $\mathbf{v} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.

Theorem 2.9. Let $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ be generalized coupled upper and lower solution of (2.41), and let $\mathbf{f}(x, t, \mathbf{u}) \equiv (f_1(x, t, \mathbf{u}), f_2(x, t, \mathbf{u}))$ satisfy the Global Lipschitz condition. Then the problem (2.41) has a unique solution $\mathbf{u}^* \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.

Proof. For any $\mathbf{z} = (z_1, z_2) \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and any $i = 1, 2$, let u_i be the solution of the linear equation

$$(u_i)_t - D_i \nabla^2 u_i + K_i u_i = K_i z_i + f_i(x, t, \mathbf{z}) \quad \text{in } D_T \tag{2.95}$$

under the boundary and initial conditions in (2.41), where K_i is the Lipschitz constant of f_i. We show that $\mathbf{u} = (u_1, u_2) \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. Let $w_i = \tilde{u}_i - u_i$, $i = 1, 2$. In view of Definition (2.6) and (2.95),

$$(w_i)_t - D_i \nabla^2 w_i + K_i w_i \geq K_i(\tilde{u}_i - z_i) + f_i(x, t, \mathbf{v}) - f_i(x, t, \mathbf{z}),$$

and $B_i w_i \geq 0$, $w_i(0, x) \geq 0$, where $\mathbf{v}$ is any function in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ with $v_i = \tilde{u}_i$. Choose $\mathbf{v} = (v_1, v_2)$ with $v_j = z_j$ for all $j \neq i$. Then :

$$f_i(x, t, \mathbf{v}) - f_i(x, t, \mathbf{z}) \geq -K_i |\mathbf{v} - \mathbf{z}| = -K_i(\tilde{u}_i - z_i).$$

This implies that

$$(w_i)_t - D_i \nabla^2 w_i + K_i w_i \geq 0 \text{ in } D_T$$

By the positivity proposition 2.3, $w_i \geq 0$ in $\overline{D}_T$, which yields to $u_i \leq \tilde{u}_i$, $i = 1, 2$. This shows that $\mathbf{u} \leq \tilde{\mathbf{u}}$. A similar argument using the property of a lower solution gives $\mathbf{u} \geq \hat{\mathbf{u}}$ in $\overline{D}_T$. This proves that $\mathbf{u} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.

Definie a modified function $\hat{\mathbf{f}} = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_1)$ such that $\hat{\mathbf{f}}(x, t, \mathbf{u}) = \mathbf{f}(x, t, \mathbf{u})$ for $\mathbf{u} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and $\hat{\mathbf{f}}$ satisfies the global Lipschitz condition (2.73). Then for any $\mathbf{u}^{(0)} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, Theorem (2.7) implies that the sequence $\left\{\mathbf{u}^{(k)}\right\} = \left\{\mathbf{u}_1^{(k)}, \mathbf{u}_2^{(k)}\right\}$ determined from the equation

$$(u_i^{(k)})_t - D_i \nabla^2 u_i^{(k)} + K_i u_i^{(k)} = K_i u_i^{(k-1)} + \hat{\mathbf{f}}(x, t, \mathbf{z}), \tag{2.96}$$

converges uniformly to a unique solution $\mathbf{u}^*$ of the problem (2.41) where $\mathbf{f}(x, t, \mathbf{u})$ is replaced by $\hat{\mathbf{f}}(x, t, \mathbf{u})$. Since for $\mathbf{u}^{(0)} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ the first iteration $\left\{\mathbf{u}^{(1)}\right\}$ is governed by (2.95) with $\mathbf{z} = \mathbf{u}^{(0)}$, the above conclusion shows that $\mathbf{u}^{(1)} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. It follows by an induction argument that $\mathbf{u}^{(k)} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ for every $k = 1, 2, ...$ This implies that $\mathbf{u}^* \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ and therefore $\hat{\mathbf{f}}(x, t, \mathbf{u}^*) = \mathbf{f}(x, t, \mathbf{u}^*)$. Hence $\mathbf{u}^*$ is the solution of the original problem (2.41). This proves the theorem. □

CHAPTER THREE

AN INTEGRO-DIFFERENTIAL REACTION-DIFFUSION SYSTEM FOR PATTERN FORMATION IN HUMID SAVANNAS

Abstract[1]

For about twenty years, the question about the essential factors promoting the long-lasting coexistence of trees and grasses in humid savannas is at the center of several mathematical works, by the construction of deterministic and/or stochastic mathematical models. A closely related topic is coexistence of open savanna and forest patches at landscape scales, which raises the challenge of accounting for contrasted spatial patterns under similar climate conditions through fire mediated tree-grass interaction models. In this chapter, we propose and study a deterministic spatio-temporal fire-mediated tree-grass interactions model. The model is based on two nonlocal reaction-diffusion equations with kernels of intra and inter-specific interactions, corresponding to woody and grassy biomasses. A novelty in this work, is the consideration of a kernel-based nonlocal facilitation of trees by other trees to promote growth of seedlings/shrubs and, indirectly, limit fire propagation and its impact. We also take into account a kernel-based nonlocal competition of trees on grasses for light availability and nutrients. A qualitative analysis of the model is carried out and it reveals several ecological thresholds that shape the overall dynamics of the system. Depending on these thresholds, monostability of the forest, grassland or savanna space-homogeneous stationary state and multistabilities (i.e. involving more that one space-homogeneous stationary state) are proven possible. Thanks to the nonlocal biomasses interactions, our model accounts for the occurrence of space inhomogeneous solutions, including a possibly periodic spatial structuring sometimes observed in the humid savanna zone. Specifically, linear stability analyses, performed in the vicinity of space-homogeneous stationary states, provides conditions for the appearance of space inhomogeneous solutions including spatially periodic or aperiodic ones. Finally, numerical simulations are presented to illustrate our theoretical results. Notably, we verify that the computed spatial wavelengths were in good agreement

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[1] This chapter is an edited version of *Spatio-temporal modelling of tree-grass dynamics in humid savannas: interplay between nonlocal competition and nonlocal facilitation*, published on Applied Mathematical Modelling, Elsevier (Tega II et al. [103]).

with the predictions from the theoretical analysis.

3.1 Introduction

Savannas are complex ecosystems characterized by the co-occurrence of trees and grasses without one lifeforms excluding the other (Higgins and Bond [50]). They are also defined as a biome that corresponds to warm mean annual temperatures ($> 20°C$) and a broad range of intermediate mean annual rainfall (Yatat Djeumen et al. [122], Sarmiento [86]). Covering $ca.$ 12% of the global land surface (February and Higgins [41]), savannas occupy in Africa, $ca.$ 50% of the land area.

Within specific stretches of the rainfall gradient, vegetation may sometimes exhibit plausibly self-organized physiognomies also termed as patchy vegetation or vegetation mosaics. Indeed, as pointed out by Yatat Djeumen et al. [123], there are several empirical evidences that highlight the existence of vegetation mosaics. Patches of vegetation display dense clusters of shrubs, grasses or trees and can be interpreted as regular spot structures or localized structures (Tlidi et al. [106]). These mosaics involve either bare soil ("desert") versus vegetation (herbaceous or woody) in arid, semi-arid regions (Lefever and Lejeune [62]; Lefever et al. [64]; Lefever and Turner [63]; Couteron and Lejeune [19]; Couteron et al. [20]; HilleRisLambers et al. [52]; Rietkerk et al. [81]; Gilad et al. [43]; Pueyo et al. [79, 80]; Deblauwe et al. [24]), or grasslands/savannas versus forests in temperate as well as humid tropical regions (Youta Happi [126]; Hirota et al. [53]; Jeffery et al. [55]; Xu et al. [120]; Stall et al. [95] and references therein; see also figure 3.1). Empirical evidences suggest that vegetation mosaics in humid regions barely feature periodic patterns. Most often, they are aperiodic but, with quite sharp boundaries like isolated groves or savanna patches encircled by forests.

Observation of these mosaics further motivated several researches that aimed to study and understand how these patterns may arise and the modalities of transitions between vegetation states that could substantiate or not the theory of abrupt shifts or catastrophic transitions in vegetation ecology (see for instance Scheffer et al. [88, 89]; Scheffer and Carpenter [87]; Staver et al. [98]; Favier et al. [39] for more details). It is well-known that at biome scale, vegetation cover displays complex interactions with climate. For instance, any shift from savanna to forest vegetation not only means increase in vegetation biomass and carbon sequestration but also may translate into changes in the regional patterns of rainfall (Yatat Djeumen et al. [124]). Therefore, being able to predict or understand the process that shapes savanna dynamics and possible transitions within vegetation patterns can help to figure out global distribution of savannas, orient their evolution in the face of recurring climatic changes in Africa (Dohn et al. [30]) and sustainably manage the natural resources provided by savanna ecosystems.

To understand such self-organized vegetation formations and associated dynamics along the rainfall gradient, theoretical approaches are required. Mathematical modelling is a useful tool to describe dynamics of complex systems and has been used since decades in various fields that include finance, biology, epidemiology, agronomy, ecology. Despite field observations that point out spatial patterns of vegetation or vegetation mosaics (see e.g. figure 3.1), how tree-grass interactions proceed in space and make vegetation propagate has insufficiently been taken into account in the study of savanna dynamics, in contrast to the insights provided by modelling regarding bare soil-vegetation mosaics in drylands. Indeed, tree-grass interactions in savanna ecosystems (fire-prone or not) have been very often modelled through frameworks that implicitly acknowledge space (see the review of Yatat Djeumen et al. [123]). According to Borgogno et al. [16], the modelling of spatial mechanisms of tree-grass interactions includes discrete kernel-based and partial differential equations (PDE) frameworks. Discrete kernel-based frameworks include cellular automaton (CA) models. CA models have been use in ecology, to explain formation of patterns in fire-prone savannas (Accatino et al. [3]), in arid and semi-arid savannas (Borgogno et al. [16], Feagin et al. [40]). Accatino et al. [3] developed a CA model to investigate how trees can invade the grass stratum in humid savannas despite repeated fires. Their results show that trees can invade the grass stratum and finally suppress fire spread because one of the following occurs: (a) trees may frequently resprout and form a population that persists despite repeated effective fires; (b) trees may be fire-resistant; (c) if trees are fire-vulnerable they may cluster and grow in density until grass growth is suppressed and fire prevented. One should note that, only (c) may require spatially-explicit modelling of tree-grass interactions. However, they also show that fire may be effective in preventing the initiation of the invasion process in the grass stratum. But once the invasion process has begun, fire alone is not able to reverse it because of the combinated strategies employed by trees i.e. resprouting, fire resistance or clumping (see also Yatat Djeumen et al. [123]).

However, since CA models are simulation-based and generally involve a fairly large number of parameters, it is not easy/possible to assess how model parameter variations may influence the model outcomes. In many cases, it is not easy to use mathematical analysis to thoroughly understand the behavior and properties of CA models (Yatat Djeumen et al. [122]). Therefore, for the specific case of fire-prone savannas, it is desirable to provide insights into the dynamical properties of extensive savanna-forest areas for which data are scarce but that however need decisions in aspects such as fire management, grazing rules, or wood harvest. Spatially-explicit mathematical models that may allow mathematical tractability are thus desirable and rely on PDE frameworks.

Most of the works done using PDE, were carried out in the arid or semi-arid environmental context, using a reaction-diffusion-advection system (emphasizing the dynamics of vegetation and water) or using an integro-differential equation expressing kernel-based modelling of interactions between plants (see the review of Borgogno et al. [16]). The goal of that type of modelling is to understand the mechanisms that produce spatial patterns in arid and semi-arid savannas. In reaction-diffusion-advection systems, authors attribute pattern formation to positive feedback between vegetation (trees and grasses) and water availability (Klausmeier [58], Gilad et al. [43], Meron et al. [71], Sherratt [92]). Two main processes are identified as responsible for this positive feedback. The first one is the flow and infiltration of surface water into vegetated areas and the second feedback process is water up-take by the plant roots that is longer for larger plants (Meron et al. [71]). Such feedback is central to another framework to address vegetation patterns in arid and semi-arid savannas and that is entirely based on kernels that express nonlocal interactions between plants. Two types of non-local mechanisms received a particular attention: facilitative interactions between plants, that promote water infiltration and reduce evapotranspiration, and competitive interactions among them for water and nutrients. It is now acknowledged that pattern formation in arid systems can be explained by a combination of long distance competition and short distance facilitation (Lefever and Lejeune [62], Lejeune et al. [65], Lefever et al. [64], Lefever and Turner [63], Couteron et al. [20]). A common point of these two classes of studies is the view that the pattern formation phenomenon is a symmetry-breaking process that induces instability in an uniform vegetation state.

(a) Pattern of spots of forest vegetation within a grassland matrix as observable in Zambia (image from 01/05/2014 accessed on Google Earth).

(b) Pattern of spots of forest vegetation within a savanna matrix as observable in Cameroon (Mpem-Djim National Park) from an UAV-borne photograph taken on 16/12/2019, P. Couteron).

Figure 3.1: Some vegetation mosaics of trees and grasses in Zambia and in Cameroon.

Only a few mathematically tractable and space-explicit tree-grass interactions models have been designed for humid environments. For instance, Yatat Djeumen et al. [122] studied a PDE-based model where dynamics of a forest-grassland pattern were studied by the mean of a bistable travelling wave. Notably, they showed that depending on the fire frequency, forest could either invade grassland (i.e. forest encroachment) or recede. Goel et al. [45] examined, using a reaction-diffusion model, the contribution of dispersal to determining savanna and forest distributions. Their reaction-diffusion model considered a one-variable (scalar) equation describing the dynamics of tree cover and took into account fire and mean annual rainfall. Their 2D reaction-diffusion model was able to reproduce the spatial aggregation of biomes with a stable savanna-forest boundary.

In the same vein, Wuyts et al. [119] proposed a reaction-diffusion model of Amazonian tree cover. Their model was able to reproduce some observations of spatial distribution of forest versus savanna. However, as pointed out in Yatat Djeumen et al. [122], modelling biomasses, instead of covers like in [45, 119], helps to take into account the fact that plant types are not mutually exclusive at a given point in space since field studies suggested that grass often develops under scattered tree crowns (see Yatat Djeumen et al. [122] and references therein). Moreover, [45, 119] emphasized the effect of precipitation on possible vegetation transitions while Yatat Djeumen et al. [124] suggested that, interplay between fire and water availability may give more realistic scenarios of vegetation distribution or transitions. Recently, Patterson et al. [78] proposed to bridge the gap between ecological models with macroscopic viewpoints (deterministic models) and microscopic descriptions of stochastic transitions (stochastic models). They studied a spatial extension of the tropical cover model of Staver and Levin [97], characterized by nonlocal interactions describing the evolution of the probability for a patch of landscape to be in a given state (to be understood as, small spatial areas of the typical size of a single tree, allowing growth of new trees). From an ecological stand point, the analysis of their model enabled a more thorough understanding of the determinant of forest-savanna boundary, particularly in the presence of precipitation, resources limitation and climate changes. Notwithstanding notable exceptions, like Patterson et al. [78], a common point of some of these models is that authors mainly relied on numerical simulations to render some spatial structures and relate them to processes. However, due to the absence of qualitative analyses, it is quite difficult to assess how model outcomes respond to model parameter variations.

In the context of humid savannas, patterns approaching regularity are fairly scarce, but not absent (see Figure 3.1 pannel (a) and also Lejeune et al. [65] or Tlidi et al. [106]). Another class of patterns is made of clearly aperiodic groves in the context of a mosaic that often corresponds to savannas transiting to forests (e.g. see Figure 3.1 pannel (b)).

Our objective in this chapter, is therefore to build a mathematically tractable space-explicit PDE-like model in order to study dynamics of spatial structuring of vegetation in wet savanna zones (Figure 3.1, pannel (b)). Tractability is an important property because it allows an efficient exploration of all parts of the parameter space ensuring that interesting situations, notably linked to multistability, are not missed as it might happen if only relying on computer simulations like in CA-based models (Yatat Djeumen et al. [124]). Another aim is to identify key mechanisms and bifurcation parameters that may shape possible transitions of vegetation physiognomy and trigger spatial pattern emergence in wet savannas. Therefore, based on a mathematical model, we aim to give new insights for the development of relevant management plans of forest-savanna mosaics. Our model is based on, and therefore extend, the recent ODE model of Yatat Djeumen et al. [124]. Indeed, based on a minimalistic (in terms of state variables and parameters) ODE model, Yatat Djeumen et al. [124] analysed fire-mediated tree-grass interactions and obtained a stability map within the fire vs. mean annual rainfall parameters space. They delineated regions of monostabilities (i.e. where desert, forest, grassland or savanna is stable), regions of multistabilities involving forest, grassland and savanna as well as multistabilities involving several savanna states. In addition, for all levels of rainfall, decreasing woody biomass with increasing fire frequency was verified contrary to almost all recent works of the same complexity or less (e.g. Accatino et al. [2]). Our model takes into account the fire resistance strategy of trees, and the main processes present in Yatat's model, such as the grass-fire feed-back and decreasing fire impact with woody biomass. In addition, we incorporate nonlocal interaction terms of intra and interspecific competition. In fact, intraspecific competition influences the growth of species (either trees or grasses) and ultimately changes the dynamics of the entire population (Kothari et al. [60]) and interspecific competition (i.e. asymmetric competition of trees on grasses) leads to a reduction in grass cover and therefore a reduction in the spread and intensity of fires. Though this paper puts emphasis on the conditions for stable, spatially regular patterns, it opens prospects for studying transient and metastable patterns. The rest of the chapter is organized as follows: section 3.2 presents the construction of the model, section 3.3 deals with the theoretical analyses of the model including the existence and uniqueness of solutions and linear stability analysis of homogeneous stationary solutions. Section 3.4 deals with numerical illustrations of theoretical results.

3.2 Model construction

Our model is based on Yatat Djeumen et al. [124] (see chapter 1 at section 1.2.1.7,page 18), where authors considered two state variables, $G(t)$ and $T(t)$ that stand for the grassy biomass and the woody biomass at time t, respectively (G in $t.ha^{-1}$ and T in $t.ha^{-1}$). In Yatat Djeumen et al. [124], the following hypotheses are done

- Trees and grasses biomasses have a logistic growth.
- Grass biomass mortality or suppression may result from natural mortality, external factors (grazing, termites, human actions, etc), interactions with tree biomass and fire.
- Tree biomass mortality may result from natural mortality, external factors (browsers, human actions, etc) or is fire-induced. In fact, fire momentum is an increasing nonlinear function of G, while its impact on woody vegetation is a decreasing nonlinear function of woody biomass T.

Starting from these assumptions, we incorporated a spatial component on state variables. Precisely, $G(x,t)$ and $T(x,t)$ denote the normalized densities (by grass and tree carrying capacities K_G and K_T, in $t.ha^{-1}$) of biomass of grass and tree, respectively, at a spatial point x and at a time t. Then, $0 \leq T(x,t) \leq 1$ and $0 \leq G(x,t) \leq 1$. We consider the following assumptions:

- Tree and grass biomasses, have a logistic growth but with an intraspecific competition which takes place in a nonlocal way, through the respective root systems of the two lifeforms. In fact, a tree (respectively grass) located at a point x, can consume resources (water, nutriment) at a point y where, another tree (respectively grass) is located or where its roots are present. Then,

$$\gamma_T T(x,t)\left(1-\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)T(y,t)dy\right) \text{ and } \gamma_G G(x,t)\left(1-\int_{-\infty}^{+\infty}\phi_{M_1}(x-y)G(y,t)dy\right), \tag{3.1}$$

  describe the logistic growth with intraspecific competition where, for $i = 1, 2$, the kernel $\phi_{M_i}(x-y)$ represents, the level of consumption of resources in the area $[-M_i; M_i]$ of the space domain, γ_G (respectively, γ_T) denotes the intrinsic growth rate of grasses (respectively, trees).

- According to Craine and Dybzinski [21], trees facilitate the germination and the recruitment of new trees by improving the conditions under or around the canopy. In fact, sapling establishment for example depends on tree cover, not just because of seed production but also by local facilitation of seedlings and saplings by other trees via hydrological facilitation and shading (Li. et al. [66]). Then, we assume that there is a factor of cooperation Ω between trees that promotes regrowth and growth of young trees, helping them to reach a fire and/or browser non-vulnerability height. Hence the γ_T coefficient of exponential growth in equation (3.1) is substituted by $\gamma_T(1+\Omega T)$.

- Trees negatively impact the dynamics of grass biomass in a nonlocal way. Indeed, a tree located at a point y can, either by its root system or by the shade created by its crown, reduces the density of grasses located at a point x by reducing the resources (light availability, nutrients) in x. Then, the term
$$\gamma_{TG} G(x,t) \int_{-\infty}^{+\infty} \phi_{M_2}(x-y) T(y,t) dy, \tag{3.2}$$
describes this nonlocal interspecific impact where $\gamma_{TG} = K_T \eta_{TG}$ and η_{TG} is the tree-grass interaction parameter in $ha.t^{-1}.yr^{-1}$. The consequence here is the reduction of the grass continuum on the ground, which will reduce the spread of fire. This term will depress grass biomass growth.

- The function describing the impact of fires, $\omega(G)$, on tree biomass depends on G. Indeed, in savanna ecology it is widely admitted that dried-up grass biomass is the main factor controlling both fire intensity and spreading capacity. For simplicity, we combined these two properties of fire in a single (fire momentum), increasing function of grass-biomass, expressing that when the average herbaceous biomass is in its highest range, fires simultaneously display the highest intensity and affect all the landscape. Conversely, low grass biomass due to aridity, grazing or tree competition, will make fires of low intensity and/or unable to reach all locations in a given year thereby decreasing the actual average frequency (see for instance Yatat Djeumen et al. [124]). Following Tchuinte et al. [101], Yatat Djeumen et al. [122, 124], we consider a Holling Type III function
$$\omega(G) = \frac{G^2}{G^2 + g_0^2}, \tag{3.3}$$
where $g_0 = \dfrac{\mu}{K_G}$ and μ is the grass biomass at which fires reach its half maximal momentum.

- We consider a function of fire-induced tree mortality that decreases with the cumulated woody biomass around any space point x. If trees are numerous and/or tall, then their mortality due to fire will be reduced. Indeed, tree parts above the flame zone are immune to topkills. This function is therefore, a decreasing function of tree biomass. In analogy with the work of Martinez-Garcia et al. [68], we consider a function of the form :
$$\mathcal{V}_T(x) = \exp\left(-p \int_{-\infty}^{+\infty} \phi_{M_2}(x-y) T(y,t) dy\right), \tag{3.4}$$
where $p = K_T \delta$ and δ is a parameter proportional to the inverse of biomass destroyed at intermediate level of mortality, in $t^{-1}.ha$, see also Yatat Djeumen et al. [124] for a nonspatial

version of $\mathcal{V}_T$.

- We also assume, according to Yatat Djeumen et al. [122], that grass biomass and tree biomass, display local isotropic biomass diffusion in space with the coefficient D_G and D_T respectively, that are modelled with Laplace operators. Here, as a first approximation, we consider local diffusion of biomasses and neglect the long-range seed dispersal.

All of this leads to the following model:

$$\left\{\begin{array}{rcl}\dfrac{\partial G}{\partial t} & = & D_G\dfrac{\partial^2 G}{\partial x^2}+\gamma_G G\left(1-\displaystyle\int_{-\infty}^{+\infty}\phi_{M_1}(x-y)G(y,t)dy\right)-\delta_G G\\ & & -\gamma_{TG}G\left(\displaystyle\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)T(y,t)dy\right)-\lambda_{fG}fG,\\ \dfrac{\partial T}{\partial t} & = & D_T\dfrac{\partial^2 T}{\partial x^2}+\gamma_T T(1+\Omega T)\left(1-\displaystyle\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)T(y,t)dy\right)-\delta_T T\\ & & -\lambda_{fT}f\omega(G)\exp\left(-p\displaystyle\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)T(y,t)dy\right)T,\end{array}\right. \tag{3.5}$$

where $x\in K=(-l,l)$ and $t>0$. Parameters are defined in table 3.1 bellow. The initial data are

$$0\le T(x,0)=T_0(x)\ \text{ and }\ 0\le G(x,0)=G_0(x), \tag{3.6}$$

where $G_0(x)$ and $T_0(x)$ are bounded and sufficiently smooth functions. In addition, we also consider homogeneous Neumann boundary condition:

$$\frac{\partial T(x,t)}{\partial x}=\frac{\partial G(x,t)}{\partial x}=0\ \text{ at }\ x=-l\ \text{ and }\ x=l,\ \ l>0. \tag{3.7}$$

We assume that the kernels ϕ_{M_i}, $(i=1,2)$ are nonnegative even functions with compact support in the interval $[-M_i,M_i]$. Then, for $0\le M_i\le l$, we consider the step function kernels:

$$\phi_{M_i}(x)=\left\{\begin{array}{lll}\dfrac{1}{2M_i} & , & |x|\le M_i,\\ 0 & , & |x|>M_i,\end{array}\right.\qquad i=1,2,$$

with ϕ_0 the δ-function and $\displaystyle\int_{-\infty}^{+\infty}\phi_{M_i}(y)dy=1$. For the chosen kernel function ϕ_{M_i}, the strength of nonlocal interaction is the same with the range $[x-M_i,x+M_i]$. However, other forms of kernels have been considered in the literature dedicated to pattern formation, notably Gaussian kernels and Laplace kernels (see for instance Lefever and Lejeune [62], Lefever et al. [64], Lefever and Turner [63]).

The choice of the step function kernels in this work was mainly motivated by the type of nonlinearities in our model and model's mathematical analysis. Indeed, we found that Gaussian and Laplace kernels are not able to induce patterns with our model (see also Remark 3.4, page 105 or Remark 3.6, page 107).

Following Yatat Djeumen et al. [124], the f (in yr^{-1}) parameter is taken as constant multiplier of $\omega(G)$ in system (3.5), and we interpret it as a man-induced ‘targeted’ fire frequency (as for instance in a fire management plan), which will not automatically translate everywhere into actual frequency of fires of notable intensity (because of $\omega(G)$). With this interpretation, the actual fire regime may substantially differ from the targeted one, as frequently observed in the field (see for instance Diouf et al. [27] in southern Niger). We therefore distinguish fire frequency from fire intensity because grass biomass controls fire spread (see e.g. Govender et al. [46], McNaughton [70], Yatat Djeumen et al. [122] and references therein).

Symbols	Description	Units
γ_G	Intrinsic growth of grasses	yr^{-1}
δ_G	Grass biomass loss due to human activities and herbivory	yr^{-1}
λ_{fG}	Portion of grass biomass loss due to fire	
γ_{TG}	Tree grass interaction parameter	yr^{-1}
γ_T	Intrinsic growth of trees	yr^{-1}
δ_T	Tree biomass loss due to human activities	yr^{-1}
λ_{fT}	Portion of tree biomass loss due to fire	
p	proportional to the inverse of biomass destroyed at intermediate level of mortality	
Ω	Cooperation factor	
f	fire frequency	yr^{-1}
D_G	Grass biomass diffusion rate	$ha^2.yr^{-1}$
D_T	Tree biomass diffusion rate	$ha^2.yr^{-1}$
M_1	Range of grass spatial nonlocal interaction	m
M_2	Range of tree spatial nonlocal interaction	m

Table 3.1: Definition of parameters used in the model.

3.3 Mathematical analysis

3.3.1 Existence and uniqueness of solutions of system (3.5)-(3.7)

Let $\overline{K} = [-l, l]$ be the closure of K, and for any $\tau > 0$, we set:

$$D_\tau = K \times (0, \tau], \;\; \overline{D}_\tau = \overline{K} \times [0, \tau], \quad S_\tau = \partial K \times (0, \tau]. \tag{3.8}$$

Denote by:

- $C^\alpha(D_\tau)$ the set of Hölder continuous functions in D_τ with the exponent $\alpha \in (0; 1)$,
- $C(D_\tau)$, the set of continuous functions in D_τ.
- $C^{2,1}(D_\tau)$ the set of functions that are twice continuously differentiable in x and once continuously differentiable in t.

The norm in $C(D_\tau)$ is defined by:

$$\|u\|_0^{D_\tau} = \sup_{(x,t)\in D_\tau} |u(x,t)|. \tag{3.9}$$

Similar norm is defined for $C(\overline{D}_\tau)$.

The above spaces for vector-valued functions are denoted respectively $\mathcal{C}^\alpha(D_\tau)$, $\mathcal{C}(D_\tau)$, $\mathcal{C}^{2,1}(D_\tau)$ and the norm used in $\mathcal{C}(D_\tau)$ for any $\mathbf{u} = (u_1, u_2)$ is :

$$\|\mathbf{u}\|_0^{D_\tau} = \sup_{(x,t)\in D_\tau} \max\left\{|u_1(x,t)|; |u_2(x,t)|\right\}. \tag{3.10}$$

We consider the following functions defined on $\mathbb{R}^4$ by:

$$\begin{array}{rcl} f_1(u_1,u_2,u_3,u_4) &=& \gamma_G u_1(1-u_3) - \delta_G u_1 - \gamma_{TG} u_1 u_4 - \lambda_{fG} f u_1, \\ f_2(u_1,u_2,u_3,u_4) &=& \gamma_T u_2(1+\Omega u_2)(1-u_4) - \delta_T u_3 - \lambda_{fT} f \omega(u_1)\exp(-p u_4) u_2, \end{array} \tag{3.11}$$

then system (3.5) can be rewrite in the form:

$$\left\{ \begin{array}{rcl} \dfrac{\partial G}{\partial t} &=& D_G \dfrac{\partial^2 G}{\partial x^2} + f_1(G(x,t), T(x,t), \phi_{M_1} * G(x,t), \phi_{M_2} * T(x,t)), \\ \\ \dfrac{\partial T}{\partial t} &=& D_T \dfrac{\partial^2 T}{\partial x^2} + f_2(G(x,t), T(x,t), \phi_{M_1} * G(x,t), \phi_{M_2} * T(x,t)), \end{array} \right. \tag{3.12}$$

with $(x,t) \in D_\tau$ and

$$\left(\phi_{M_1} * G\right)(x,t) = \int_{-\infty}^{+\infty} \phi_{M_1}(x-y) G(y,t) dy \text{ and } \left(\phi_{M_2} * T\right)(x,t) = \int_{-\infty}^{+\infty} \phi_{M_2}(x-y) T(y,t) dy \tag{3.13}$$

where ϕ_{M_i} is a spatial kernel function satisfying:

$$\int_{-\infty}^{+\infty} \phi_{M_i}(y) dy = 1, \quad i = 1, 2. \tag{3.14}$$

Definition 3.1. (Tian et al. [104])

A pair of nonnegative functions $\tilde{\mathbf{U}} = (\tilde{G}, \tilde{T})'$ and $\hat{\mathbf{U}} = (\hat{G}, \hat{T})' \in \mathcal{C}(\overline{D}_\tau) \cap \mathcal{C}^{2,1}(D_\tau)$ is called upper and lower solutions of (3.5) if $\tilde{\mathbf{U}} \geq \hat{\mathbf{U}}$ and if for $(x,t) \in D_\tau$,

$$\begin{cases} \dfrac{\partial \tilde{G}(x,t)}{\partial t} - D_G \dfrac{\partial^2 \tilde{G}(x,t)}{\partial x^2} \geq \gamma_G \tilde{G}(x,t)\left(1 - \phi_{M_1} * \tilde{G}(x,t)\right) - \delta_G \tilde{G}(x,t) - \gamma_{TG}\tilde{G}(x,t)\left(\phi_{M_2} * \hat{T}(x,t)\right) - \\ \lambda_{fG} f \tilde{G}(x,t), \\ \\ \dfrac{\partial \tilde{T}(x,t)}{\partial t} - D_T \dfrac{\partial^2 \tilde{T}(x,t)}{\partial x^2} \geq \gamma_T \tilde{T}(x,t)(1+\Omega \tilde{T}(x,t))\left(1 - \phi_{M_2} * \tilde{T}(x,t)\right) - \delta_T \tilde{T}(x,t) - \lambda_{fT} f \omega(\hat{G}(x,t)) \times \\ \exp\left(-p\phi_{M_2} * \tilde{T}(x,t)\right)\tilde{T}(x,t), \\ \\ \dfrac{\partial \hat{G}(x,t)}{\partial t} - D_G \dfrac{\partial^2 \hat{G}(x,t)}{\partial x^2} \leq \gamma_G \hat{G}(x,t)\left(1 - \phi_{M_1} * \hat{G}(x,t)\right) - \delta_G \hat{G}(x,t) - \gamma_{TG}\hat{G}(x,t)\left(\phi_{M_2} * \tilde{T}(x,t)\right) - \\ \lambda_{fG} f \hat{G}(x,t), \\ \\ \dfrac{\partial \hat{T}(x,t)}{\partial t} - D_T \dfrac{\partial^2 \hat{T}(x,t)}{\partial x^2} \leq \gamma_T \hat{T}(x,t)(1+\Omega \hat{T}(x,t))\left(1 - \phi_{M_2} * \hat{T}(x,t)\right) - \delta_T \hat{T}(x,t) - \lambda_{fT} f \omega(\tilde{G}(x,t)) \times \\ \exp\left(-p\phi_{M_2} * \hat{T}(x,t)\right)\hat{T}(x,t), \end{cases} \tag{3.15}$$

and:

$$\begin{cases} \dfrac{\partial \hat{G}(x,t)}{\partial t}, \dfrac{\partial \hat{T}(x,t)}{\partial t} \leq 0, \dfrac{\partial \tilde{G}(x,t)}{\partial t}, \dfrac{\partial \tilde{T}(x,t)}{\partial t} \geq 0 \text{ on } S_\tau. \\ \tilde{G}(x,0) \geq G(x,0),\ \ \tilde{T}(x,0) \geq T(x,0),\ \ \hat{G}(x,0) \leq G(x,0)\ \ \hat{T}(x,0) \leq T(x,0)\ \ x \in K. \end{cases} \tag{3.16}$$

The ordering relation $\tilde{\mathbf{U}} \geq \hat{\mathbf{U}}$ means that $\tilde{G}(x,t) \geq \hat{G}(x,t)$ and $\tilde{T}(x,t) \geq \hat{T}(x,t)$ for $(x,t) \in D_\tau$.

For a given pair of ordered upper and lower solutions $\tilde{\mathbf{U}}$ and $\hat{\mathbf{U}}$, we set:

$$\langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle = \left\{ \mathbf{U} = (G,T)' \in \mathcal{C}(\overline{D}_\tau) : \hat{\mathbf{U}} \leq \mathbf{U} \leq \tilde{\mathbf{U}} \right\}. \tag{3.17}$$

Definition 3.2. (Tian et al. [104])

f_i is called Lipschitz continuous with respect to $\langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$ if there exist a constant $k_i > 0$ for any $\mathbf{U} = (G_1, T_1)'$ $\mathbf{V} = (G_2, T_2)' \in \langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$ such that:

$$\left| f_i(G_1(x,t), T_1(x,t), \phi_{M_1} * G_1(x,t), \phi_{M_2} * T_1(x,t)) - f_i(G_2(x,t), T_2(x,t), \phi_{M_1} * G_2(x,t), \phi_{M_2} * T_2(x,t)) \right| \leq$$

$$k_i \left(\left| G_1(x,t) - G_2(x,t) \right| + \left| T_1(x,t) - T_2(x,t) \right| + \phi_{M_1} * \left| G_1(x,t) - G_2(x,t) \right| + \phi_{M_2} * \left| T_1(x,t) - T_2(x,t) \right| \right). \tag{3.18}$$

for any $(x,t) \in D_\tau$.

Furthermore, if f_1 and f_2 are Lipschitz continuous with respect to $\langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$, then we call $\mathbf{f} = (f_1; f_2)'$ is Lipschitz continuous with respect to $\langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$.

Proposition 3.1. (Lipschitz constants)

If $\hat{\mathbf{U}}$ and $\tilde{\mathbf{U}}$ are bounded, direct calculations show that there exists constants k_1 and k_2 such that $\mathbf{f} = (f_1; f_2)'$ defined in (3.11) is Lipschitz continuous with respect to $\langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$, with:

$$\begin{aligned}
k_1 &= (\gamma_G + \delta_G + \lambda_{fG} f) + (\gamma_G + \gamma_{TG})\|G_1\|_0 + \gamma_G \|G_2\|_0 + \gamma_{TG}\|T_2\|_0, \\
k_2 &= (\gamma_T + \delta_T) + \gamma_T(1+\Omega)(\|T_1\|_0 + \|T_2\|_0) + \Omega\gamma_T\left[\|T_1\|_0^2 + \|T_2\|_0^2\left(\|T_1\|_0 + \|T_2\|_0\right)\right] \\
&\quad + \lambda_{fT}\left(1 + (\theta_1+\theta_2)\|T_2\|_0\right),
\end{aligned}$$

where θ_1 and θ_2 are respectively the Lipschitz constants of the function $\omega(G)$ and $\exp(-pT)$ and $\|G_i\|_0 = \sup_{\overline{D}_\tau} |G_i|$, $\|T_i\|_0 = \sup_{\overline{D}_\tau} |T_i|$, $i = 1, 2$.

Proof. Let $\mathbf{U} = (G_1, T_1)$ and $\mathbf{V} = (G_2, T_2) \in \langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$, with $(x,t) \in D_\tau$. Set:

$$\eta_{x,t} = |G_1(x,t) - G_2(x,t)| + |T_1(x,t) - T_2(x,t)| + \phi_{M_1} * |G_1(x,t) - G_2(x,t)| + \phi_{M_2} * |T_1(x,t) - T_2(x,t)| \tag{3.19}$$

we have:

$$f_1(G_1(x,t), T_1(x,t), \phi_{M_1} * G_1(x,t), \phi_{M_2} * T_1(x,t)) - f_1(G_2(x,t), T_2(x,t), \phi_{M_1} * G_2(x,t), \phi_{M_2} * T_2(x,t)) = A - B - C - D - E,$$

where:

$$\begin{aligned}
A &= \gamma_G(G_1(x,t) - G_2(x,t), \\
B &= \gamma_G\Big(G_1(x,t)(\phi_{M_1} * G_1(x,t)) - G_2(x,t)(\phi_{M_1} * G_2(x,t))\Big), \\
C &= \delta_G(G_1(x,t) - G_2(x,t)), \\
D &= \lambda_{fG} f(G_1(x,t) - G_2(x,t)), \\
E &= \gamma_{TG}\Big(G_1(x,t)(\phi_{M_2} * T_1(x,t)) - G_2(x,t)(\phi_{M_2} * T_2(x,t))\Big).
\end{aligned}$$

Therefore:

$$\begin{aligned}
|A| &\leq \gamma_G |G_1(x,t) - G_2(x,t)|, \\
&\leq \gamma_G \eta_{x,t},
\end{aligned}$$

$$\begin{aligned}
|C| &\leq \delta_G |G_1(x,t) - G_2(x,t)|, \\
&\leq \delta_G \eta_{x,t},
\end{aligned}$$

$$\begin{aligned}
|D| &\leq \lambda_{fG} f |G_1(x,t) - G_2(x,t)|, \\
&\leq \lambda_{fG} f \eta_{x,t}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
B &= \gamma_G\Big[G_1(x,t)(\phi_{M_1}*G_1(x,t)) - G_2(x,t)(\phi_{M_1}*G_2(x,t))\Big],\\
&= \gamma_G\Big[G_1(x,t)\Big(\phi_{M_1}*(G_1(x,t)-G_2(x,t))\Big) + (G_1(x,t)-G_2(x,t))\phi_{M_1}*G_2(x,t)\Big].
\end{aligned}$$

and,

$$\begin{aligned}
E &= \gamma_{TG}\Big[G_1(x,t)(\phi_{M_1}*T_1(x,t)) - G_2(x,t)(\phi_{M_2}*T_2(x,t))\Big],\\
&= \gamma_{TG}\Big[G_1(x,t)\Big(\phi_{M_2}*(T_1(x,t)-T_2(x,t))\Big) + (G_1(x,t)-G_2(x,t))\phi_{M_2}*T_2(x,t)\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
|B| &= \gamma_G\Big|G_1(x,t)(\phi_{M_1}*G_1(x,t)) - G_2(x,t)(\phi_{M_1}*G_2(x,t))\Big|,\\
&= \gamma_G\Big|G_1(x,t)\Big(\phi_{M_1}*(G_1(x,t)-G_2(x,t)\Big) + (G_1(x,t)-G_2(x,t))\phi_{M_1}*G_2(x,t)\Big|,\\
&\leq \gamma_G\Big[\|G_1\|_0(\phi_{M_1}*|G_1(x,t)-G_2(x,t)|) + |G_1(x,t)-G_2(x,t)|\|G_2\|_0\Big],\\
&\leq \gamma_G(\|G_1\|_0\eta_{x,t} + \|G_2\|_0\eta_{x,t}),\\
&= \gamma_G\Big(\|G_1\|_0 + \|G_2\|_0\Big)\eta_{x,t}.
\end{aligned}$$

In the same way:

$$\begin{aligned}
|E| &= \gamma_{TG}\Big|G_1(x,t)(\phi_{M_2}*T_1(x,t)) - G_2(x,t)(\phi_{M_2}*T_2(x,t))\Big|,\\
&= \gamma_{TG}\Big|G_1(x,t)\Big(\phi_{M_2}*(T_1(x,t)-T_2(x,t)\Big) + (G_1(x,t)-G_2(x,t))\phi_{M_2}*T_2(x,t)\Big|,\\
&\leq \gamma_{TG}\Big[\|G_1\|_0\,(\phi_{M_2}*|T_1(x,t)-T_2(x,t)|) + |G_1(x,t)-G_2(x,t)|\|T_2\|_0\Big],\\
&\leq \gamma_{TG}\left((\|G_1\|_0)\eta + (\|T_2\|_0)\eta_{x,t}\right),\\
&= \gamma_{TG}\Big(\|G_1\|_0 + \|T_2\|_0\Big)\eta_{x,t}.
\end{aligned}$$

Therefore,

$$\Big| f_1(G_1(x,t), T_1(x,t), \phi_{M_1} * G_1(x,t), \phi_{M_2} * T_1(x,t)) - f_1(G_2(x,t), T_2(x,t), \phi_{M_1} * G_2(x,t), \phi_{M_2} * T_2(x,t)) \Big| \leq |A| + |B| + |C| + |D| + |E|,$$

$$\leq \left(\gamma_G + \delta_G + \lambda_{fG} f\right) \eta_{x,t} + \gamma_G \left(\|G_1\|_0 + \|G_2\|_0\right) \eta_{x,t} + \gamma_{TG} \left(\|G_1\|_0 + \|T_2\|_0\right) \eta_{x,t},$$

$$= \Bigg[\left(\gamma_G + \delta_G + \lambda_{fG} f\right) + \left(\gamma_G + \gamma_{TG}\right) \|G_1\|_0 + \gamma_G \|G_2\|_0 + \gamma_{TG} \|T_2\|_0 \Bigg] \eta_{x,t}.$$

$$= k_1 \eta_{x,t}, \text{ where } k_1 = \Bigg[\left(\gamma_G + \delta_G + \lambda_{fG} f\right) + \left(\gamma_G + \gamma_{TG}\right) \|G_1\|_0 + \gamma_G \|G_2\|_0 + \gamma_{TG} \|T_2\|_0.$$

In the same manner,

$$f_2(G_1(x,t), T_1(x,t), \phi_{M_1} * G_1(x,t), \phi_{M_2} * T_1(x,t)) - f_2(G_2(x,t), T_2(x,t), \phi_{M_1} * G_2(x,t), \phi_{M_2} * T_2(x,t) = A' + B' - C' - D' - E' - F',$$

where,

$$\begin{aligned}
A' &= \gamma_T (T_1(x,t) - T_2(x,t)), \\
B' &= \Omega \gamma_T (T_1^2(x,t) - T_2^2(x,t)), \\
C' &= \gamma_T \Bigg(T_1(x,t) \left(\phi_{M_2} * T_1(x,t)\right) - T_2(x,t) \left(\phi_{M_2} * T_2(x,t)\right) \Bigg), \\
D' &= \Omega \gamma_T \Bigg(T_1^2(x,t) \left(\phi_{M_2} * T_1(x,t)\right) - T_2^2(x,t) \left(\phi_{M_2} * T_2(x,t)\right) \Bigg), \\
E' &= \lambda_{fT} f \Bigg(\omega(G_1(x,t)) \exp(-p \phi_{M_2} * T_1(x,t)) T_1(x,t) - \omega(G_2(x,t)) \exp(-p \phi_{M_2} * T_2(x,t)) T_2(x,t) \Bigg), \\
F' &= \delta_T (T_1(x,t) - T_2(x,t)).
\end{aligned}$$

We have:

$$\begin{aligned}
|A'| &= \gamma_T|T_1(x,t)-T_2(x,t)|,\\
&\leq \gamma_T\eta_{x,t},
\end{aligned}$$

$$\begin{aligned}
|B'| &= \Omega\gamma_T|(T_1^2(x,t)-T_2^2)(x,t)|,\\
&= \Omega\gamma_T|T_1(x,t)-T_2(x,t)|\times|T_1(x,t)+T_2(x,t)|,\\
&\leq \Omega\gamma_T\left(\|T_1\|_0+\|T_2\|_0\right)|T_1(x,t)-T_2(x,t)|,\\
&\leq \Omega\gamma_T\left(\|T_1\|_0+\|T_2\|_0\right)\eta_{x,t},
\end{aligned}$$

$$\begin{aligned}
|C'| &= \gamma_T\Bigg(|T_1(x,t)\left(\phi_{M_2}*T_1(x,t)\right)-T_2(x,t)\left(\phi_{M_2}*T_2(x,t)\right)|\Bigg),\\
&= \gamma_T\Bigg(\Bigg|T_1(x,t)\left(\phi_{M_2}*\left(T_1(x,t)-T_2(x,t)\right)\right)+\left(T_1(x,t)-T_2(x,t)\right)\phi_{M_2}*T_2(x,t)\Bigg|\Bigg),\\
&\leq \gamma_T\Bigg(\|T_1\|_0\left(\phi_{M_2}*\left(T_1(x,t)-T_2(x,t)\right)\right)+\|T_2\|_0|T_1(x,t)-T_2(x,t)|\Bigg),\\
&\leq \gamma_T\Bigg((\|T_1\|_0)\eta+(\|T_2\|_0)\eta_{x,t}\Bigg),\\
&= \gamma_T\Bigg(\left(\|T_1\|_0+\|T_2\|_0\right)\Bigg)\eta_{x,t}.
\end{aligned}$$

$$\begin{aligned}
|D'| &= \Omega\gamma_T\Bigg(|T_1^2(x,t)\left(\phi_{M_2}*T_1(x,t)\right)-T_2^2(x,t)\left(\phi_{M_2}*T_2(x,t)\right)|\Bigg),\\
&= \Omega\gamma_T\Bigg(|T_1^2(x,t)\left(\phi_{M_2}*\left(T_1(x,t)-T_2(x,t)\right)\right)+\left(T_1^2(x,t)-T_2^2(x,t)\right)\phi_{M_2}*T_2(x,t)|\Bigg),\\
&\leq \Omega\gamma_T\Bigg((\|T_1\|_0)^2\phi_{M_2}*|T_1(x,t)-T_2(x,t)|+|T_1(x,t)-T_2(x,t)|\left(\|T_1\|_0+\|T_2\|_0\right)\|T_2\|_0\Bigg),\\
&\leq \Omega\gamma_T\Bigg((\|T_1\|_0)^2\eta_{x,t}+\left(\|T_1\|_0+\|T_2\|_0\right)\|T_2\|_0\eta_{x,t}\Bigg),\\
&\leq \Omega\gamma_T\Bigg((\|T_1\|_0)^2+\left(\|T_1\|_0+\|T_2\|_0\right)\|T_2\|_0\Bigg)\eta_{x,t},
\end{aligned}$$

$$\begin{aligned}
|F'| &= \delta_T|T_1(x,t)-T_2(x,t)|,\\
&\leq \delta_T\eta_{x,t}.
\end{aligned}$$

and

$$\begin{aligned}
|E'| &= \lambda_{fT}f\Bigg|\omega(G_1(x,t))\exp(-p\phi_{M_2}*T_1(x,t))T_1(x,t)-\omega(G_2(x,t))\exp(-p\phi_{M_2}*T_2(x,t))T_2(x,t)\Bigg|,\\
&= \lambda_{fT}f\Bigg|\omega(G_1(x,t))\exp(-p\phi_{M_2}*T_1(x,t))\left(T_1-T_2\right)(x,t)+\\
&\quad T_2(x,t)\Bigg[\omega(G_1(x,t))\exp(-p\phi_{M_2}*T_1(x,t))-\omega(G_2(x,t))\exp(-p\phi_{M_2}*T_2(x,t))\Bigg]\Bigg|.
\end{aligned}$$

We have:

$$\begin{aligned}
\Big|\omega(G_1(x,t))\exp(-p\phi_{M_2}*T_1(x,t))\,(T_1-T_2)\,(x,t)\Big| &= \Big|\omega(G_1(x,t))\Big|\times\Big|\exp(-p\phi_{M_2}*T_1(x,t))\Big|\times \\
&\quad \Big|(T_1-T_2)(x,t)\Big|, \\
&\leq \Big|T_1(x,t)-T_2(x,t)\Big|, \\
&\leq \eta_{x,t}.
\end{aligned}$$

$$\Big|\omega(G_1(x,t))\exp(-p\phi_{M_2}*T_1(x,t))-\omega(G_2(x,t))\exp(-p\phi_{M_2}*T_2(x,t))\Big| = \Big|\,(\omega(G_1(x,t))-\omega(G_2(x,t)))$$
$$\exp(-p\phi_{M_2}*T_1(x,t))+\omega(G_2(x,t))\Big[\exp(-p\phi_{M_2}*T_1(x,t))-\exp(-p\phi_{M_2}*T_2(x,t))\Big]\Big|,$$

$$\leq \Big|\,(\omega(G_1(x,t))-\omega(G_2(x,t)))\,\Big|\times\Big|\exp(-p\phi_{M_2}*T_1(x,t))\Big|+\Big|\omega(G_2(x,t))\Big|\times$$
$$\Big|\exp(-p\phi_{M_2}*T_1(x,t))-\exp(-p\phi_{M_2}*T_2(x,t))\Big|,$$

$$\begin{aligned}
&\leq \theta_1\Big|G_1(x,t)-G_2(x,t)\Big|+\theta_2\Big|\phi_{M_2}*T_1(x,t)-\phi_{M_2}*T_2(x,t)\Big|, \\
&\leq \Big(\theta_1+\theta_2\Big)\eta_{x,t}.
\end{aligned}$$

where θ_1 and θ_2 are respectively the Lipschitz constant of functions $\omega(G)$ and $\exp(-pT)$. Then,

$$\Big|f_2(G_1(x,t),T_1(x,t),\phi_{M_1}*G_1(x,t),\phi_{M_2}*T_1(x,t))-f_2(G_2(x,t),T_2(x,t),\phi_{M_1}*G_2(x,t),\phi_{M_2}*T_2(x,t))\Big|$$
$$\leq |A'|+|B'|+|C'|+|D'|+|E'|+|F'|,$$

$$\leq \Big[(\gamma_T+\delta_T)+\gamma_T(1+\Omega)\,(\|T_1\|_0+\|T_2\|_0)+\Omega\gamma_T\Big(\,\|T_1\|_0^2+\|T_2\|_0(\|T_1\|_0+\|T_2\|_0)\Big)+$$
$$\lambda_{fT}\,(1+(\theta_1+\theta_2)\|T_2\|_0)\,\Big]\eta_{x,t}.$$

$$= k_2\eta_{x,t},\ \text{ where }\ k_2=(\gamma_T+\delta_T)+\gamma_T(1+\Omega)(\|T_1\|_0+\|T_2\|_0)+\Omega\gamma_T\left[\|T_1\|_0^2+\|T_2\|_0^2\,(\|T_1\|_0+\|T_2\|_0)\right]$$
$$+\lambda_{fT}\,(1+(\theta_1+\theta_2)\|T_2\|_0)\,.$$

Therefore, $\mathbf{f}=(f_1;f_2)'$ is Lipschitz continuous with respect to $\langle\hat{\mathbf{U}},\tilde{\mathbf{U}}\rangle$. □

In addition, we define on $\mathcal{C}^{2,1}(D_\tau)\cap\mathcal{C}(\overline{D}_\tau)$, the following operators $\mathbf{L}=(L_1,L_2)'$ and $\mathbf{F}=$

$(F_1, F_2)'$:

$$\begin{aligned} L_1 G &= \frac{\partial G}{\partial t} - D_G \frac{\partial^2 G}{\partial x^2} + k_1 G, \\ L_2 T &= \frac{\partial T}{\partial t} - D_T \frac{\partial^2 G}{\partial x^2} + k_2 T, \\ F_1(G,T) &= k_1 G + f_1(G, T, \phi_{M_1} * G, \phi_{M_2} * T), \\ F_2(G,T) &= k_2 T + f_2(G, T, \phi_{M_1} * G, \phi_{M_2} * T). \end{aligned} \tag{3.20}$$

Then the system (3.5) can be reformulated as follows:

$$\begin{cases} L_1 G = F_1(G,T) & \text{in} \quad D_\tau, \\ L_2 T = F_2(G,T) & \text{in} \quad D_\tau, \\ \dfrac{\partial G}{\partial x} = \dfrac{\partial T}{\partial x} = 0 & \text{on} \quad S_\tau, \\ G(x,0) = G_{10}(x), \;\; T(x,0) = T_{20}(x) & \text{in} \quad K. \end{cases} \tag{3.21}$$

Now we are in position to show that the system (3.21) has a unique global solution. To this aim, we construct a sequence $\left\{\mathbf{U}^{(m)}\right\} \equiv \left\{G^{(m)}, T^{(m)}\right\}$ according to the following iteration process:

$$\begin{cases} L_1 G^{(m)} = F_1(G^{(m-1)}, T^{(m-1)}) & \text{in} \quad D_\tau, \\ L_2 T^{(m)} = F_2(G^{(m-1)}, T^{(m-1)}) & \text{in} \quad D_\tau, \\ \dfrac{\partial G^{(m)}}{\partial x} = \dfrac{\partial T^{(m)}}{\partial x} = 0 & \text{on} \quad S_\tau, \\ G^{(m)}(x,0) = G_{10}(x), \;\; T^{(m)}(x,0) = T_{20}(x) & \text{in} \quad K, \end{cases} \tag{3.22}$$

with $\mathbf{U}^{(0)} \in \mathcal{C}^\alpha(D_\tau) \cap \langle \hat{\mathbf{U}}, \tilde{\mathbf{U}} \rangle$.

To show the convergence of the sequence $\left\{\mathbf{U}^{(m)}\right\}$, set:

$$\begin{cases} w_1 = e^{-\gamma t} G, \\ w_2 = e^{-\gamma t} T, \end{cases} \tag{3.23}$$

where γ is a positive constant. The system (3.21) is equivalent to the following system:

$$\begin{cases} L_i w_i + \gamma w_i = H_i(w_1, w_2) \text{ for } i = 1,2 & \text{in} \quad D_\tau, \\ \dfrac{\partial w_1}{\partial x} = \dfrac{\partial w_2}{\partial x} = 0 & \text{on} \quad S_\tau, \\ w_1(x,0) = w_{10}(x), \;\; w_2(x,0) = w_{20}(x) & \text{in} \quad K, \end{cases} \tag{3.24}$$

where,

$$\begin{aligned} w_{10}(x) &= G_{10}(x), \;\; w_{20}(x) = T_{20}(x), \\ H_1(w_1, w_2) &= k_1 w_1 + \gamma_G w_1 (1 - e^{\gamma t} \phi_{M_1} * w_1) - \delta_G w_1 - \gamma_{TG} w_1 e^{\gamma t} \phi_{M_2} * w_2 - \lambda_{fG} f w_1, \\ H_2(w_1, w_2) &= k_2 w_2 + \gamma_T w_2 (1 + \Omega e^{\gamma t} w_2)(1 - e^{\gamma t} \phi_{M_2} * w_2) - \delta_T w_2 - \lambda_{fT} f \omega(w_1) \exp(-p e^{\gamma t} \phi_{M_2} * w_2) w_2, \end{aligned} \tag{3.25}$$

with $\omega(w_1) = \dfrac{w_1^2}{w_1^2 + (g_0 e^{-\gamma t})^2}$.

According to (3.24), we can construct sequences $\mathbf{w}^{(m)} = (w_1^{(m)}, w_2^{(m)})$ with $\mathbf{w}^{(0)} = e^{-\gamma t} \mathbf{U}^{(0)}$ via the following iteration process:

$$\begin{cases} L_i w_i^{(m)} + \gamma w_i^{(m)} = H_i(w_1^{(m-1)}, w_2^{(m-1)}) \text{ for } i = 1,2 & \text{in} \quad D_\tau, \\ \dfrac{\partial w_1^{(m)}}{\partial x} = \dfrac{\partial w_2^{(m)}}{\partial x} = 0 & \text{on} \quad S_\tau, \\ w_1^{(m)}(x,0) = w_{10}(x), \;\; w_2^{(m)}(x,0) = w_{20}(x) & \text{in} \quad K. \end{cases} \tag{3.26}$$

In term of the integral representation theory for linear parabolic boundary-value problems as we seen in chapter 2, the sequence $\mathbf{w}^{(m)}$ can be expressed as:

$$\begin{aligned} w_i^{(m)}(x,t) &= \int_0^t \int_K \Gamma_i(x-y, t-\tau)(H_i(\mathbf{w}^{(m-1)}))(y,\tau)\,dy d\tau \\ &+ \int_0^t \int_{\partial K} \Gamma_i(x-y, t-\tau)(\psi_i(\mathbf{w}^{(m-1)}))(y,\tau)\,dy d\tau + \int_K \Gamma_i(x-y,t) w_{i0}(y)\,dy, \end{aligned} \tag{3.27}$$

where Γ_i is the fundamental solution of the Linear parabolic operator $L_i + \gamma$ and defined like in chapter 2 by the relation (2.18) page 28, ψ_i a density function defined by the relation (2.23) in page 29.

We now show that the sequence $\{\mathbf{w}^{(m)}\}$ converges in $\mathcal{C}(\overline{D}_\tau)$ to a unique solution of the associated integral in (3.27). Set $\mathcal{X} = X_1 \times X_2$, where :

$$X_i = \left\{ w_i \in C^\alpha(D_\tau) \cap C(\overline{D}_\tau) : w_i(0,x) = w_{i0}(x) \in K \right\} \text{ for } i = 1,2. \tag{3.28}$$

We also consider the vector valued function defined on $\mathcal{C}^{\alpha}(D_\tau) \cap \mathcal{C}(\overline{D}_\tau)$ by $\mathcal{H} = (H_1, H_2)$ where H_i $(i = 1, 2)$ are given in (3.25). We have the following properties:

Lemma 3.1. If w and w′ $\in \mathcal{X}$, then $\mathcal{H}(\mathbf{w}) \in \mathcal{C}^{\alpha}(D_\tau) \cap \mathcal{C}(\overline{D}_\tau)$ and:

$$\|\mathcal{H}(\mathbf{w}) - \mathcal{H}(\mathbf{w}')\|_0 \leq k\|\mathbf{w} - \mathbf{w}'\|_0, \tag{3.29}$$

where $k = 8\max\{k_1, k_2\}$.

Proof. We first show that $\mathcal{H}(\mathbf{w}) = (H_1(\mathbf{w}), H_2(\mathbf{w})) \in \mathcal{C}^{\alpha}(D_\tau) \cap \mathcal{C}(\overline{D}_\tau)$. Due to the definition of H_i in (3.25) and without loss of generality, we only need to treat for the case of spatial convolution and the case for $exp(pe^{\gamma t}\phi_{M_2} * w_2)$, because the plus and multiplication do not change the Hölder continuous property of the functions.
Recall that :

$$\phi_{M_i} * w_i(x,t) = \int_{\mathbb{R}} \phi_{M_i}(x-y)w_i(y,t)dy = \int_{\mathbb{R}} \phi_{M_i}(y)w_i(x-y,t)dy, \text{ for } (x,t) \in D_\tau.$$

Let $(x,t),\ (x',t') \in D_\tau$ and θ_2 be the Lipschitz constant of the function defined on $\mathbb{R}$ by $z \mapsto \ exp(-pz)$. By the property of w_i, we have:

$$\begin{aligned}
|\phi_{M_i} * w_i(x,t) - \phi_{M_i} * w_i(x',t')| &\leq \int_{\mathbb{R}} \phi_{M_i}(y)|w_i(x-y,t) - w_i(x'-y,t')|dy,\\
&\leq \int_{\mathbb{R}} c_i\left(|x-x'|^{\alpha} + |t-t'|^{\alpha}\right)\phi_{M_i}(y)dy,\\
&= c_i\left(|x-x'|^{\alpha} + |t-t'|^{\alpha}\right),
\end{aligned}$$

for some constant c_i and $i = 1, 2$. This show that $\phi_{M_i} * w_i$, is Hölder continuous in D_τ. Similarly,

$$\begin{aligned}
\left|exp\left(-pe^{\gamma t}\phi_{M_2} * w_2(x,t)\right) - exp\left(-pe^{\gamma t'}\phi_{M_2} * w_2(x',t')\right)\right| &\leq \theta_2|e^{\gamma t}\phi_{M_2} * w_2(x,t) - e^{\gamma t'}\phi_{M_2} * w_2(x',t')|,\\
&\leq \theta_2\Bigg[|e^{\gamma t}| \times |\phi_{M_2} * w_2(x,t) - \phi_{M_2} * w_2(x',t')|\\
&\quad +|\phi_{M_2} * w_2(x',t')| \times |e^{\gamma t} - e^{\gamma t'}|\Bigg],\\
&\leq \theta_2\Bigg[e^{\gamma\tau}c_2\left(|x-x'|^{\alpha} + |t-t'|^{\alpha}\right) + \beta\|w_2\|_0 \times\\
&\quad \left(|x-x'|^{\alpha} + |t-t'|^{\alpha}\right)\Bigg],\\
&\leq \theta_2\max\left\{e^{\gamma\tau}c_2; \beta\|w_2\|_0\right\}(|x-x'|^{\alpha} + |t-t'|
\end{aligned}$$

It follows that $exp(pe^{\gamma t}\phi_{M_2} * w_2)$ is Hölder continuous in D_τ. Then $H_i(w_1, w_2)$ is Hölder continuous in $(x,t) \in D_\tau$. Then $H_i(\mathbf{w})$ is Hölder continuous in D_τ. Secondly,

$$\begin{aligned}
|H_i(\mathbf{w}) - H_i(\mathbf{w}')| &= \Bigg| k_i(w_i - w_i') + e^{-\gamma t}\Bigg(f_i(e^{\gamma t}w_1, e^{\gamma t}w_2, e^{\gamma t}\phi_{M_1} * w_1, e^{\gamma t}\phi_{M_2} * w_2) \\
&\quad - f_i(e^{\gamma t}w_1', e^{\gamma t}w_2', e^{\gamma t}\phi_{M_1} * w_1', e^{\gamma t}\phi_{M_2} * w_2')\Bigg)\Bigg|, \\
&\leq k_i|w_i - w'i| + k_i\Bigg[|w_1 - w_1'| + |w_2 - w_2'|\Bigg] + k_i\Bigg[\phi_{M_1} * |w_1 - w_1'| + \phi_{M_2} * |w_2 - w_2'|\Bigg], \\
&\leq 2k_i\Bigg(|w_1 - w_1'| + |w_2 - w_2'| + \phi_{M_1} * |w_1 - w_1'| + \phi_{M_2} * |w_2 - w_2'|\Bigg).
\end{aligned}$$

Then for any $(x,t) \in D_\tau$, we have:

$$|H_i(\mathbf{w})(x,t) - H_i(\mathbf{w}')(x,t)| \leq 8k_i\|\mathbf{w} - \mathbf{w}'\|_0, \;\; i = 1,2. \tag{3.30}$$

Consequently,

$$\|\mathcal{H}(\mathbf{w}) - \mathcal{H}(\mathbf{w}')\|_0 \leq 8\max\{k_1, k_2\}\|\mathbf{w} - \mathbf{w}'\|_0, \tag{3.31}$$

□

Theorem 3.1. Let $(\tilde{\mathbf{U}}, \hat{\mathbf{U}})$ be a pair of coupled upper and lower solutions of system (3.5). Then the system (3.5) has a unique solution $\mathbf{U}^*(x,t)$ Moreover, if $\mathbf{U}^{(0)} \in \mathcal{C}^\alpha(D_\tau) \cap \langle \tilde{\mathbf{U}}, \hat{\mathbf{U}} \rangle$ with $\mathbf{U}^{(0)} = (G_{10}(x), T_{20}(x))$ in K, the sequence obtained from (3.22) converges to $\mathbf{U}^*$ as $m \to \infty$ and $\mathbf{U}^* \in \langle \tilde{\mathbf{U}}, \hat{\mathbf{U}} \rangle$.

Proof. The proof is based on the contraction mapping theorem in the Banach space $\mathcal{C}(\overline{D}_\tau)$. For each $i = 1,2$, we define the operators $A_i : D(A_i) \to R(A_i)$ and $H_i : \mathcal{X} \to C^\alpha(D_\tau) \cap C(\overline{D}_\tau)$ by:

$$\begin{aligned}
A_i w_i &= L_i w_i + \gamma w_i \;\; (w_i \in D(A_i)), \\
H_i(\mathbf{w}) &= H_i(w_1, w_2) \;\; (\mathbf{w} \in \mathcal{X})
\end{aligned} \tag{3.32}$$

where $D(A_i)$ is the domain of A_i given by:

$$D(A_i) = \left\{ w_i \in C^{2,1}(D_\tau) \cap C(\overline{D}_\tau) : \;\; \frac{\partial w_i}{\partial x} = 0 \;\text{ on }\; S_\tau, \;\; w_i(0,x) = w_{i0}(x) \;\text{ in }\; K \right\}. \tag{3.33}$$

$R(A_i)$ is the range of A_i, and $H_i(w_1, w_2)$ is given by (3.25). In terms of the operators A_i and H_i, the iteration process in (3.26) can be written as:

$$A_i w_i^{(m)} = H_i(w_1^{(m-1)}, w_2^{(m-1)}) \;\; (w_i^{(m-1)} \in D(A_i)) \;\text{ for }\; i = 1,2, \tag{3.34}$$

and in vector form it becomes:

$$\mathcal{A}\mathbf{w}^{(m)} = \mathcal{H}(\mathbf{w}^{(m-1)}) \;\; (\mathbf{w}^{(m)} \in D(\mathcal{A})). \tag{3.35}$$

From the standard parabolic theorem the inverse operator $\mathcal{A}^{-1}$ exists and possesses the property:

$$\|\mathcal{A}^{-1}\mathbf{w} - \mathcal{A}^{-1}\mathbf{w}'\|_0 \leq \gamma^{-1}\|\mathbf{w} - \mathbf{w}'\|_0, \text{ for } \mathbf{w}, \mathbf{w}' \in \mathcal{C}^{\alpha}(D_\tau) \cap \mathcal{C}(\overline{D}_\tau) \tag{3.36}$$

This implies that (3.35) is equivalent to:

$$\mathbf{w}^{(m)} = \mathcal{A}^{-1}\mathcal{H}(\mathbf{w}^{(m-1)}), \ (\mathbf{w}^{(m-1)} \in D(\mathcal{A})), \tag{3.37}$$

which can be considered as a compact form for the integral representation (3.27) in the space $\mathcal{C}^{\alpha}(D_\tau) \cap \mathcal{C}(\overline{D}_\tau)$. In term of lemma (3.1), there exists a constant $k = 3\max\{k_1, k_2\}$, independent of γ, such that:

$$\|\mathcal{H}(\mathbf{w}) - \mathcal{H}(\mathbf{w}')\|_0 \leq k\|\mathbf{w} - \mathbf{w}'\|_0, \text{ for } \mathbf{w}, \mathbf{w}' \in \mathcal{X}. \tag{3.38}$$

Combining (3.36) and (3.38), we have:

$$\|\mathcal{A}^{-1}\mathcal{H}(\mathbf{w}) - \mathcal{A}^{-1}\mathcal{H}(\mathbf{w}')\|_0 \leq k(\gamma)^{-1}\|\mathbf{w} - \mathbf{w}'\|_0, \text{ for } \mathbf{w}, \mathbf{w}' \in \mathcal{X}. \tag{3.39}$$

By choosing $\gamma > k$, we have $\|\mathcal{A}^{-1}\mathcal{H}(\mathbf{w}) - \mathcal{A}^{-1}\mathcal{H}(\mathbf{w}')\|_0 \leq k(\gamma)^{-1}\|\mathbf{w} - \mathbf{w}'\|_0$ for $\mathbf{w}, \mathbf{w}' \in \mathcal{X}$. Thus, the operator $\mathcal{A}^{-1}\mathcal{H}$ possesses a contraction property in $\mathcal{X}$. This ensures that the sequence $\{\mathbf{w}^{(m)}\}$ converges in $\mathcal{C}(\overline{D}_\tau)$ to a unique solution $\mathbf{w}^* = (w_1^*, w_2^*)$ of

$$\mathcal{A}^{-1}\mathcal{H}(\mathbf{w}) = \mathbf{w}. \tag{3.40}$$

By the equivalence between (3.37) and (3.27) the sequence $\left\{w_i^{(m)}\right\}$ given by (3.27) converges in $C(\overline{D}_\tau)$ to w_i^* for $i = 1, 2$.

To show that $\mathbf{w}^*$ is the unique solution of (3.24), we need to raise the regularity of $\mathbf{w}^*$. Letting $m \to +\infty$ in (3.27) , Proposition 2.2 in chapter 2, page 29 shows that the density $\psi_i^{(m)}$ converges to some continuous function ψ_i^* and then, w_i^* satisfies by the dominated convergence the integral equation:

$$\begin{aligned} w_i^*(x,t) &= \int_0^t \int_K \Gamma_i(x-y, t-\tau)(H_i(\mathbf{w}^*))(y,\tau)dyd\tau \\ &\quad + \int_0^t \int_{\partial K} \Gamma_i(x-y, t-\tau)(\psi_i(\mathbf{w}^*))(y,\tau)dyd\tau + \int_K \Gamma_i(x-y,t)w_{i0}(y)dy. \end{aligned} \tag{3.41}$$

Since w_i^* is continuous in $\overline{D}_\tau$, the function $H_i(\mathbf{w}^*)$ is also continuous in $\overline{D}_\tau$. The continuity of $H_i(\mathbf{w}^*)$ ensures that the volume potential in (3.41) is Hölder continuity in $\overline{D}_T$ which lead to the Hölder continuity of $\mathbf{w}^*$. It follow that $H_i(\mathbf{w}^*)$ is Hölder continuity in $\overline{D}_T$. Theorem 2.1 in chapter 2, page 29 implies that $\mathbf{w}^*$ is the unique solution of (3.24). Since $\mathbf{U}^{(m)} = e^{\gamma t}\mathbf{w}^{(m)}$, the sequence $\mathbf{U}^{(m)}$ governed by (3.22) converges to a unique solution $\mathbf{U}^* = e^{\gamma t}\mathbf{w}^*$ of the equation (3.21). By the equivalence between (3.21) and (3.5), $\mathbf{U}^*$ is the unique solution of the system (3.5). Since $\mathbf{U}^{(\mathbf{0})} \in \langle \tilde{\mathbf{U}}, \hat{\mathbf{U}} \rangle$ and by Theorem 2.9 in page 49, $\mathbf{U}^* \in \langle \tilde{\mathbf{U}}, \hat{\mathbf{U}} \rangle$. ☐

Let us consider the following thresholds:

$$\begin{cases} \mathcal{R}_G = \dfrac{\gamma_G}{\delta_G + f\lambda_{fG}}, \\ \mathcal{R}_{G,0} = \dfrac{\gamma_G}{\delta_G}, \\ \mathcal{R}_{T,0} = \dfrac{\gamma_T}{\delta_T}. \end{cases} \tag{3.42}$$

Our model is designed for humid savannas where we assume that rainfall is sufficient to ensure that

$$\mathcal{R}_{G,0} > 1, \quad \text{and} \quad \mathcal{R}_{T,0} > 1. \tag{3.43}$$

Hence, in the rest of the chapter, we assume that (3.43) holds true.

Theorem 3.2 (Existence and uniqueness of global solution)**.** Assume that the following three conditions are valid.

- $\mathcal{R}_G > 1$,
- the initial functions $G(x,0)$ and $T(x,0) \in C^{\alpha}(D_\tau) \cap C(\overline{D}_\tau)$ and
- $0 \leq (G_0(x), T_0(x))' \leq 1$.

Then, the nonlocal reaction-diffusion system (3.5)-(3.7) admits a unique global solution $\mathbf{U}^*(x,t) = (G^*(x,t), T^*(x,t))'$ for $(x,t)' \in K \times (0,+\infty)$ and

$$0 \leq G^*(x,t) \leq W_1, \quad 0 \leq T^*(x,t) \leq W_2, \tag{3.44}$$

where

$$\begin{aligned} W_1 &= \max\left\{\sup\nolimits_K G(0,x), 1 - \frac{1}{\mathcal{R}_G}\right\}, \\ W_2 &= \max\left\{\sup\nolimits_K T(0,x), 1 - \frac{1}{\mathcal{R}_{T,0}}\right\}, \text{ if } \Omega = 0, \\ W_2 &= \max\left\{\sup\nolimits_K T(0,x), \frac{\sqrt{(1-\Omega)^2 + 4\Omega\left(1 - \dfrac{1}{\mathcal{R}_{T,0}}\right)} - (1-\Omega)}{2\Omega}\right\}, \text{ if } \Omega > 0. \end{aligned}$$

Proof. In theorem (3.1) we prove that to show the existence and the uniqueness of the solution to the system (3.5), we only need to find a pair of coupled upper and lower solution $\tilde{\mathbf{U}}$ and $\hat{\mathbf{U}}$ which satisfy the Lipschitz condition. If we choose $\tilde{\mathbf{U}}$ and $\hat{\mathbf{U}}$ to be constant vectors $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2)$ and $\hat{\mathbf{c}} = (\hat{c}_1, \hat{c}_2)$, these constant need to satisfy:

$$\begin{array}{rcl} 0 & \geq & \gamma_G\tilde{c}_1(1-\tilde{c}_1)-\delta_G\tilde{c}_1-\gamma_{TG}\tilde{c}_1\hat{c}_2-\lambda_{fG}f\tilde{c}_1, \\ 0 & \geq & \gamma_T\tilde{c}_2(1+\Omega\tilde{c}_2)(1-\tilde{c}_2)-\delta_T\tilde{c}_2-\lambda_{fT}f\omega(\tilde{c}_1)\exp(-p\hat{c}_2)\tilde{c}_2, \\ 0 & \leq & \gamma_G\hat{c}_1(1-\hat{c}_1)-\delta_G\hat{c}_1-\gamma_{TG}\hat{c}_1\tilde{c}_2-\lambda_{fG}f\hat{c}_1, \\ 0 & \leq & \gamma_T\hat{c}_2(1+\Omega\hat{c}_2)(1-\hat{c}_2)-\delta_T\hat{c}_2-\lambda_{fT}f\omega(\tilde{c}_1)\exp(-p\hat{c}_2)\hat{c}_2 \end{array} \tag{3.45}$$

and

$$\begin{array}{ll} \tilde{c}_1 & \geq \sup_K G(x,0), \\ \tilde{c}_2 & \geq \sup_K T(x,0), \\ \hat{c}_1 & \leq \inf_K G(x,0), \\ \hat{c}_2 & \leq \inf_K T(x,0). \end{array} \tag{3.46}$$

We choose $\hat{c}_1=\hat{c}_2=0$. Then $\tilde{c}_1=\max\left\{\sup_K G(0,x), 1-\dfrac{\delta_G+\lambda_{fG}f}{\gamma_G}\right\}$ and

$$\begin{array}{rcl} \tilde{c}_2 & = & \max\left\{\sup_K T(0,x), \dfrac{\gamma_T-\delta_T}{\gamma_T}\right\}, \text{ if } \Omega=0, \\ \tilde{c}_2 & = & \max\left\{\sup_K T(0,x), \dfrac{\sqrt{(1-\Omega)^2+4\Omega\left(1-\frac{\delta_T}{\gamma_T}\right)}-(1-\Omega)}{2\Omega}\right\}, \text{ if } \Omega>0. \end{array}$$

Consequently, system (3.5) has in D_τ an unique local solution which is uniformly bounded. We therefore end with the conclusion that system (3.5) admits a unique global solution for $(x,t)\in K\times(0,+\infty)$. ☐

3.3.2 Space homogeneous steady states and linear stability analysis

Our aim in this section is to derive a condition on spatial convolution such that an equilibrium or space homogeneous steady state is locally stable in the case $M_1=M_2=0$ but unstable for some $M_i>0, i=1,2$.

3.3.2.1 The local case: $M_1 = M_2 = 0$

Proposition 3.2. The local spatio-temporal model (LSTM) associated to the system (3.5) is given by the following system of two reaction-diffusion equations:

$$\begin{cases} \dfrac{\partial G}{\partial t} & = & D_G\dfrac{\partial^2 G}{\partial x^2} + \varphi_1(G,T), \;\; x \in (-l, l) \\ \dfrac{\partial T}{\partial T} & = & D_G\dfrac{\partial^2 T}{\partial x^2} + \varphi_2(G,T), \;\; x \in (-l, l) \\ \dfrac{\partial G}{\partial x} & = & \dfrac{\partial T}{\partial x} = 0 \;\text{ at }\; x = -l \;\text{ and }\; x = l, \end{cases} \tag{3.47}$$

where: $G(x,0) = G_0(x),\;\; T(x,0) = T_0(x)$, $x \in K$ and:

$$\begin{aligned} \varphi_1(G,T) &= \gamma_G G(1-G) - \delta_G G - \gamma_{TG} TG - \lambda_{fG} fG, \\ \varphi_2(G,T) &= \gamma_T T\left(1+\Omega T\right)\left(1-T\right) - \delta_T T - \lambda_{fT} f\omega(G)\exp(-pT)T. \end{aligned} \tag{3.48}$$

Moreover for $G \geq 0$ and $T \geq 0$, system (3.47) is quasi-monotone decreasing (or nonincreasing).

For the existence of space homogeneous solution of (3.47) it suffices to remark by the proof of Theorem 3.2 that $\mathbf{G} = \left(1-\dfrac{1}{\mathcal{R}_G};\dfrac{\sqrt{\left(1-\Omega\right)^2 + 4\Omega\left(1-\dfrac{1}{\mathcal{R}_{T,0}}\right)} - (1-\Omega)}{2\Omega}\right)$ and $\mathbf{T} = (0,0)$ are respectively upper and lower solution of the following system (3.49)

$$\begin{cases} -D_G\dfrac{\partial^2 G}{\partial x^2} & = & \varphi_1(G,T), \\ -D_T\dfrac{\partial^2 T}{\partial x^2} & = & \varphi_2(G,T), \end{cases} \tag{3.49}$$

when $\Omega > 0$ and $\mathbf{G} = \left(1-\dfrac{1}{\mathcal{R}_G};1-\dfrac{1}{\mathcal{R}_{T,0}}\right)$, $\mathbf{T} = (0,0)$ when $\Omega = 0$.

Due to the fact that LSTM associated to the system (3.5) is quasi-monotone decreasing (Smith [94]), we have the two following consequences. First, the LSTM can not lead to pattern formation (see the remark .5, page 178 in the Appendix .4 dedicated to mathematical process for pattern formation , or see e.g Kishimoto and Weinberger [57], Banerjee and Volpert [7], [10]) and second, the linear stability analysis of homogeneous steady states associated to LSTM is the same as for the space-implicit model i.e., the ODE model associated to system (3.5). The space-implicit ODE model corresponding to system (3.5) is:

$$\begin{cases} \dfrac{dG}{dt} & = & \gamma_G G(1-G) - \delta_G G - \gamma_{TG} TG - \lambda_{fG} fG, \\ \dfrac{dT}{dt} & = & \gamma_T T\left(1+\Omega T\right)\left(1-T\right) - \delta_T T - \lambda_{fT} f\omega(G)\exp(-pT)T, \end{cases} \tag{3.50}$$

with non negative initial data.

In this part, we are interested in the behavior of steady states of system (3.50), notably in the characterization of their stability properties. Recall that steady states of system (3.50) are also space homogeneous steady states of system (3.5) and/or system (3.47). Steady states of (3.50) are solutions of system (3.51):

$$\left\{\begin{array}{lcl}\gamma_G G(1-G)-\delta_G G-\gamma_{TG}TG-\lambda_{fG}fG & = & 0,\\ \gamma_T T(1+\Omega T)(1-T)-\delta_T T-\lambda_{fT}f\omega(G)\exp(-pT)T & = & 0.\end{array}\right. \tag{3.51}$$

Recall that we assumed that (3.43) is valid meaning that, the desert (the state with absence of vegetation) can not be stable. The following result is valid.

Proposition 3.3. (Trivial and semi-trivial steady states of system (3.50))
The system (3.5) admits three homogeneous steady states.

$a)$ a desert steady state $E_0=(0,0)'$.

$b)$ a forest steady state such that:

- When $\Omega=0$, then $E_{T_1}=\left(0,1-\dfrac{1}{\mathcal{R}_{T,0}}\right)'$ is the forest steady state. This is the case of no tree-tree facilitation.
- When $\Omega>0$, then $E_{T_2}=(0,T_2)'=\left(0,\dfrac{\sqrt{(1-\Omega)^2+4\Omega\left(1-\dfrac{1}{\mathcal{R}_{T,0}}\right)}-(1-\Omega)}{2\Omega}\right)'$ is the forest steady state. This is the case of tree-tree facilitation.

$c)$ a grassland steady state:

$$E_{G_e}=(G_e,0)'=\left(1-\frac{1}{\mathcal{R}_G},0\right)'.$$

Proof. Trivial and semi-trivial steady states of system (3.50) are those for which at least one of the components is zero. They satisfy the following system:

$$\left\{\begin{array}{lcl}G & = & 0,\\ T & = & 0,\end{array}\right. \text{ or } \left\{\begin{array}{lcl}G & = & 0,\\ \gamma_T(1+\Omega T)(1-T)-\delta_T & = & 0,\end{array}\right. \text{ or } \left\{\begin{array}{lcl}\gamma_G(1-G)-\delta_G-\gamma_{TG}T-\lambda_{fG}f & = & 0,\\ T & = & 0,\end{array}\right.$$

The first system give the trivial steady state $E_0=(0;0)$. The second system leads to:

$$\left\{\begin{array}{lcl}G & = & 0,\\ \Omega\gamma_T T^2+\gamma_T(1-\Omega)T-(\gamma_T-\delta_T) & = & 0,\end{array}\right. \tag{3.52}$$

and for the the second equation of system (3.52) we get the following case:

- If $\Omega = 0$ we have :
$$\begin{cases} G &= 0, \\ T &= \dfrac{\gamma_T - \delta_T}{\gamma_T}, \end{cases}$$
and $E_{T_1} = \left(0; \dfrac{\gamma_T - \delta_T}{\gamma_T}\right)'$ is the forest steady state.
- If $\Omega > 0$, the second equation of the system (3.52) have a unique positive solution:
$$T_2 = \frac{\sqrt{(1-\Omega)^2 + 4\Omega\left(1 - \frac{\delta_T}{\gamma_T}\right)} - (1-\Omega)}{2\Omega},$$
and for all $\Omega > 0$, $T_2 < 1$. Then, $E_{T_2} = \left(0; \dfrac{\sqrt{(1-\Omega)^2 + 4\Omega\left(1 - \frac{\delta_T}{\gamma_T}\right)} - (1-\Omega)}{2\Omega}\right)'$ is the forest steady state.

The system
$$\begin{cases} \gamma_G(1-G) - \delta_G - \gamma_{TG}T - \lambda_{fG}f &= 0, \\ T &= 0, \end{cases}$$
leads to :
$$\begin{cases} G_e &= 1 - \dfrac{1}{\mathcal{R}_G}, \\ T &= 0, \end{cases}$$
with $\mathcal{R}_G = \dfrac{\gamma_G}{\delta_G + \lambda_{fG}f}$. Then, if $\mathcal{R}_G < 1$ then $G_e < 0$ and we don't have grassland steady state. If $\mathcal{R}_G > 1$ we have $0 < G_e < 1$ and then grassland steady state:
$$E_{G_e} = \left(1 - \frac{\delta_G}{\gamma_G} - \frac{f\lambda_{fG}}{\gamma_G}; 0\right)' = \left(1 - \frac{1}{\mathcal{R}_G}; 0\right)'.$$

□

Remark 3.1. It is straightforward to observe that E_{T_2} is an increasing function of Ω.

We are now interested in the coexistence steady state (savanna steady state); set:
$$\begin{aligned} a &= -\frac{\lambda_{fG}f + \delta_G}{\gamma_{TG}}, \\ b &= \frac{\gamma_G}{\gamma_{TG}}, \\ \theta &= 2(a+b)b\Omega\gamma_T + \gamma_T(1-\Omega)b, \\ \alpha &= \Omega\gamma_T b^2, \\ q &= (\gamma_T - \delta_T) + \gamma_T(\Omega - 1)(a+b) - \Omega\gamma_T(a+b)^2, \\ m &= \lambda_{fT} f \exp\left(-p(a+b)\right), \\ \theta^* &= \frac{24\alpha + mpb\left((pb)^2 + 6(pb) + 6\right)\exp(pb)}{6}, \end{aligned}$$

and

$$\mathcal{R}_T = \frac{\gamma_T}{\delta_T + \lambda_{fT} f \omega(G_e)}, \quad \mathcal{R}_{F,f} = \frac{\gamma_G}{\delta_G + \lambda_{fG} f + \gamma_{TG} T_i}, \quad \mathcal{R}^1_{\Omega} = \frac{\gamma_T(1-\Omega)}{p\lambda_{fT} f \omega(G_e)}. \tag{3.53}$$

Proposition 3.4. (Savanna steady state)

- **case I**: $f = 0$.
  If $\mathcal{R}_{F,f=0} > 1$, then we have a unique savanna steady state $E_s = (G^*, T^*)'$ such that

  $$G^* = 1 - \frac{1}{\mathcal{R}_{F,f=0}} \text{ and } T^* = T_i, i = 1, 2. \tag{3.54}$$

- **case II**: $f > 0$ and $\gamma_{TG} = 0$.
  The savanna steady state $E^* = (G^*, T^*)'$ verifies:

  $$\begin{cases} G^* = G_e \\ \Omega\gamma_T(T^* - T_2)(T^* - T_{2-}) + \lambda_{fT} f \omega(G_e) \exp(-pT^*) \quad = \quad 0 \end{cases} \tag{3.55}$$

  where $T_{2-} = -\dfrac{(1-\Omega) + \sqrt{(1-\Omega)^2 + 4\Omega(1 - \frac{\delta_T}{\gamma_T})}}{2\Omega}$. Hence:
  - $*$ if $\mathcal{R}^1_{\Omega} > 1$, then there may exist 0 or 1 savanna steady state.
  - $*$ if $\mathcal{R}^1_{\Omega} < 1$, then there may exist 0, 1 or 2 savanna steady states.

- **case III**: $f > 0$ and $\gamma_{TG} \neq 0$.
  The savanna steady state $E_s = (G^*, T^*)'$ must satisfy these two relations:

  $$-\alpha(G^*)^4 + \theta(G^*)^3 - m\exp(pbG^*)(G^*)^2 + (q - \alpha g_0^2)(G^*)^2 + \theta g_0^2 G^* + q g_0^2 = 0, \tag{3.56}$$

  and

  $$T^* = (a + b) - bG^*. \tag{3.57}$$

  Moreover G^* must satisfy the inequality

  $$\max\left\{G_e - \frac{\gamma_{TG}}{\gamma_G}; 0\right\} < G^* < G_e. \tag{3.58}$$

  We can therefore summarize the maximum number of savanna steady states according to the following cases:

  - **Case 1**: $\theta < mpb$

Condition	$q < m + \alpha g_0^2$	$q > m + \alpha g_0^2$
Maximal number of savanna steady states	2	3

Table 3.2: Maximal number of savanna steady states of system (3.50) with $\theta < mpb$

- **Case 2**: $\theta > mpb$,

Condition	$\theta < \theta^*$	$\theta > \theta^*$
Maximal number on savanna steady states	4	3

Table 3.3: Maximal number of savanna steady states of system (3.50) with $\theta > mpb$

Proof. A savanna steady state $(G^*, T^*)'$ is a solution of the following system of two equations in G and T:

$$\begin{cases} \gamma_G(1-G) - \delta_G - \gamma_{TG}T - \lambda_{fG}f = 0, \\ \gamma_T(1+\Omega T)(1-T) - \delta_T - \lambda_{fT}f\omega(G)\exp(-PT) = 0. \end{cases} \tag{3.59}$$

If $f = 0$, savanna steady state $(G^*, T^*)'$ is a solution of

$$\begin{cases} \gamma_G(1-G) - \delta_G - \gamma_{TG}T &= 0, \\ \gamma_T(1+\Omega T)(1-T) - \delta_T &= 0. \end{cases} \tag{3.60}$$

The second equation of system (3.60) give :

$$T^* = T_i, i = 1, 2 \text{ (depending on the values of } \Omega).$$

The first system of (3.60) leads to

$$\begin{aligned} G^* &= 1 - \frac{\delta_G + \gamma_{TG}T_i}{\gamma_G}, \\ &= 1 - \frac{1}{\mathcal{R}_{F,f=0}}. \end{aligned}$$

If $f > 0$ and $\gamma_{TG} = 0$, then the savanna equilibrium $(G^*; T^*)'$, satisfies:

$$\begin{cases} G^* = G_e, \\ \Omega\gamma_T(T^* - T_2)(T^* - T_{2-}) + \lambda_{fT}f\omega(G_e)\exp(-pT^*) &= 0. \end{cases} \tag{3.61}$$

Let us set $J(T) = \Omega\gamma_T(T - T_2)(T - T_{2-}) + \lambda_{fT}f\omega(G_e)\exp(-pT)$, then:

$$\lim_{T\to 0} J(T) = (\delta_T + \lambda_{fT}f\omega(G_e))\left(1 - \mathcal{R}_T\right). \tag{3.62}$$

We have also the first derivative of J:

$$\begin{aligned} J'(T) &= \Omega\gamma_T\left[2T - T_2 - T_{2-}\right] - p\lambda_{fT}f\omega(G_e)\exp(-pT), \\ \lim_{T\to 0} J'(T) &= p\lambda_{fT}f\omega(G_e)[\mathcal{R}^1_\Omega - 1], \\ \lim_{T\to 1} J'(T) &= p\lambda_{fT}f\omega(G_e)\exp(-p)[\mathcal{R}^2_\Omega - 1], \end{aligned} \tag{3.63}$$

where $\mathcal{R}^2_\Omega = \dfrac{\gamma_T(1+\Omega)}{p\lambda_{fT}f\omega(G_e)\exp(-p)}$.

The second derivative:

$$J"(T) = 2\Omega\gamma_T + p^2\lambda_{fT} f\omega(G_e)\exp(-pT) > 0. \tag{3.64}$$

Therefore, J' is increasing on $[0;1]$.

(I) if $\mathcal{R}_\Omega^1 > 1$, then $J'(T) > 0$ on $[0;1]$, and J is increasing on $[0;1]$;

- (a) if $\mathcal{R}_T < 1$, then $J(T) > 0$ on $[0;1]$.
- (b) if $\mathcal{R}_T > 1$, then there exists at most one savanna steady state.

(II) if $\mathcal{R}_\Omega^1 < 1$, then $\lim_{T\to 0} J'(T) < 0$ and due to J' increasing, we have:

- (a) if $\mathcal{R}_\Omega^2 < 1$, then $J'(T) < 0$ on $[0;1]$ and J is decreasing on that interval. Then
  - (a_1) if $\mathcal{R}_T > 1$, then $J(T) < 0$ on $[0;1]$.
  - (a_2) if $\mathcal{R}_T < 1$, we have at most one savanna steady state.
- (b) if $\mathcal{R}_\Omega^2 > 1$, then by the intermediate value theorem, there exist $T_0 \in [0;1]$ such that $J'(T_0) = 0$. Then:
  - (b_1) if $J(T_0) > 0$, then $J(T) > 0$ on $[0;1]$.
  - (b_2) if $J(T_0) < 0$, we have at most two savanna steady states $(G_e, T_i^*)'$,$i = 1,2$ where $T_1^* \in [0;T_0]$ and $T_2^* \in [T_0;1]$

If $f > 0$ and $\gamma_{TG} \neq 0$, savanna equilibrium is a solution of the system:

$$\left\{\begin{array}{lcl} \gamma_G(1-G) - \delta_G - \gamma_{TG}T - \lambda_{fG}f & = & 0, \\ \gamma_T(1+\Omega T)(1-T) - \delta_T - \lambda_{fT}f\omega(G)\exp(-pT) & = & 0, \end{array}\right. \tag{3.65}$$

the first equation of system (3.65) gives:

$$T = -\frac{\delta_G + \lambda_{fG}f}{\gamma_{TG}} + \frac{\gamma_G}{\gamma_{TG}}(1-G)$$

set:

$$a = -\frac{\delta_G + \lambda_{fG}f}{\gamma_{TG}} \text{ and } b = \frac{\gamma_G}{\gamma_{TG}}$$

then

$$T = (a+b) - bG. \tag{3.66}$$

Using the fact that $G, T \in]0;1]$, (3.66) gives that:

$$\frac{a-1}{b} + 1 < G < \frac{a}{b} + 1.$$

Note that $\dfrac{a-1}{b}+1=(1-\dfrac{1}{\mathcal{R}_G})-\dfrac{\gamma_{TG}}{\gamma_G}$ and $\dfrac{a}{b}+1=1-\dfrac{1}{\mathcal{R}_G}$. Therefore, because $\mathcal{R}_G>1$ then

$$G_e-\frac{\gamma_{TG}}{\gamma_G}<G<G_e. \tag{3.67}$$

The second equation of system (3.65) gives :

$$\lambda_{fT}f\omega(G)\exp(-pT)=(\gamma_T-\delta_T)+\gamma_T(\Omega-1)T-\gamma_T\Omega T^2. \tag{3.68}$$

Substituting (3.66) in (3.68) we obtain first:

$$\begin{aligned}(\gamma_T-\delta_T)+\gamma_T(\Omega-1)T-\gamma_T\Omega T^2 &= (\gamma_T-\delta_T)+\gamma_T(\Omega-1)(a+b)-\gamma_T\Omega(a+b)^2+\\ &\quad\left(2(a+b)b\gamma_T\Omega-b\gamma_T(\Omega-1)\right)G-\gamma_T\Omega b^2G^2\end{aligned}$$

then,

$$\begin{aligned}\lambda_{fT}f\omega(G)\exp(-pT) &= (\gamma_T-\delta_T)+\gamma_T(\Omega-1)(a+b)-\gamma_T\Omega(a+b)^2+\\ &\quad\left(2(a+b)b\gamma_T\Omega-b\gamma_T(\Omega-1)\right)G-\gamma_T\Omega b^2G^2.\end{aligned} \tag{3.69}$$

Set:

$$\begin{aligned}q &= (\gamma_T-\delta_T)+\gamma_T(\Omega-1)(a+b)-\gamma_T\Omega(a+b)^2,\\ \theta &= 2(a+b)b\gamma_T\Omega-b\gamma_T(\Omega-1),\\ \alpha &= \gamma_T\Omega b^2.\end{aligned}$$

Then, we obtain in (3.69)

$$\lambda_{fT}f\omega(G)\exp(-pT)=q+\theta G-\alpha G^2. \tag{3.70}$$

Substituting (3.66) in (3.70), we obtain:

$$\lambda_{fT}f\exp\left(-p(a+b)\right)\exp(pbG)G^2=qg_0^2+\theta g_0^2G+(q-\alpha g_0^2)G^2+\theta G^3-\alpha G^4. \tag{3.71}$$

Set

$$m=\lambda_{fT}f\exp\left(-p(a+b)\right).$$

Hence,

$$-\alpha G^4+\theta G^3-m\exp(pbG)G^2+(q-\alpha g_0^2)G^2+\theta g_0^2G+qg_0^2=0. \tag{3.72}$$

Define the function f by:

$$f(G)=-\alpha G^4+\theta G^3-m\exp(pbG)G^2+(q-\alpha g_0^2)G^2+\theta g_0^2G+qg_0^2 \tag{3.73}$$

and fine the roots of f in the interval $[0;1]$.

$$\begin{cases} \lim_{G\to 0} f(G) &= qg_0^2, \\ \lim_{G\to +\infty} f(G) &= -\infty, \\ \lim_{G\to 1} f(G) &= (\theta-\alpha+q)(g_0^2+1) - m\exp(pb). \end{cases} \tag{3.74}$$

The first derivative of f is :

$$f'(G) = -4\alpha G^3 + 3\theta G^2 - mpb\exp(pbG)G^2 - 2m\exp(pbG)G + 2(q-\alpha g_0^2)G + \theta g_0^2 \tag{3.75}$$

and

$$\begin{cases} \lim_{G\to 0} f'(G) &= \theta g_0^2, \\ \lim_{G\to +\infty} f'(G) &= -\infty, \\ \lim_{G\to 1} f'(G) &= -4\alpha + 3\theta + 2(q-\alpha g_0^2) + \theta g_0^2 - m\exp(pb)[pb+2]. \end{cases} \tag{3.76}$$

The second derivative of f is given by:

$$f''(G) = -12\alpha G^2 + 6\theta G - m(pb)^2\exp(pbG)G^2 - 4mpb\exp(pbG)G - 2m\exp(pbG) + 2(q-\alpha g_0^2) \tag{3.77}$$

and:

$$\begin{cases} \lim_{G\to 0} f''(G) &= 2\left[q-(m+\alpha g_0^2)\right], \\ \lim_{G\to +\infty} f''(G) &= -\infty, \\ \lim_{G\to 1} f''(G) &= 6\theta + 2(q-\alpha g_0^2) - 12\alpha - m\exp(pb)\left[(pb)^2+4pb+2\right]. \end{cases} \tag{3.78}$$

The third derivative of f is given by:

$$f'''(G) = -24\alpha G + 6\theta - m(pb)^3\exp(pbG)G^2 - 6m(pb)^2\exp(pbG)G - 6m(pb)\exp(pbG) \tag{3.79}$$

and:

$$\begin{cases} \lim_{G\to 0} f'''(G) &= 6(\theta - mpb), \\ \lim_{G\to +\infty} f'''(G) &= -\infty, \\ \lim_{G\to 1} f'''(G) &= 6\theta - \left[mpb\left((pb)^2+6pb+6\right)\exp(pb) + 24\alpha\right]. \end{cases} \tag{3.80}$$

The fourth derivative of f is given by:

$$f''''(G) = -m(pb)^4\exp(pbG)G^2 - 8m(pb)^3\exp(pbG)G - 6m(pb)^2\exp(pbG) - 24\alpha. \tag{3.81}$$

$f''''(G) < 0$ on $[0,+\infty[$, therefore on $[0;1]$. Then f''' decreases on $[0;1]$.

(I) if $\theta < mpb$, then $f'''(G) < 0$ in $[0;+\infty[$ and then f'' strictly decreases on $[0;+\infty[$. According to (3.78) we have:

$(I.1)$ if $q < m + \alpha g_0^2$ then $f''(G) < 0$ on $[0; +\infty[$. Therefore f' strictly decreases on $[0; +\infty[$. According to (3.76) we have:

$(I.1.1)$ if $\theta < 0$ then $f'(G) < 0$ on $[0; +\infty[$ and then f strictly decreases on $[0; +\infty[$. According to (3.74)

$(I.1.1.1)$ If $q < 0$ then $f(G) < 0$ on $[0; +\infty[$.

$(I.1.1.2)$ If $q > 0$ then,

1. if $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.
2. if $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G_1^* \in [0; 1]$ such that $f(G_1^*) = 0$.

$(I.1.2)$ If $\theta > 0$ then $f'(G)$ has a positive root on $[0; +\infty[$.

$(I.1.2.1)$ If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $f'(G) > 0$ on $[0; 1]$ and f strictly increases on $[0; 1]$.

$(I.1.2.1.1)$ If $q > 0$, then $f(G) > 0$ on $[0; 1]$.

$(I.1.2.1.2)$ If $q < 0$, then:

(a) if $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $f(G) < 0$ on $[0; 1]$.

(b) If $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G_2^* \in [0; 1]$ such that $f(G_2^*) = 0$.

$(I.1.2.2)$ If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $\exists G^{(0)} \in [0; 1]$ such that $f'(G^{(0)}) = 0$ on and $f'(G) > 0$ on $[0; G^{(0)}]$ and $f'(G) < 0$ on $[G^{(0)}; 1]$.

$(I.1.2.2.1)$ If $f(G^{(0)}) < 0$, then $f(G) < 0$ on $[0; 1]$.

$(I.1.2.2.2)$ If $f(G^{(0)}) > 0$, then:

(a) If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

(b) If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G_3^* \in [0; G^{(0)}]$ is the unique root of f.

(c) If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $G_4^* \in [G^{(0)}; 1]$ is the unique root of f on $[0; 1]$.

(d) If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G_3^* \in [0; G^{(0)}]$ and $G_4^* \in [G^{(0)}; 1]$ such that $f(G_3^*) = f(G_4^*) = 0$.

$(I.2)$ If $q > m + \alpha g_0^2$ then $f''(G)$ has a unique positive root on $[0; +\infty[$.

$(I.2.1)$ If $q > 6\alpha - 3\theta + \alpha g_0^2 + \dfrac{1}{2}m\left[(pb)^2 + 4(pb) + 2\right]\exp(pb)$ then $f''(G) > 0$ on $[0; 1]$, then f' strictly increases on $[0; 1]$.

$(I.2.1.1)$ If $\theta > 0$ then $f'(G) > 0$ on $[0; 1]$. So, f strictly increases on $[0; 1]$.

$(I.2.1.1.1)$ If $q > 0$, then $f(G) > 0$ on $[0; 1]$.

$(I.2.1.1.1)$ If $q < 0$, then:

(a) If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

(b) If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_5^* \in [0;1]$ such that $f(G_5^*) = 0$.

$(I.2.1.2)$ If $\theta < 0$, then because f' is strictly increasing on $[0;1]$ we have:

$(I.2.1.2.1)$ If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $f'(G) < 0$ on $[0;1]$. Then, f strictly decreases on $[0;1]$.

(a) If $q < 0$ then, $f(G) < 0$ on $[0;1]$.

(b) If $q > 0$ then,

$(b.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_6^* \in [0;1]$ such that $f(G_6^*) = 0$.

$(I.2.1.2.2)$ If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\exists G^{(00)} \in [0;1]$ such that $f'(G^{(00)}) = 0$ and then $f'(G) < 0$ on $[0;G^{(00)}]$ and $f'(G) > 0$ on $[G^{(00)};1]$.

(a) If $f(G^{(00)}) > 0$ then $f(G) > 0$ on $[0;1]$.

(b) If $f(G^{(00)}) < 0$ then,

$(b.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_7^* \in [0;G^{(00)}]$ such that $f(G_7^*) = 0$.

$(b.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_8^* \in [G^{(00)};1]$ such that $f(G_8^*) = 0$.

$(b.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G_7^* \in [0;G^{(00)}]$ and $G_8^* \in [G^{(00)};1]$ are the two roots of f.

$(I.2.2)$ If $q < 6\alpha - 3\theta + \alpha g_0^2 + \dfrac{1}{2}m\left[(pb)^2 + 4(pb) + 2\right]\exp(pb)$, then $\exists\ G^{(000)} \in [0;1]$ such that $f''(G^{(000)}) = 0$. Then, $f''(G) > 0$ on $[0;G^{(000)}]$ and $f''(G) < 0$ on $[G^{(000)};1]$.

$(I.2.2.1)$ If $f'(G^{(000)}) < 0$, then $f'(G) < 0$ on $[0;1]$ and f strictly decreases on that interval.

(a) If $q < 0$ then $f(G) < 0$ on $[0;1]$.

(b) If $q > 0$, then:

$(b.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then, $\exists G_9^* \in [0;1]$ such that $f(G_9^*) = 0$.

$(I.2.2.2)$ If $f'(G^{(000)}) > 0$, then:

$(I.2.2.2.1)$ If $\theta > 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $f'(G) > 0$ on $[0;1]$ and f is increasing on $[0;1]$.

(a) If $q > 0$ then $f(G) > 0$ on $[0;1]$.

(b) If $q < 0$, then :

$(b.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{10} \in [0;1]$ such that $f(G^*_{10}) = 0$.

$(I.2.2.2.2)$ If $\theta < 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\exists G^{(0000)} \in [0;G^{(000)}]$ such that $f'(G^{(0000)}) = 0$. Therefore $f'(G) < 0$ on $[0;G^{(0000)}]$ and $f'(G) > 0$ on $[G^{(0000)};1]$.

(a) If $f(G^{(0000)}) > 0$, then $f(G) > 0$ on $[0;1]$.

(b) If $f(G^{(0000)}) < 0$, then:

$(b.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{11} \in [0;G^{(0000)}]$ such that $f(G^*_{11}) = 0$.

$(b.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$ then $\exists G^*_{12} \in [G^{(0000)};1]$ such that $f(G^*_{12}) = 0$.

$(b.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$ then $G^*_{11} \in [0;G^{(0000)}]$ and $G^*_{12} \in [G^{(0000)};1]$ are the two roots of f.

$(I.2.2.2.3)$ If $\theta > 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\exists G^{(00000)} \in [G^{(000)};1]$ such that $f'(G^{(00000)}) = 0$. Therefore, $f'(G) > 0$ on $[0;G^{(00000)}]$ and $f'(G) < 0$ on $[G^{(00000)};1]$.

(a) If $f(G^{(00000)} < 0$, then $f(G) < 0$ on $[0;1]$.

(b) If $f(G^{(00000)} > 0$, then:

$(b.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$ then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{13} \in [0;G^{(00000)}]$ such that $f(G^*_{13}) = 0$.

$(b.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{14} \in [G^{(00000)};1]$ such that $f(G^*_{14}) = 0$.

$(b.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G^*_{13} \in [0;G^{(00000)}]$ and $G^*_{14} \in [G^{(00000)};1]$ are the two roots of f.

$(I.2.2.2.4)$ If $\theta < 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\exists G^{(0000)} \in [0;G^{(000)}]$ and $G^{(00000)} \in [G^{(000)};1]$ such that $f'(G^{(0000)}) = f'(G^{(00000)}) = 0$. Therefore $f'(G) < 0$ on $[0;G^{(0000)}] \cup [G^{(00000)};1]$ and $f'(G) > 0$ on $[G^{(0000)};G^{(00000)}]$.

(a) If $f(G^{(0000)}) > 0$, then

$(a.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(a.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, $G^*_{14} \in [G^{(00000)};1]$ is the unique root of f.

(b) If $f(G^{(0000)}) < 0$ and $f(G^{(00000)}) < 0$ then

$(b.1)$ $q < 0$ on then $f(G) < 0$ on $[0;1]$.

$(b.2)$ $q > 0$ then $\exists G^*_{11} \in [0; G^{(0000)}]$ is the unique root of f.

(c) If $f(G^{(0000)}) < 0$ and $f(G^{(00000)}) > 0$ then $\exists G^*_{15} \in [G^{(0000)}; G^{(00000)}]$ such that $f(G^*_{13}) = 0$.

$(c.1)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$ then G^*_{15} is the unique root of f in the interval $[0;1]$.

$(c.2)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G^*_{15} we have also $G^*_{14} \in [G^{(00000)};1]$ roots of f. Therefore,$f(G^*_{15}) = f(G^*_{14}) = 0$.

$(c.3)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,then with G^*_{15} we have also $G^*_{11} \in [0; G^{(0000)}]$ roots of f. Therefore,$f(G^*_{15}) = f(G^*_{11}) = 0$.

$(c.4)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,then with G^*_{15} we have also $G^*_{11} \in [0; G^{(0000)}]$ and $G^*_{14} \in [G^{(00000)};1]$ roots of f. Therefore,$f(G^*_{15}) = f(G^*_{11}) = f(G^*_{14}) = 0$.

(II) we suppose that $\theta > mpb$, because f''' is decreasing on $[0;1]$, by the intermediate value theorem f''' has a unique positive root on $[0;+\infty[$.

$(II.1)$ If $\theta > \dfrac{1}{6}\left[24\alpha + mpb\left((pb)^2 + 6(pb) + 6\right)\exp(pb)\right]$, then $f'''(G) > 0$ on $[0;1]$ and therefore, f'' is increasing on $[0;1]$.

$(II.1.1)$ If $q > m + \alpha g_0^2$, then $f''(G) > 0$ on $[0;1]$ therefore, f' strictly increases on $[0;1]$.

$(II.1.1.1)$ If $\theta > 0$, then $f'(G) > 0$ on $[0;1]$ and therefore f strictly increases on $[0;1]$.

(a) If $q > 0$, then $f(G) > 0$ on $[0;1]$.

(b) If $q < 0$, then:

$(b.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$, on $[0;1]$.

$(b.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{17} \in [0;1]$ such that $f(G^*_{17}) = 0$.

$(II.1.1.2)$ If $\theta < 0$, then :

$(II.1.1.2.1)$ If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $f'(G) < 0$ on $[0;1]$ and then f decrease strictly on $[0;1]$.

(a) If $q < 0$ then $f(G) < 0$ on $[0;1]$.

(b) If $q > 0$ then:

$(b.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{18}^* \in [0;1]$ such that $f(G_{18}^*) = 0$.

$(II.1.1.2.2)$ If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$, then $\exists G_{(0)} \in [0;1]$ such that $f'(G_{(0)}) = 0$. Therefore, $f'(G) < 0$ on $[0;G_{(0)}]$ and $f'(G) > 0$ on $[G_{(0)};1]$.

(a) If $f(G_{(0)}) > 0$ then $f(G) > 0$ on $[0;1]$.

(b) If $f(G_{(0)}) < 0$, then:

$(b.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$ then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G_{19}^* \in [0;G_{(0)}]$ such that $f(G_{19}^*) = 0$.

$(b.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{20}^* \in [G_{(0)};1]$ such that $f(G_{20}^*) = 0$.

$(b.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G_{19}^* \in [0;G_{(0)}]$ and $G_{20}^* \in [G_{(0)};1]$ are the two roots of f.

$(II.1.2)$ If $q < m + \alpha g_0^2$, then we have the following situations:

$(II.1.2.1)$ If $q < 6\alpha - 3\theta + \alpha g_0^2 + \dfrac{1}{2}m\exp(pb)\left[(pb)^2 + pb + 2\right]$, then $f''(G) < 0$ on $[0;1]$. Then f' decrease strictly on $[0;1]$.

$(II.1.2.1.1)$ If $\theta < 0$, then $f'(G) < 0$ on $[0;1]$ and f is therefore decreasing on $[0;1]$.

(a) If $q < 0$, then $f(G) < 0$ on $[0;1]$.

(b) If $q > 0$, then:

$(b.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{21}^* \in [0;1]$ such that $f(G_{21}^*) = 0$.

$(II.1.2.1.2)$ If $\theta > 0$, then, we have the following situations:

(a) If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$,then $f'(G) > 0$ on $[0;1]$ and f increase strictly on $[0;1]$.

$(a.1)$ If $q > 0$, then $f(G) > 0$ on $[0;1]$.

$(a.2)$ If $q < 0$, then:

$(a.2.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,then $f(G) < 0$ on $[0;1]$.

$(a.2.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{22}^* \in [0;1]$ such that $f(G_{22}^*) = 0$.

(b) If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$. We use the fact that f' is decreasing (strictly) on $[0;1]$. By the intermediate values theorem: $\exists G_{(00)} \in [0;1]$ such that $f'(G_{(00)}) = 0$. Therefore, $f'(G) > 0$ on $[0;G_{(00)}]$ and $f'(G) < 0$ on $[G_{(00)};1]$.

$(b.1)$ If $f(G_{(00)}) < 0$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $f(G_{(00)}) > 0$, then:

$(b.2.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$ then $f(G) > 0$ on $[0;1]$.

$(b.2.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{23}^* \in [0;G_{(00)}]$ such that $f(G_{23}^*) = 0$.

$(b.2.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G_{24}^* \in [G_{(00)};1]$ such that $f(G_{24}^*) = 0$.

$(b.2.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G_{23}^* \in [0;G_{(00)}]$ and $G_{24}^* \in [G_{(00)};1]$ are the two roots of f.

$(II.1.2.2)$ If $q > 6\alpha - 3\theta + \alpha g_0^2 + \dfrac{1}{2}m\exp(pb)\left[(pb)^2 + pb + 2\right]$, because f'' increase strictly on $[0;1]$, by the intermediate values theorem $\exists G_{(000)} \in [0;1]$ such that $f''(G_{(000)}) = 0$. So, $f''(G) < 0$ on $[0;G_{(000)}]$ and there $f''(G) > 0$ on $[G_{(000)};1]$.

$(II.1.2.2.1)$ If $f'(G_{(000)}) > 0$, then $f'(G) > 0$ on $[0;1]$, therefore f is increasing on that interval.

(a) If $q > 0$, then $f(G) > 0$ on $[0;1]$.

(b) If $q < 0$, then

$(b.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{25}^* \in [0;1]$ such that $f(G_{25}^*) = 0$.

$(II.1.2.2.2)$ If $f'(G_{(000)}) < 0$, then:

(a) If $\theta < 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$, then $f'(G) < 0$ on $[0;1]$. Therefore f strictly decreases on $[0;1]$.

$(a.1)$ If $q < 0$, then $f(G) < 0$ on $[0;1]$.

$(a.2)$ If $q > 0$, then:

$(a.2.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(a.2.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{26}^* \in [0;1]$ such that $f(G_{26}^*) = 0$.

(b) If $\theta > 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$, then $\exists G_{(0000)} \in [0; G_{(000)}]$ such that $f'(G_{(0000)}) = 0$. Therefore $f'(G) > 0$ on $[0; G_{(0000)}]$ and $f'(G) < 0$ on $[G_{(0000)}; 1]$.

$(b.1)$ If $f(G_{(0000)}) < 0$, then $f(G) < 0$ on $[0; 1]$.

$(b.2)$ If $f(G_{(0000)}) > 0$, then:

$(b.2.1)$ If $q > 0$, and $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

$(b.2.2)$ If $q < 0$, and $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G^*_{27} \in [0; G_{(0000)}]$ such that $f(G^*_{27}) = 0$.

$(b.2.3)$ If $q > 0$, and $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G^*_{28} \in [G_{(0000)}; 1]$ such that $f(G^*_{28}) = 0$.

$(b.2.4)$ If $q < 0$, and $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $^*_{27} \in [0; G_{(0000)}]$ and $G^*_{28} \in [G_{(0000)}; 1]$ are the two roots of f.

(c) If $\theta < 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\,[pb + 2]\right]$, then $\exists G_{(00000)} \in [G_{(000)}; 1]$ such that $f'(G_{(00000)}) = 0$. Therefore $f'(G) < 0$ on $[0; G_{(00000)}]$ and $f'(G) > 0$ on $[G_{(00000)}; 1]$.

$(c.1)$ If $f(G_{(00000)}) > 0$, then $f(G) > 0$ on $[0; 1]$.

$(c.2)$ If $f(G_{(00000)}) < 0$:

$(c.2.1)$ If $q < 0$, and $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $f(G) < 0$ on $[0; 1]$.

$(c.2.2)$ If $q > 0$, and $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G^*_{29} \in [0; G_{(00000)}]$ such that $f(G^*_{29}) = 0$.

$(c.2.3)$ If $q < 0$, and $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $\exists G^*_{30} \in [0; G_{(00000)}]$ such that $f(G^*_{30}) = 0$.

$(c.2.4)$ If $q > 0$, and $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then $G^*_{29} \in [0; G_{(00000)}]$ and $G^*_{30} \in [0; G_{(00000)}]$ are the two roots of f.

(d) If $\theta > 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\,[pb + 2]\right]$, then: $\exists G_{(0000)} \in [0; G_{(000)}]$ and $G_{(00000)} \in [G_{(000)}; 1]$ such that $f'(G_{(0000)}) = f'(G_{(00000)}) = 0$. Therefore $f'(G) > 0$ on $[0; G_{(0000)}] \cup [G_{(00000)}; 1]$ and $f'(G) > 0$ on $[G_{(0000)}; G_{(0000)}]$.

$(d.1)$ If $f(G_{(0000)}) < 0$,then

$(d.1.1)$ if $q < \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$ then $f(G) < 0$ on $[0; 1]$.

$(d.1.2)$ if $q > \dfrac{m\exp(pb)}{g_0^2 + 1} + \alpha - \theta$, then G^*_{30} is the unique root of f.

$(d.2)$ If $f(G_{(0000)}) > 0$ and $f(G_{(00000)}) > 0$, then:

$(d.2.1)$ If $q > 0$, then $f(G) > 0$ on $[0; 1]$.

$(d.2.2)$ If $q < 0$, then $G^*_{27} \in [0; G_{(0000)}]$ is the unique root of f.

$(d.3)$ If $f(G_{(0000)}) > 0$ and $f(G_{(00000)}) < 0$, then: $\exists G^*_{31} \in [G_{(0000)}; G_{(00000)}]$ such that $f(G^*_{31}) = 0$.

$(d.3.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G^*_{31}, we have also $G^*_{30} \in [G_{(00000)}; 1]$ such that $f(G^*_{31}) = f(G^*_{30}) = 0$.

$(d.3.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{31} is the unique root of f. Therefore $f(G^*_{31}) = 0$.

$(d.3.3)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G^*_{31}, we have also $G^*_{27} \in [0; G_{(0000)}]$ such that $f(G^*_{31}) = f(G^*_{27}) = 0$.

$(d.3.4)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G^*_{31} and $G^*_{30} \in [G_{(00000)}; 1]$, we have also $G^*_{27} \in [0; G_{(0000)}]$ such that $f(G^*_{29}) = f(G^*_{30}) = f(G^*_{32}) = 0$

$(II.2)$ If $\theta < \dfrac{1}{6}\left[24\alpha + mpb\left((pb)^2 + 6(pb) + 6\right)\exp(pb)\right]$, then because of the decreasing of f''' on $[0; 1]$ and by using the intermediate values theorem, $\exists \tilde{G}^{(0)} \in [0; 1]$ such that $f'''(\tilde{G}^{(0)}) = 0$. Then $f'''(G) > 0$ on $[0; \tilde{G}^{(0)}]$ and $f'''(G) < 0$ on $[\tilde{G}^{(0)}; 1]$.

$(II.2.1)$ If $f''(\tilde{G}^{(0)}) < 0$, then $f''(G) < 0$ on $[0; 1]$. Therefore f is decreasing on that interval.

$(II.2.1.1)$ If $\theta < 0$, then $f'(G) < 0$ on $[0; 1]$ and f is decreasing on $[0; 1]$.

$(II.2.1.1.1)$ If $q < 0$, then $f(G) < 0$ on $[0; 1]$.

$(II.2.1.1.2)$ If $q > 0$, then:

(a) If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

(b) If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{33} \in [0; 1]$ such that $f(G^*_{33}) = 0$.

$(II.2.1.2)$ If $\theta > 0$, then we have the following cases:

$(II.2.1.2.1)$ If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $f'(G) > 0$ on $[0; 1]$. Therefore f is increasing on $[0; 1]$.

(a) If $q > 0$, then $f(G) > 0$ on $[0; 1]$.

(b) If $q < 0$, then:

$(b.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0; 1]$.

$(b.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{34} \in [0; 1]$ such that $f(G^*_{34}) = 0$.

$(II.2.1.2.2)$ If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$, then $\exists \tilde{G}^{(1)} \in [0;1]$ such that $f'(\tilde{G}^{(1)}) = 0$.Therefore $f'(G) > 0$ on $[0;\tilde{G}^{(1)}]$ and $f'(G) > 0$ on $[\tilde{G}^{(1)};1]$.

(a) If $f(\tilde{G}^{(1)}) < 0$, then $f(G) < 0$ on $[0;1]$.

(b) If $f(\tilde{G}^{(1)}) > 0$, then:

$(b.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{35}^* \in [0;\tilde{G}^{(1)}]$ such that $f(G_{35}^*) = 0$.

$(b.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{36}^* \in [\tilde{G}^{(1)};1]$ such that $f(G_{36}^*) = 0$.

$(b.3)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{35}^* and G_{36}^* are the two roots on $[0;1]$ of f.

$(II.2.2)$ If $f''(\tilde{G}^{(0)}) > 0$, then:

$(II.2.2.1)$ If $q > m + \alpha g_0^2$ and $q > 6\alpha - 3\theta + \alpha g_0^2 + \dfrac{1}{2}m\exp(pb)\left[(pb)^2 + pb + 2\right]$, then $f''(G) > 0$ on $[0;1]$.Therefore f' is increasing on $[0;1]$.

$(II.2.2.1.1)$ If $\theta > 0$, then $f'(G) > 0$ on $[0;1]$. Therefore f is increasing on $[0;1]$.

(a) If $q > 0$, then $f(G) > 0$ on $[0;1]$.

(b) If $q < 0$, then:

$(b.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{37}^* \in [0;1]$ such that $f(G_{37}^*) = 0$.

$(II.2.2.1.2)$ If $\theta < 0$, then:

(a) If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$, then $f'(G) < 0$ on $[0;1]$. Therefore f is decreasing on $[0;1]$.

$(a.1)$ If $q < 0$, then $f(G) < 0$ on $[0;1]$.

$(a.2)$ If $q > 0$, then:

$(a.2.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(a.2.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{38}^* \in [0;1]$ such that $f(G_{38}^*) = 0$.

(b) If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\left[pb+2\right]\right]$, then $\exists \tilde{G}^{(2)} \in [0;1]$ such that $f'(\tilde{G}^{(2)}) = 0$. Therefore $f'(G) < 0$ on $[0;\tilde{G}^{(2)}]$ and $f'(G) > 0$ on $[\tilde{G}^{(2)};1]$.

$(b.1)$ If $f(\tilde{G}^{(2)}) > 0$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $f(\tilde{G}^{(2)}) < 0$, then we have the following cases:

$(b.2.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{39}^* \in [0;\tilde{G}^{(2)}]$ such that $f(G_{39}^*) = 0$.

$(b.2.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{40}^* \in [\tilde{G}^{(2)};1]$ such that $f(G_{40}^*) = 0$.

$(b.2.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{39}^* and G_{40}^* are the two roots on $[0;1]$ of f.

$(II.2.2.2)$ If $q < m + \alpha g_0^2$ and $q > 6\alpha - 3\theta + \alpha g_0^2 + \frac{1}{2} m\exp(pb)\left[(pb)^2 + pb + 2\right]$, then: $\exists \tilde{G}^{(3)} \in [0;\tilde{G}^{(0)}]$ such that $f''(\tilde{G}^{(3)}) = 0$. Therefore $f''(G) < 0$ on $[0;\tilde{G}^{(3)}]$ and $f''(G) > 0$ on $[\tilde{G}^{(3)};1]$.

$(II.2.2.2.1)$ If $f'(\tilde{G}^{(3)}) > 0$, then $f'(G) > 0$ on $[0;1]$. Therefore f is increasing on $[0;1]$.

(a) If $q > 0$, then $f(G) > 0$ on $[0;1]$.

(b) If $q < 0$, then:

$(b.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{41}^* \in [0;1]$ such that $f(G_{41}^*) = 0$.

$(II.2.2.2.2)$ If $f'(\tilde{G}^{(3)}) < 0$, then we have the following cases:

(a) If $\theta < 0$ and $q < \frac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $f'(G) < 0$ on $[0;1]$. Therefore f is decreasing on $[0;1]$.

$(a.1)$ If $q < 0$, then $f(G) < 0$ on $[0;1]$.

$(a.2)$ If $q > 0$, then:

$(a.2.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(a.2.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{42}^* \in [0;1]$ such that $f(G_{42}^*) = 0$.

(b) If $\theta > 0$ and $q < \frac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $\exists \tilde{G}^{(4)} \in [0;\tilde{G}^{3}]$ such that $f'(\tilde{G}^{(4)}) = 0$. Therefore $f'(G) > 0$ on $[0;\tilde{G}^{(4)}]$ and $f'(G) < 0$ on $[\tilde{G}^{(4)};1]$.

$(b.1)$ If $f(\tilde{G}^{(4)}) < 0$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $f(\tilde{G}^{(4)}) > 0$, then:

$(b.2.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{43} \in [0;\tilde{G}^{(4)}]$ such that $f(G^*_{43}) = 0$.

$(b.2.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{44} \in [\tilde{G}^{(4)};1]$ such that $f(G^*_{44}) = 0$.

$(b.2.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{44} and G^*_{43} are the two roots on $[0;1]$ of f.

(c) If $\theta < 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\exists \tilde{G}^{(5)} \in [\tilde{G}^3;1]$ such that $f'(\tilde{G}^{(5)}) = 0$. Therefore $f'(G) < 0$ on $[0;\tilde{G}^{(5)}]$ and $f'(G) > 0$ on $[\tilde{G}^{(5)};1]$.

$(c.1)$ If $f(\tilde{G}^{(5)}) > 0$, then $f(G) > 0$ on $[0;1]$.

$(c.2)$ If $f(\tilde{G}^{(5)}) < 0$, then:

$(c.2.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(c.2.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{45} \in [0;\tilde{G}^{(5)}]$ such that $f(G^*_{45}) = 0$.

$(c.2.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{46} \in [\tilde{G}^{(5)};1]$ such that $f(G^*_{46}) = 0$.

$(c.2.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{45} and G^*_{46} are the two roots of f on $[0;1]$.

(d) If $\theta > 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\tilde{G}^{(4)}$ and $\tilde{G}^{(5)}$ are the two roots of f'. Therefore $f'(G) > 0$ on $[0;\tilde{G}^{(4)}] \cup [\tilde{G}^{(5)};1]$ and $f'(G) < 0$ on $[\tilde{G}^{(4)};\tilde{G}^{(5)}]$.

$(d.1)$ If $f(\tilde{G}^{(4)}) < 0$, then we have the following cases:

$(d.1.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(d.1.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{46} is the unique roots of f.

$(d.2)$ If $f(\tilde{G}^{(4)}) > 0$ and $f(\tilde{G}^{(5)}) > 0$, then:

$(d.2.1)$ If $q > 0$, then $f(G) > 0$ on $[0;1]$.

$(d.2.2)$ If $q < 0$, then G^*_{43} is the unique roots of f.

$(d.3)$ If $f(\tilde{G}^{(4)}) > 0$ and $f(\tilde{G}^{(5)}) < 0$, then: $\exists G^*_{47} \in [\tilde{G}^{(4)};\tilde{G}^{(5)}]$ such that $f(G^*_{47}) = 0$.

$(d.3.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{47}^* and G_{46}^* are the two roots of f.

$(d.3.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{47}^* is the unique root of f.

$(d.3.3)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{47}^* and G_{43}^* are the two roots of f.

$(d.3.4)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{47}^* ,G_{46}^* and G_{43}^* are the three roots of f.

$(II.2.2.3)$ If $q > m+\alpha g_0^2$ and $q < 6\alpha-3\theta+\alpha g_0^2+\dfrac{1}{2}m\exp(pb)\left[(pb)^2+pb+2\right]$, then: $\exists \tilde{G}^{(6)} \in [\tilde{G}^{(0)};1]$ such that $f''(\tilde{G}^{(6)}) = 0$. Therefore, $f''(G) > 0$ on $[0;\tilde{G}^{(6)}]$ and $f''(G) < 0$ on $[\tilde{G}^{(6)};1]$.

$(II.2.2.3.1)$ If $f'(\tilde{G}^{(6)}) < 0$, then $f'(G) < 0$ on $[0;1]$. Therefore f is decreasing on $[0;1]$.

(a) If $q < 0$, then $f(G) < 0$ on $[0;1]$.

(b) If $q > 0$, then:

$(b.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{48}^* \in [0;1]$ such that $f(G_{48}^*) = 0$.

$(II.2.2.3.2)$ If $f'(\tilde{G}^{(6)}) > 0$, then

(a) If $\theta > 0$ and $q > \dfrac{1}{2}\left[4\alpha-3\theta+(2\alpha-\theta)g_0^2+m\exp(pb)\left[pb+2\right]\right]$, then $f'(G) > 0$ on $[0;1]$ and therefore f is increasing on $[0;1]$.

$(a.1)$ If $q > 0$, then $f(G) > 0$ on $[0;1]$.

$(a.2)$ If $q < 0$, then:

$(a.2.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(a.2.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{49}^* \in [0;1]$ such that $f(G_{49}^*) = 0$.

(b) If $\theta < 0$ and $q > \dfrac{1}{2}\left[4\alpha-3\theta+(2\alpha-\theta)g_0^2+m\exp(pb)\left[pb+2\right]\right]$, then $\exists \tilde{G}^{(7)} \in [0;\tilde{G}^{(6)}]$ such that $f'(\tilde{G}^{(7)}) = 0$. Therefore $f'(G) < 0$ on $[0;\tilde{G}^{(7)}]$ and $f'(G) > 0$ on $[\tilde{G}^{(7)};1]$

$(b.1)$ If $f(\tilde{G}^{(7)}) > 0$, then $f(G) > 0$ on $[0;1]$.

$(b.2)$ If $f(\tilde{G}^{(7)}) < 0$, then:

$(b.2.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.2.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{50} \in [0; \tilde{G}^{(7)}]$ such that $f(G^*_{50}) = 0$.

$(b.2.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{51} \in [\tilde{G}^{(7)}; 1]$ such that $f(G^*_{51}) = 0$.

$(b.2.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{50} and G^*_{51} are the two roots on $[0; 1]$ of f.

(c) If $\theta > 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\,[pb + 2]\right]$, then $\exists \tilde{G}^{(8)} \in [\tilde{G}^{(6)}; 1]$ such that $f'(\tilde{G}^{(8)}) = 0$. Therefore $f'(G) > 0$ on $[0; \tilde{G}^{(8)}]$ and $f'(G) < 0$ on $[\tilde{G}^{(8)}; 1]$

$(c.1)$ If $f(\tilde{G}^{(8)}) < 0$, then $f(G) < 0$ on $[0; 1]$.

$(c.2)$ If $f(\tilde{G}^{(8)}) > 0$, then:

$(c.2.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

$(c.2.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{52} \in [0; \tilde{G}^{(8)}]$ such that $f(G^*_{52}) = 0$.

$(c.2.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{53} \in [\tilde{G}^{(8)}; 1]$ such that $f(G^*_{53}) = 0$.

$(c.2.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{52} and G^*_{53} are the two roots on $[0; 1]$ of f.

(d) If $\theta < 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\,[pb + 2]\right]$, then $\tilde{G}^{(8)}$ and $\tilde{G}^{(7)}$ are the two positive roots of f'. Therefore $f'(G) > 0$ on $[\tilde{G}^{(7)}; \tilde{G}^{(8)}]$ and $f'(G) < 0$ on $[0; \tilde{G}^{(7)}] \cup [\tilde{G}^{(8)}; 1]$.

$(d.1)$ If $f(\tilde{G}^{(7)}) > 0$, then:

$(d.1.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

$(d.1.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{53} is the unique root of f.

$(d.2)$ If $f(\tilde{G}^{(7)}) < 0$ and $f(\tilde{G}^{(8)}) < 0$, then :

$(d.2.1)$ If $q < 0$, then $f(G) < 0$ on $[0; 1]$.

$(d.2.2)$ If $q > 0$, then G^*_{50} is the unique root of f.

$(d.3)$ If $f(\tilde{G}^{(7)}) < 0$ and $f(\tilde{G}^{(8)}) > 0$, then :$\exists G^*_{54} \in [\tilde{G}^{(7)}; \tilde{G}^{(8)}]$ such that $f(G^*_{54}) = 0$.

$(d.3.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G^*_{54} we have also G^*_{53} as a root of f.

$(d.3.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{54} is the unique root of f. .

$(d.3.3)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G^*_{54} we have also G^*_{50} as a root of f.

$(d.3.4)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{50}, G^*_{54} and G^*_{53} are the three roots on $[0; 1]$ of f.

$(II.2.2.4)$ If $q < m + \alpha g_0^2$ and $q < 6\alpha - 3\theta + \alpha g_0^2 + \dfrac{1}{2} m\exp(pb)\left[(pb)^2 + pb + 2\right]$, then:$\tilde{G}^{(3)}$ and $\tilde{G}^{(6)}$ are the two roots of f''. Therefore, $f''(G) > 0$ on $[\tilde{G}^{(3)}; \tilde{G}^{(6)}]$ and $f''(G) < 0$ on $[0; \tilde{G}^{(3)}] \cup [\tilde{G}^{(6)}; 1]$.

$(II.2.2.4.1)$ If $f'(\tilde{G}^{(3)}) > 0$, then:

(a) If $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $f'(G) > 0$ on $[0; 1]$, therefore f is increasing on that interval.

$(a.1)$ If $q > 0$, then $f(G) > 0$ on $[0; 1]$.

$(a.2)$ If $q < 0$, then:

$(a.2.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0; 1]$.

$(a.2.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{41} \in [\tilde{G}^{(6)}; 1]$ such that $f(G^*_{41}) = 0$.

(b) If $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\left[pb + 2\right]\right]$, then $\tilde{G}^{(8)}$ is the unique root of f'. Therefore

$(b.1)$ If $f(\tilde{G}^{(8)}) < 0$, then $f(G) < 0$ on $[0; 1]$.

$(b.2)$ If $f(\tilde{G}^{(8)}) > 0$, then:

$(b.2.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

$(b.2.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{52} \in [0; \tilde{G}^{(8)}]$ such that $f(G^*_{52}) = 0$.

$(b.2.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G^*_{53} \in [\tilde{G}^{(8)}; 1]$ such that $f(G^*_{53}) = 0$.

$(b.2.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G^*_{52} and G^*_{53} are the two roots on $[0; 1]$ of f.

$(II.2.2.4.2)$ If $f'(\tilde{G}^{(3)}) < 0$ and $f'(\tilde{G}^{(6)}) < 0$, then we have the following cases:

(a) If $\theta < 0$, then $f'(G) < 0$ on $[0; 1]$. Therefore f is decreasing on $[0; 1]$.

$(a.1)$ If $q < 0$, then $f(G) < 0$ on $[0; 1]$.

$(a.2)$ If $q > 0$, then :

$(a.2.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(a.2.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{42}^* is the unique root of f.

(b) If $\theta > 0$, then $\tilde{G}^{(4)}$ is the unique root of f'. Therefore:

$(b.1)$ If $f(\tilde{G}^{(4)}) < 0$, then $f(G) < 0$ on $[0;1]$.

$(b.2)$ If $f(\tilde{G}^{(4)}) > 0$, then:

$(b.2.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0;1]$.

$(b.2.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{43}^* \in [0;\tilde{G}^{(4)}]$ such that $f(G_{43}^*) = 0$.

$(b.2.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{44}^* \in [\tilde{G}^{(4)};1]$ such that $f(G_{44}^*) = 0$.

$(b.2.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{44}^* and G_{43}^* are the two roots on $[0;1]$ of f.

$(II.2.2.4.3)$ If $f'(\tilde{G}^{(3)}) < 0$ and $f'(\tilde{G}^{(6)}) > 0$, then $\exists \tilde{G}^9 \in [\tilde{G}^3;\tilde{G}^6]$ such that $f'(\tilde{G}^{(9)}) = 0$.

(a) If $\theta < 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then $\tilde{G}^9$ is the unique roots of f'.

$(a.1)$ If $f(\tilde{G}^9) > 0$, then $f(G) > 0$ on $[0;1]$.

$(a.2)$ If $f(\tilde{G}^9) < 0$, then:

$(a.2.1)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(a.2.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{55}^* \in [0;\tilde{G}^{(9)}]$ such that $f(G_{55}^*) = 0$.

$(a.2.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $\exists G_{56}^* \in [\tilde{G}^{(9)};1]$ such that $f(G_{56}^*) = 0$.

$(a.2.4)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{55}^* and G_{56}^* are the roots of f.

(b) If $\theta > 0$ and $q > \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then with $\tilde{G}^9$ we have also $\tilde{G}^4$ root of f'. Therefore $f'(G) < 0$ on $[\tilde{G}^4;\tilde{G}^9]$ and $f'(G) > 0$ on $[0;\tilde{G}^4] \cup [\tilde{G}^9;1]$.

$(b.1)$ If $f(\tilde{G}^4) < 0$, then:

$(b.1.1)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) < 0$ on $[0;1]$.

$(b.1.2)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $G_{56}^* \in [\tilde{G}^{(9)}; 1]$ is the unique root of f.

$(b.2)$ If $f(\tilde{G}^4) > 0$ and $f(\tilde{G}^9) > 0$, then:

$(b.2.1)$ If $q > 0$, then $f(G) > 0$ on $[0; 1]$.

$(b.2.1)$ If $q < 0$, then G_{43}^* is the unique root of f.

$(b.3)$ If $f(\tilde{G}^4) > 0$ and $f(\tilde{G}^9) < 0$, then $\exists G_{57}^* \in [\tilde{G}^4; \tilde{G}^9]$ such that $f(G_{57}^*) = 0$.

$(b.3.1)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,then G_{57}^* is the unique root of f.

$(b.3.2)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G_{57}^* we have also G_{43}^* roots of f.

$(b.3.3)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G_{57}^* we have also G_{56}^* roots of f.

$(b.3.4)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G_{57}^* we have also G_{56}^* and G_{43}^* roots of f.

(c) If $\theta < 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha - \theta)g_0^2 + m\exp(pb)\,[pb + 2]\right]$, then with $\tilde{G}^9$ we have also $\tilde{G}^8$ roots of f'. Therefore $f'(G) > 0$ on $[\tilde{G}^9; \tilde{G}^8]$ and $f'(G) < 0$ on $[0; \tilde{G}^9] \cup [\tilde{G}^8; 1]$.

$(c.1)$ If $f(\tilde{G}^9) > 0$, then:

$(c.1.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

$(c.1.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{53}^* is the unique root of f.

$(c.2)$ If $f(\tilde{G}^9) < 0$ and $f(\tilde{G}^8) < 0$ then

$(c.2.1)$ If $q < 0$, then $f(G) < 0$ on $[0; 1]$.

$(c.2.2)$ If $q > 0$, then G_{55}^* is the unique root of f.

$(c.3)$ If $f(\tilde{G}^9) < 0$ and $f(\tilde{G}^8) > 0$, then $\exists G_{58}^* \in [\tilde{G}^9; \tilde{G}^8]$ such that $f(G_{58}^*) = 0$.

$(c.3.1)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{58}^* is the unique root of f.

$(c.3.2)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,then with G_{58}^* we have also G_{55}^* root of f.

$(c.3.3)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G_{58}^* we have also G_{53}^* root of f.

$(c.3.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then with G_{58}^* we have also G_{53}^* and G_{55}^* root of f.

(d) If $\theta > 0$ and $q < \dfrac{1}{2}\left[4\alpha - 3\theta + (2\alpha-\theta)g_0^2 + m\exp(pb)\,[pb+2]\right]$, then with $\tilde{G}^9$ we have also $\tilde{G}^4$ and $\tilde{G}^8$ roots of f'. Therefore $f'(G) > 0$ on $[0; \tilde{G}^4] \cup [\tilde{G}^9; \tilde{G}^8]$ and $f'(G) < 0$ on $[\tilde{G}^4; \tilde{G}^9] \cup [\tilde{G}^8; 1]$.

$(d.1)$ If $f(\tilde{G}^4) < 0$ and $f(\tilde{G}^8) < 0$, then $f(G) < 0$ on $[0; 1]$.

$(d.2)$ If $f(\tilde{G}^4) > 0$ and $f(\tilde{G}^8) < 0$, then $f(\tilde{G}^9) < 0$ and therefore $\exists G_{59}^* \in [\tilde{G}^4; \tilde{G}^9]$ such that $f(G_{59}^*) = 0$. Then:

$(d.2.1)$ If $q > 0$, then G_{59}^* is the unique root of f.

$(d.2.2)$ If $q < 0$, then with G_{59}^*, we have also G_{43}^* roots of f.

$(d.3)$ If $f(\tilde{G}^4) < 0$ and $f(\tilde{G}^8) > 0$, then $f(\tilde{G}^9) < 0$ and therefore $\exists G_{60}^* \in [\tilde{G}^9; \tilde{G}^8]$ such that $f(G_{60}^*) = 0$. Then:

$(d.3.1)$ If $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then G_{60}^* is the unique root of f.

$(d.3.2)$ If $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, with G_{60}^* we have also G_{53}^* roots of f.

$(d.4)$ If $f(\tilde{G}^4) > 0$ and $f(\tilde{G}^8) > 0$, then:

$(d.4.1)$ If $f(\tilde{G}^9) > 0$, then:

$(d.4.1.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then $f(G) > 0$ on $[0; 1]$.

$(d.4.1.2)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then f has an unique root G_{53}^*.

$(d.4.1.3)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, then f has an unique root G_{43}^*.

$(d.4.1.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, G_{43}^* and G_{53}^* are the two roots of f.

$(d.4.2)$ If $f(\tilde{G}^9) < 0$, then with G_{59}^* and G_{58}^*, we have also:

$(d.4.2.1)$ If $q > 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, G_{59}^* and G_{58}^* roots of f.

$(d.4.2.2)$ If $q < 0$ and $q > \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$,$G_{43}^*$. Therefore G_{59}^*,G_{58}^*,G_{43}^* are the three roots of f.

$(d.4.2.3)$ If $q > 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, G_{53}^*. Therefore G_{59}^*,G_{58}^*,G_{53}^* are the three roots of f.

$(d.4.2.4)$ If $q < 0$ and $q < \dfrac{m\exp(pb)}{g_0^2+1} + \alpha - \theta$, G_{43}^* and G_{53}^*. Therefore G_{59}^*,G_{58}^*, G_{53}^* and G_{43}^* are the fourth roots of f

□

Remark **3.2.** Let us set

$$\mathcal{R}_G^0=\frac{2\gamma_G}{2\lambda_{fG}f+2\delta_G+\gamma_{TG}},\quad \mathcal{R}_{TG}=\frac{2\gamma_{TG}}{2\lambda_{fG}f+2\delta_G+\gamma_{TG}}\quad \text{and}\quad \mathcal{R}^*=\frac{\gamma_T\left(1+\Omega\frac{\mathcal{R}_G^0}{\mathcal{R}_{TG}}\right)}{p\lambda_{fT}f\exp\left(-p(a+b)\right)}.$$

- ∗ If $\mathcal{R}^*<1$, then $\theta<mpb$ and if $\mathcal{R}^*>1$, then $\theta>mpb$.
- ∗ $\mathcal{R}^*$ is the primary production of tree including a portion of tree production due to tree-tree cooperation and asymmetric tree-grass competition relative to fire induced tree and grass loss.

Now, we want to characterize local stability property of previous steady states. System (3.50) is a planar, competitive and dissipative system. Hence, based on Smith [94] (Theorem 2.2, page 35), we deduce that solutions of system (3.50) will always converge toward an equilibrium point. That is, no stable limit cycles may exist for system (3.50).

Proposition **3.5. (Stability properties of trivial and semi trivial steady states).** The following results are valid for system (3.50).

(a) The desert steady state $E_0=(0,0)'$ is always unstable.

(b) If $\mathcal{R}_F<1$ then the forest steady state E_{T_i} is locally asymptotically stable (LAS).

(c) If $\mathcal{R}_T<1$ then the grassland steady state $E_{G_e}=(G_e;0)'$ is LAS.

Proof. The Jacobian matrix associated to the system (3.51) is given by:

$$M(G,T)=\begin{pmatrix}\delta_G-\gamma_G-2\gamma_G G-\gamma_{TG}T-\lambda_{fG}f & -\gamma_{TG}G\\ \\ -\lambda_{fT}f\omega'(G)\exp(-pT)T & \begin{array}{c}\gamma_T(1+\Omega T)(1-T)+\gamma_T\Omega T(1-T)-\gamma_T T(1+\Omega T)\\ -\delta_T+p\lambda_{fT}f\omega(G)\exp(-pT)T-\lambda_{fT}f\omega(G)\exp(-pT)\end{array}\end{pmatrix} \tag{3.82}$$

Then:

- For the desert steady state $E_0=(0;0)'$ we have:

$$\begin{aligned}M(0,0)&=\begin{pmatrix}\gamma_G-\delta_G-\lambda_{fG}f & 0\\ 0 & \gamma_T-\delta_T\end{pmatrix},\\ &=\begin{pmatrix}(\delta_G+\lambda_{fG}f)\left[\mathcal{R}_G-1\right] & 0\\ 0 & \gamma_T\left[1-\mathcal{R}_{T,0}\right]\end{pmatrix}.\end{aligned} \tag{3.83}$$

Then the eigenvalues of $M(0,0)$ are $(\delta_G+\lambda_{fG}f)\left[\mathcal{R}_G-1\right]$ and $\gamma_T\left[1-\mathcal{R}_{T,0}\right]$. But for $\mathcal{R}_{T,0}>1$, and $\mathcal{R}_G>1$, E_0 is unstable.

- For the forest steady state $E_{T_i} = (0; T_i)'$ we have:

$$\begin{aligned} M(0,T_i) &= \begin{pmatrix} \gamma_G - \delta_G - \lambda_{fG} f - \gamma_{TG} T_i & 0 \\ 0 & \gamma_T T\left[(\Omega - 1) - 2\Omega T_i\right] \end{pmatrix}, \\ &= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \end{aligned} \tag{3.84}$$

If $\Omega = 0$, then $B = -\left(\gamma_T - \delta_T\right) < 0$.
If $\Omega > 0$ then $B = -\gamma_T T_2 \sqrt{(1-\Omega)^2 + 4\Omega\left(1 - \dfrac{\delta_T}{\gamma_T}\right)} < 0$.
Then for all case $B < 0$.

$$\begin{aligned} A &= \left(\delta_G + \lambda_{fG} f + \gamma_{TG} T_i\right)\left(\frac{\gamma_G}{\delta_G + \lambda_{fG} f + \gamma_{TG} T_i} - 1\right), \\ A &= \left(\delta_G + \lambda_{fG} f + \gamma_{TG} T_i\right)\left(\mathcal{R}_F - 1\right). \\ A < 0 &\Leftrightarrow \mathcal{R}_F < 1. \end{aligned}$$

Consequently, if $\mathcal{R}_F < 1$, then $A < 0$ and therefore E_{T_i} is stable.

- For the grassland steady state $E_{G_e} = (G_e; 0)'$ we have:

$$\begin{aligned} M(G_e;0) &= \begin{pmatrix} \gamma_G - \delta_G - \lambda_{fG} f - 2\gamma_G G_e & -\gamma_{TG} G_e \\ 0 & \gamma_T - \left[\delta_T + \lambda_{fT} f \omega(G_e)\right] \end{pmatrix}, \\ &= \begin{pmatrix} C & -\gamma_{TG} G_e \\ 0 & D \end{pmatrix}. \end{aligned} \tag{3.85}$$

We have:

$$\begin{aligned} C &= \gamma_G - \delta_G - \lambda_{fG} f - 2\gamma_G G_e, \\ &= \left(\delta_G + \lambda_{fG} f\right)\left(1 - \frac{\gamma_G}{\delta_G + \lambda_{fG} f}\right), \\ &= \left(\delta_G + \lambda_{fG} f\right)\left(1 - \mathcal{R}_G\right). \end{aligned}$$

Therefore $C < 0$, due to the fact that $\mathcal{R}_G > 1$.

$$\begin{aligned} D &= \gamma_T - \left[\delta_T + \lambda_{fT} f \omega(G_e)\right], \\ &= \left[\delta_T + \lambda_{fT} f \omega(G_e)\right]\left(\frac{\gamma_T}{\delta_T + \lambda_{fT} f \omega(G_e)} - 1\right), \\ &= \left[\delta_T + \lambda_{fT} f \omega(G_e)\right]\left(\mathcal{R}_T - 1\right). \end{aligned}$$

Then, if $\mathcal{R}_T < 1$, then $D < 0$. Therefore if $\mathcal{R}_T < 1$, E_{G_e} is LAS.

□

Now we deal with conditions of stability of a savanna steady state when its exists. Set:

$$\begin{aligned}\mathcal{R}_1^* &= \frac{\gamma_T\left[(1-\Omega)+2\Omega T^*\right]}{p\lambda_{fT}f\omega(G^*)\exp(-pT^*)},\\ \mathcal{R}_2^* &= \frac{\gamma_{TG}\omega'(G^*)}{p\gamma_G\omega(G^*)}.\end{aligned} \tag{3.86}$$

Proposition 3.6. (Stability condition of a savanna steady state). The stability conditions of a coexistence steady state, of system (3.50), when it exists are given by the following cases:

- **case 1**:Assume $f=0$, then $(G^*,T^*)'$ is LAS.
- **case 2**: Assume $f>0$. If:

$$\mathcal{R}_1^*-\mathcal{R}_2^*>1, \tag{3.87}$$

then $(G^*,T^*)'$ is LAS.

Proof. Set:

$$\begin{aligned}a_{11} &= -\gamma_G G^*,\\ a_{12} &= -\gamma_{TG}G^*,\\ a_{21} &= -\lambda_{fT}f\omega'(G^*)\exp(-pT^*)T^*,\\ a_{22} &= -\gamma_T\left[(1-\Omega)T^*+2\Omega(T^*)^2\right]+p\lambda_{fT}f\omega(G^*)\exp(-pT^*)T^*.\end{aligned}$$

For the savanna steady state, we have the Jacobian Matrix:

$$M(G^*;T^*) = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}. \tag{3.88}$$

If $f=0$, then:

$$\begin{aligned}a_{11} &= -\gamma_G G^*,\\ a_{12} &= -\gamma_{TG}G^*,\\ a_{21} &= 0,\\ a_{22} &= -\gamma_T\left[(1-\Omega)T^*+2\Omega(T^*)^2\right]=-\gamma_T T^*\left[(1-\Omega)+2\Omega T^*\right].\end{aligned}$$

Therefore:

(a) If $\Omega=0$, then $a_{22}<0$. Consequently $a_{11}<0$ and $a_{22}<0$. So because $a_{21}=0$, $(G^*;T^*)$ is LAS.

(b) If $\Omega>0$, then

$$a_{22} = -\gamma_T T^*\sqrt{(1-\Omega)^2+4\Omega\left(1-\frac{\delta_T}{\gamma_T}\right)}<0,$$

then $a_{11}<0$ and $a_{22}<0$. So, $(G^*;T^*)$ is LAS.

If $f \neq 0$,

$$\begin{aligned} Tr(M(G^*;T^*)) &= a_{11} + a_{22}, \\ Det((G^*;T^*)) &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{3.89}$$

$Det(M) > 0 \Leftrightarrow a_{11}a_{22} - a_{12}a_{21} > 0$

$$\begin{aligned} a_{11}a_{22} - a_{21}a_{12} > 0 \quad &\Leftrightarrow \quad \gamma_G\gamma_T G^*T^*\left[(1-\Omega) + 2\Omega T^*\right] - p\gamma_G\lambda_{fT}f\omega(G^*)\exp(-pT^*)G^*T^* \\ &\qquad -\gamma_{TG}\lambda_{fT}f\omega'(G^*)\exp(-pT^*)G^*T^* > 0, \\ &\Leftrightarrow \quad \frac{\gamma_T\left[(1-\Omega)+2\Omega T^*\right]}{p\lambda_{fT}f\omega(G^*)\exp(-pT^*)} - \frac{\gamma_{TG}\omega'(G^*)}{p\gamma_G\omega(G^*)} > 1. \end{aligned}$$

Second $Tr(M(G^*,T^*)) < 0 \Leftrightarrow \dfrac{\gamma_G G^*}{p\lambda_{fT}f\omega(G^*\exp(-pT^*)T^*} + \dfrac{\gamma_T\left[(1-\Omega)+2\Omega T^*\right]}{p\lambda_{fT}f\omega(G^*\exp(-pT^*)} > 1.$

But,

$$\frac{\gamma_T\left[(1-\Omega)+2\Omega T^*\right]}{p\lambda_{fT}f\omega(G^*)\exp(-pT^*)} - \frac{\gamma_{TG}\omega'(G^*)}{p\gamma_G\omega(G^*)} > 1 \Rightarrow \frac{\gamma_G G^*}{p\lambda_{fT}f\omega(G^*\exp(-pT^*)T^*} + \frac{\gamma_T\left[(1-\Omega)+2\Omega T^*\right]}{p\lambda_{fT}f\omega(G^*\exp(-pT^*)} > 1.$$

Consequently, if $\dfrac{\gamma_T\left[(1-\Omega)+2\Omega T^*\right]}{p\lambda_{fT}f\omega(G^*)\exp(-pT^*)} - \dfrac{\gamma_{TG}\omega'(G^*)}{p\gamma_G\omega(G^*)} > 1$, then $(G^*;T^*)$ is stable. □

Remark 3.3. (thresholds ecological interpretation)

(i) $\mathcal{R}_{T,0} = \dfrac{\gamma_T}{\delta_T}$ denotes the primary production of tree biomass relative to tree biomass loss due to human activities and herbivory.

(ii) $\mathcal{R}_{G,0} = \dfrac{\gamma_G}{\delta_G}$ is the primary production of grass biomass relative to grass biomass loss due to human activities and herbivory.

(iii) $\mathcal{R}_G = \dfrac{\gamma_G}{\delta_G + f\lambda_{fG}}$ denotes the primary production of grass biomass relative to grass biomass loss due to grazing or human action and additional fire induced biomass loss.

(iv) $\mathcal{R}_T = \dfrac{\gamma_T}{\delta_G + \lambda_{fT}f\omega(G_e)}$ is the primary production of tree biomass relative to fire-induced biomass loss at the grassland equilibrium and the additional loss due to herbivory (grazing) or human action.

(v) $\mathcal{R}_{F,f} = \dfrac{\gamma_G}{\delta_G + f\lambda_{fG} + \gamma_{TG}T_i}$ represents the primary production of grass biomass, relative to grass biomass loss induced by fire, herbivory (grazing) or human action and additional grass suppression due to tree competition, at the closed forest equilibrium.

3.3.2.2 The nonlocal case (M_1 or $M_2 > 0$)

Our aim now is to derive a condition on spatial convolution such that a steady state $(G_s, T_s)' \in \{(G_e, 0)'; (0, T_2)'; (G^*, T^*)'\}$ is locally asymptotically stable in the case $M_1 = M_2 = 0$, but unstable for some $M_i > 0,\ \ i = 1, 2$.

In fact, the spatial patterns appearing in the nonlocal savanna model (3.5) can be studied by performing a linear stability analysis of the stationary homogeneous solution of (3.5) given by the system (3.51). Linearizing the integro-differential system (3.5) around $(G_s;T_s)'$, leads to the following results:

Proposition 3.7. (linearized system)
Set: $g(x,t) = G(x,t) - G_s \ \ and \ \ h(x,t) = T(x,t) - T_s$ two perturbations around a non trivial homogeneous steady state. The system obtained after linearization is:

$$\begin{cases} \dfrac{\partial g}{\partial t} &= D_G\dfrac{\partial^2 g}{\partial x^2} + \left[\gamma_G(1-G_s) - \delta_G - \gamma_{TG}T_s - \lambda_{fG}f\right] g - \gamma_G G_s\displaystyle\int_{-\infty}^{+\infty} \phi_{M_1}(x-y)g(y,t)dy \\ & \quad -\gamma_{TG}G_s\displaystyle\int_{-\infty}^{+\infty} \phi_{M_2}(x-y)h(y,t)dy, \\ \dfrac{\partial h}{\partial t} &= D_T\dfrac{\partial^2 h}{\partial x^2} + \left[(\gamma_T(1+\Omega T_s)(1-T_s) - \delta_T - \lambda_{fT}f\omega(G_s)\exp(-pT_s)) + \gamma_T\Omega T_s(1-T_s)\right] h \\ & \quad + \left(p\lambda_{fT}f\omega(G_s)\exp(-pT_s)T_s - \gamma_T T_s(1+\Omega T_s)\right)\displaystyle\int_{-\infty}^{+\infty} \phi_{M_2}(x-y)h(y,t)dy \\ & \quad -\lambda_{fT}f\omega'(G_s)\exp(-pT_s)T_s g. \end{cases} \tag{3.90}$$

Proof. $h = T - T_s$ and $g = G - G_s$, then $\dfrac{\partial h}{\partial t} = \dfrac{\partial T}{\partial t}$ and $\dfrac{\partial^2 h}{\partial x^2} = \dfrac{\partial T^2}{\partial x^2}$
In the same way $\dfrac{\partial g}{\partial t} = \dfrac{\partial G}{\partial t}$ and $\dfrac{\partial^2 g}{\partial x^2} = \dfrac{\partial G^2}{\partial x^2}$, Then:

$$\begin{aligned} \frac{\partial h}{\partial t} &= D_T\frac{\partial^2 h}{\partial x^2} + \gamma_T\left(h+T_s\right)\left(1+\Omega\left(h+T_s\right)\right)\left(1 - \int_{-\infty}^{+\infty}\phi_{M_2}(x-y)(h(t,y)+T_s)dy\right) \\ & \quad -\delta_T(h+T_s) - \lambda_{fT}f\omega(g+G_s)\exp\left(-p\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)(h(t,y)+T_s)dy\right)(h+T_s). \end{aligned} \tag{3.91}$$

Developing the right-hand side of equation (3.91) and neglecting the nonlinear expressions in h we get:

$$\begin{aligned} \frac{\partial h}{\partial t} &= D_T\frac{\partial^2 h}{\partial x^2} + \left(\gamma_T(1+\Omega T_s) + \gamma_T\Omega T_s\right)(1-T_s)h - \delta_T h - \lambda_{fT}f\omega(G_s)\exp(-pT_s)h \\ & \quad -\lambda_{fT}f\omega'(G_s)\exp(-pT_s)T_s g - \gamma_T T_s(1+\Omega T_s)\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)h(y,t)dy \\ & \quad +\lambda_{fT}fp\omega(G_s)\exp(-pT_s)T_s\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)h(y,t)dy, \\ &= D_T\frac{\partial^2 h}{\partial x^2} + \left[(\gamma_T(1+\Omega T_s)(1-T_s) - \delta_T - \lambda_{fT}f\omega(G_s)\exp(-pT_s)) + \gamma_T\Omega T_s(1-T_s)\right] h \\ & \quad + \left(p\lambda_{fT}f\omega(G_s)\exp(-pT_s)T_s - \gamma_T T_s(1+\Omega T_s)\right)\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)h(y,t)dy \\ & \quad -\lambda_{fT}f\omega'(G_s)\exp(-pT_s)T_s g. \end{aligned}$$

In the same way we have :

$$\begin{aligned}\frac{\partial g}{\partial t} &= D_G\frac{\partial^2 g}{\partial x^2}+\gamma_G(g+G^*)\left(1-\int_{-\infty}^{+\infty}\phi_{M_1}(x-y)(g(y,t)+G^*)dy\right)-\delta_G(g+G^*)-\lambda_{fG}f(g+G^*)\\ &\quad -\gamma_{TG}\left(\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)(h(y,t)+T^*)dy\right)(g+G^*).\end{aligned} \tag{3.92}$$

Developing the right-hand side of equation (3.92) and neglecting the nonlinear expressions in g we get:

$$\begin{aligned}\frac{\partial g}{\partial t} &= D_G\frac{\partial^2 g}{\partial x^2}+\left[\gamma_G(1-G_s)-\delta_G-\gamma_{TG}T_s-\lambda_{fG}f\right]g-\gamma_G G_s\int_{-\infty}^{+\infty}\phi_{M_1}(x-y)g(y,t)dy\\ &\quad -\gamma_{TG}G_s\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)h(y,t)dy.\end{aligned}$$

□

Now we are in position to study linear stability around all non trivial homogeneous steady state.

3.3.2.2.1 Linear stability analysis around the grassland homogeneous steady state $E_G=(G_e,0)'$

Set :

$$\begin{aligned} b_{11} &= \gamma_G G_e,\\ b_{12} &= \gamma_{TG}G_e,\\ b_{22} &= \gamma_T-\delta_T-\lambda_{fT}f\omega(G_e).\end{aligned} \tag{3.93}$$

The following results hold:

Proposition 3.8. (Linearized system around the grassland homogeneous steady state)
Let $g(x,t)=G(x,t)-G_e$ *and* $h(x,t)=T(x,t)$ be two perturbations around the grassland homogeneous steady state. The system obtained after linearization is:

$$\begin{cases}\dfrac{\partial g}{\partial t} = D_G\dfrac{\partial^2 g}{\partial x^2}-b_{11}\displaystyle\int_{-\infty}^{+\infty}\phi_{M_1}(x-y)g(y,t)dy-b_{12}\int_{-\infty}^{+\infty}\phi_{M_2}(x-y)h(y,t)dy,\\ \dfrac{\partial h}{\partial t} = D_T\dfrac{\partial^2 h}{\partial x^2}+b_{22}h.\end{cases} \tag{3.94}$$

By the Fourier analysis, we get the eigenvalue problem associated with system (3.94) by setting :

$$g(x,t)=n\exp(\lambda t)\exp(ikx)\ \text{ and }\ h(x,t)=q\exp(\lambda t)\exp(ikx), \tag{3.95}$$

where λ is the eigenvalue associated with the wavenumber $k\geq 0$, n and q are positive constants. Therefore, using (3.95) in (3.94), we get the following system:

$$\begin{cases}\lambda n = -D_G k^2 n-b_{11}\overline{\phi_{M_1}}(k)n-b_{12}\overline{\phi_{M_2}}(k)q,\\ \lambda q = -D_T k^2 q+b_{22}q,\end{cases} \tag{3.96}$$

where $\overline{\phi_{M_i}}(k) = \dfrac{\sin kM_i}{kM_i}$, is the Fourier transforms of the functions ϕ_{M_i}.
Therefore, the system in (3.96) can be written in the matrix form:

$$\lambda \begin{pmatrix} n \\ \\ q \end{pmatrix} = \begin{bmatrix} -D_G k^2 - b_{11}\overline{\phi_{M_1}}(k) & -b_{12}\overline{\phi_{M_2}}(k) \\ \\ 0 & -D_T k^2 + b_{22} \end{bmatrix} \begin{pmatrix} n \\ \\ q \end{pmatrix}. \tag{3.97}$$

Let us consider:

$$M = \begin{bmatrix} -D_G k^2 - b_{11}\overline{\phi_{M_1}}(k) & -b_{12}\overline{\phi_{M_2}}(k) \\ \\ 0 & -D_T k^2 + b_{22} \end{bmatrix}, \tag{3.98}$$

$$Tr(M) = -\left(D_G + D_T\right)k^2 - b_{11}\overline{\phi_{M_1}}(k) + b_{22}, \tag{3.99}$$

and

$$Det(M) = D_G D_T k^4 + \left[b_{11} D_T \overline{\phi_{M_1}}(k) - b_{22} D_G\right] k^2 - b_{11} b_{22} \overline{\phi_{M_1}}(k). \tag{3.100}$$

Therefore, the grassland homogeneous steady state is locally asymptotic stable if:

$$Tr(M) < 0, \tag{3.101}$$

and

$$Det(M) > 0. \tag{3.102}$$

If (3.102) is not satisfied then we have an inhomogeneous solution call pattern (deriving from a Turing bifurcation).
We are now in position to find Turing bifurcation threshold around the grassland homogeneous steady state. Because of the form of $\overline{\phi_{M_1}}(k)$, we set $z = kM_1$ and denote, for simplicity, $\overline{\phi_{M_1}}(k)$ by $\overline{\phi_1}(z)$.

Theorem 3.3. (Stability of the Grassland homogeneous steady state)
If $\mathcal{R}_T < 1$ and $\overline{\phi_1}(z) \geq 0$ for all z, then the grassland homogeneous steady state is locally asymptotically stable for system (3.5).

Proof. Assume that $\mathcal{R}_T < 1$. Then, $b_{22} < 0$ thanks to the stability conditions of the grassland steady state in the space-implicit model (see for instance proposition 3.5, page 99). Therefore, if $\overline{\phi_1}(z) \geq 0$, then $Tr(M) < 0$ and $Det(M) > 0$. □

Remark 3.4. The previous theorem ensures that for this model, the choice of Gaussian kernels can not lead to pattern formation around the grassland homogeneous steady state. More generally, due to the type of nonlinearities involved in our model, the class of kernel-functions called “positive-definite functions" and characterized by a positive Fourier transform (see also Bochner [12] and Tzanakis [108]) are such that the empirically evidenced vegetation patterns are not reachable with the model. Then, $\overline{\phi_1}(z) < 0$ is a necessary condition for spatial Turing instability around the grassland homogeneous steady state and this could happen if ϕ_1 has discontinuities.

Theorem 3.4. (Grassland homogeneous steady state instability)
Assume that $\mathcal{R}_T < 1$ and we have a range of positive values of z such that:

$$\overline{\phi_1}(z) < 0 \tag{3.103}$$

holds ; If there exists a critical point M_1^T such that :

$$M_1 > M_1^T \Rightarrow \frac{1}{(M_1)^2} \leq S_1(z_1), \tag{3.104}$$

where

$$S_1(z) = \frac{-\overline{\phi}_1(z)}{z^2}\left(\frac{b_{11}}{D_G}\right), \tag{3.105}$$

and z_1 is the value of z such that $S_1(z)$ takes it global maximum, then the homogeneous grassland steady state is unstable. Furthermore, system (3.5) undergoes Turing bifurcation at $M_1 = M_1^T$.

Proof. Assume that $\mathcal{R}_T < 1$ (then $b_{22} < 0$) and we have a range of positive values of z such that: $\overline{\phi_1}(z) < 0$. Writing $Det(M)$ as a binomial expression of $\dfrac{z^2}{M_1^2}$ implies:

$$\begin{aligned}
Det(M) &= D_G D_T \frac{z^4}{M_1^4} + \left[b_{11} D_T \overline{\phi_1}(z) - b_{22} D_G\right] \frac{z^2}{M_1^2} - b_{11} b_{22} \overline{\phi_1}(z), \\
&= D_G D_T \left[\left(\frac{z^2}{M_1^2} + \frac{b_{11} D_T \overline{\phi_1}(z) - b_{22} D_G}{2 D_G D_T}\right)^2 - \frac{\left(b_{11} D_T \overline{\phi_1}(z) - b_{22} D_G\right)^2}{4 (D_G D_T)^2} - \frac{b_{11} b_{22} \overline{\phi_1}(z)}{D_G D_T}\right] \\
&= \frac{1}{D_G D_T}\left[D_G D_T \frac{z^2}{M_1^2} + \frac{b_{11} D_T \overline{\phi_1}(z) - b_{22} D_G}{2}\right]^2 - \frac{1}{4 D_G D_T}\Bigg[\left(b_{11} D_T \overline{\phi_1}(z) - b_{22} D_G\right)^2 \\
&\quad + 4 D_G D_T b_{11} b_{22} \overline{\phi_1}(z)\Bigg] \\
&= \frac{1}{D_G D_T}\left[D_G D_T \frac{z^2}{M_1^2} + \frac{b_{11} D_T \overline{\phi_1}(z) - b_{22} D_G}{2}\right]^2 - \frac{1}{4 D_G D_T}\left[b_{11} D_T \overline{\phi_1}(z) + b_{22} D_G\right]^2 \\
&= \frac{1}{D_G D_T}\left[D_G D_T \frac{z^2}{M_1^2} - b_{22} D_G\right] \times \left[D_G D_T \frac{z^2}{M_1^2} + b_{11} D_T \overline{\phi_1}(z)\right].
\end{aligned} \tag{3.106}$$

$D_G D_T \dfrac{z^2}{M_1^2} - b_{22} D_G > 0$ because $b_{22} < 0$ and therefore $Det(M) \leq 0$ gives:

$$\frac{1}{(M_1)^2} \leq \frac{-\overline{\phi}_1(z)}{z^2}\left(\frac{b_{11}}{D_G}\right).$$

Consequently, $(G_e, 0)'$ is unstable. To show that system (3.5) undergoes spatial Turing bifurcation at M_1^T, we need to verify that spatial Turing bifurcation occurs prior to the temporal Hopf bifurcation (case where $Tr(M) = 0$ and $Det(M) > 0$) as M_1 increases to M_1^T. From the above argument, we only need to show that if (3.101) fails then (3.102) must have already failed as M_1 increases. When (3.101) fails, we have:

$$\overline{\phi}_1(z) = \frac{-(D_G + D_T)\frac{z^2}{M_1^2} + b_{22}}{b_{11}}. \tag{3.107}$$

Plugging (3.107) into 3.106), we see that

$$Det(M) = -\left(D_T \frac{z^2}{(M_1)^2} + b_{22}\right)^2 \leq 0. \tag{3.108}$$

Thus, (3.102) does not hold and this ends the proof. ☐

Remark 3.5. The space period σ_G of the spatial structure is given by: $\sigma_G = \dfrac{2\pi M_1}{z_1}$ (see also Genieys et al. [42], page 71) where z_1 is given in the previous theorem.

3.3.2.2.2 Linear stability analysis around the forest homogeneous steady state $E_{T_i} = (0, T_i)'$,$i = 1, 2$.

Set:

$$\begin{array}{rcl} m_{11} & = & -\gamma_G + (\delta_G + \lambda_{fG} f) + \gamma_{TG} T_i, \\ m_{22}^{*} & = & \gamma_T \Omega T_i (1 - T_i), \\ m_{22}^{**} & = & \gamma_T T_i (1 + \Omega T_i). \end{array}$$

Proposition 3.9. (linearized system around the forest homogeneous steady state)
Set: $g(x,t) = G(x,t) \ \ and \ \ h(x,t) = T(x,t) - T_i$ two perturbations around the forest homogeneous steady state. The system obtained after linearization is:

$$\left\{ \begin{array}{rcl} \dfrac{\partial g}{\partial t} & = & D_G \dfrac{\partial^2 g}{\partial x^2} - m_{11} g, \\ \dfrac{\partial h}{\partial t} & = & D_T \dfrac{\partial^2 h}{\partial x^2} + m_{22}^{*} h - m_{22}^{**} \displaystyle\int_{-\infty}^{+\infty} \phi_{M_2}(x-y) h(y,t) dy. \end{array} \right. \tag{3.109}$$

By considering the eigenvalue problem of the system (3.109) and in the same way like in proposition 3.8, we obtain the following theorem:

Theorem 3.5. (Forest homogeneous steady state stability)
If $\mathcal{R}_F < 1$ and $\overline{\phi_2}(z) \geq \dfrac{m_{22}^{*}}{m_{22}^{**}}$ for all z, then the forest homogeneous steady state is locally asymptotically stable for system (3.5), where $\overline{\phi_2}(z)$ denotes the Fourier transform of ϕ_{M_2}.

Proof. The proof is done like for theorem 3.3, page 105. Hence, we omitted it. ☐

Remark 3.6. In the case of no tree-tree facilitation, by the previous theorem, the use of Gaussian kernels can not lead to inhomogeneous patterned solution in the vicinity of forest homogeneous steady state because with $\Omega = 0$, $m_{22}^{*} = 0$ and then, the condition (with Gaussian kernels) of local stability of the forest homogeneous steady state is always verified.

Theorem 3.6. (Forest homogeneous steady state instability)
Assume that $\mathcal{R}_{F,f} < 1$ and we have a range of positive values of z such that:

$$\overline{\phi_2}(z) < \frac{m_{22}^{*}}{m_{22}^{**}} \tag{3.110}$$

holds. If there exist a critical value $M_2^T > 0$ such that:

$$M_2 > M_2^T \Rightarrow \frac{1}{(M_2)^2} \leq S_2(z_2), \tag{3.111}$$

where

$$S_2(z) = -\frac{\overline{\phi}_2(z)}{z^2}\left(\frac{m_{22}^{**}}{D_T}\right) + \frac{1}{z^2}\left(\frac{m_{22}^{*}}{D_T}\right), \tag{3.112}$$

and z_2 is the value of z such that $S_2(z)$ takes a global maximum. Then, the forest homogeneous steady state is unstable and system (3.5) undergoes a Turing bifurcation at $M_2 = M_2^T$.

Proof. The proof is similar to the proof of the theorem 3.4. Therefore, it is omitted. □

Remark 3.7. The space period of the spatial structures σ_T observed in this case is given by $\sigma_T = \dfrac{2\pi M_2}{z_2}$ (see also Genieys et al. [42]), where z_2 is given in the previous theorem.

3.3.2.2.3 Linear stability analysis around the savanna homogeneous steady state $E^* = (G^*, T^*)$

Set:

$$\begin{aligned}
a_{11} &= -\gamma_G G^*, \\
a_{12} &= -\gamma_{TG} G^*, \\
a_{21} &= -\lambda_{fT} f \omega'(G^*) \exp(-pT^*) T^*, \\
a_{22} &= -\gamma_T \left[(1-\Omega)T^* + 2\Omega(T^*)^2\right] + p\lambda_{fT} f \omega(G^*) \exp(-pT^*) T^*, \\
c &= \gamma_T \Omega T^*(1-T^*).
\end{aligned}$$

Proposition 3.10. (Linearized system around the savanna coexistence state)
Let $g(x,t) = G(x,t) - G^* \;\; and \;\; h(x,t) = T(x,t) - T^*$ be two perturbations around the savanna homogeneous steady state. The system obtained after linearization is:

$$\begin{cases}
\dfrac{\partial g}{\partial t} = D_G \dfrac{\partial^2 g}{\partial x^2} + a_{11}\displaystyle\int_{-\infty}^{+\infty} \phi_{M_1}(x-y) g(y,t)dy + a_{12}\int_{-\infty}^{+\infty} \phi_{M_2}(x-y) h(y,t)dy, \\
\dfrac{\partial h}{\partial t} = D_T \dfrac{\partial^2 h}{\partial x^2} + (a_{22} - c)\displaystyle\int_{-\infty}^{+\infty} \phi_{M_2}(x-y) h(y,t)dy + ch + a_{21} g.
\end{cases} \tag{3.113}$$

Then, proceeding in the same way as it was in system (3.94), we get the following eigenvalue problem of the system (3.113) by setting $g(x,t) = n\exp(\lambda t)\exp(ikx)$, and $h(x,t) = q\exp(\lambda t)\exp(ikx)$, where λ is the eigenvalue and k the associated wavenumber:

$$\begin{cases}
\lambda n = -D_G k^2 n + a_{11}\overline{\phi_{M_1}}(k) n + a_{12}\overline{\phi_{M_2}}(k) q, \\
\lambda q = -D_T k^2 q + cq + (a_{22} - c)\overline{\phi_{M_2}}(k) q + a_{21} n,
\end{cases} \tag{3.114}$$

where $\overline{\phi_{M_i}}(k),\ \ i=1,2$ is the Fourier transforms of the function $\phi_{M_i}(x)$.

Proposition 3.11. (Characteristic equation)
The Characteristic equation of system (3.114) is:

$$\lambda^2 - Tr(k, M_1, M_2)\lambda + Det(k, M_1, M_2) = 0, \tag{3.115}$$

where:

$$Tr(k, M_1, M_2) = -(D_G + D_T)k^2 + a_{11}\overline{\phi_{M_1}}(k) + a_{22}\overline{\phi_{M_2}}(k) + (1 - \overline{\phi_{M_2}}(k))c, \tag{3.116}$$

and

$$\begin{aligned} Det(k, M_1, M_2) \;\; = \;\; & D_G D_T k^4 - \left[a_{22}D_G\overline{\phi_{M_2}}(k) + a_{11}D_T\overline{\phi_{M_1}}(k) + cD_G(1 - \overline{\phi_{M_2}}(k))\right]k^2 + \\ & a_{11}(a_{22} - c)\overline{\phi_{M_1}}(k)\overline{\phi_{M_2}}(k) + ca_{11}\overline{\phi_{M_1}}(k) - a_{12}a_{21}\overline{\phi_{M_2}}(k). \end{aligned} \tag{3.117}$$

From the characteristic equation (3.115), we can write the stability conditions of the savanna homogeneous steady state $(G^*, T^*)'$ as follows:

$$Tr(k, M_1, M_2) < 0, \tag{3.118}$$

and

$$Det(k, M_1, M_2) > 0. \tag{3.119}$$

To determine the stability boundary, we need to determine the thresholds for k, M_1, and M_2 such that only one of the eigenvalue of the characteristic equation (3.115) crosses the origin from the left to the right and other eigenvalues have negative real parts. If (3.118) holds and (3.119) is not satisfied, then there is a real eigenvalue crossing the origin. Initially ($k = M_1 = M_2 = 0$), (3.118) and (3.119) hold. So we find the thresholds k^T, M_1^T and M_2^T so that (3.119) is not satisfied (it is call Turing Bifurcation). Therefore, we find the value of parameters for which $Det(k, M_1, M_2)$ is non-negative for all values of k, M_1 and M_2 and equals to zero at the points of its minima. Then, these thresholds correspond to the minima of the stability boundary region and satisfy:

$$Det(k, M_1, M_2) = 0, \;\; \frac{\partial Det(k, M_1, M_2)}{\partial M_1} = 0, \;\; \frac{\partial Det(k, M_1, M_2)}{\partial M_2} = 0, \;\; \frac{\partial Det(k, M_1, M_2)}{\partial k} = 0. \tag{3.120}$$

With the given conditions in (3.120) we deduce the following result:

Theorem 3.7. (Stationary pattern condition around the savanna homogeneous steady state)
Consider z_1 and z_2 two positive solutions of the equation $\tan(z) = z$ $(z_1 < z_2)$ such that: $\mu_j = \dfrac{\sin z_j}{z_j} < 0,\ \ j = 1, 2$. Then, suppose that:

$$\mathcal{R}_1^* - \mathcal{R}_2^* > 1 \;\text{ and }\; \frac{a_{11}(c - a_{22})\mu_1\mu_2}{ca_{11}\mu_1 - a_{12}a_{21}\mu_2} < 1. \tag{3.121}$$

Assume also that:

$$M_j > M_j^T := z_j \left(\frac{D_G D_T}{\left(a_{11}a_{22} - ca_{11}\right)\mu_1\mu_2 + ca_{11}\mu_1 - a_{12}a_{21}\mu_2} \right)^{1/4}, \;\; j = 1, 2, \qquad (3.122)$$

and

$$k > k^T := \left(\frac{\left(a_{11}a_{22} - ca_{11}\right)\mu_1\mu_2 + ca_{11}\mu_1 - a_{12}a_{21}\mu_2}{D_G D_T} \right)^{1/4}. \qquad (3.123)$$

Then we have the appearance of periodic solutions in space in the neighborhood of savanna homogeneous steady state.

Proof. Suppose that $\mathcal{R}_1^* - \mathcal{R}_2^* > 1$ and $\dfrac{a_{11}(c - a_{22})\mu_1\mu_2}{ca_{11}\mu_1 - a_{12}a_{21}\mu_2} < 1$, we have:

$$\begin{aligned} Det(k, M_1, M_2) \;\; = \;\; & D_G D_T k^4 - \left[a_{22}D_G\overline{\phi_{M_2}}(k) + a_{11}D_T\overline{\phi_{M_1}}(k) + cD_G(1 - \overline{\phi_{M_2}}(k))\right] k^2 + \\ & a_{11}(a_{22} - c)\overline{\phi_{M_1}}(k)\overline{\phi_{M_2}}(k) + ca_{11}\overline{\phi_{M_1}}(k) - a_{12}a_{21}\overline{\phi_{M_2}}(k) \end{aligned}$$

and

$$\overline{\phi_{M_i}}(k) = \frac{\sin(kM_i)}{kM_i}, \;\; i = 1, 2.$$

We are interested by the determination of thresholds k^T, M_1^T and M_2^T so that:

$$Det(k^T, M_1^T, M_2^T) = 0.$$

These thresholds are solutions of the equations:

$$Det(k, M_1, M_2) = 0, \;\; \frac{\partial Det(k, M_1, M_2)}{\partial M_1} = 0, \;\; \frac{\partial Det(k, M_1, M_2)}{\partial M_2} = 0 \;\; \frac{\partial Det(k, M_1, M_2)}{\partial k} = 0. \qquad (3.124)$$

Differentiating $Det(k, M_1, M_2)$ with respect to M_1 and M_2 and using the fact that: $\dfrac{\partial Det(k, M_1, M_2)}{\partial M_1} = 0$ and $\dfrac{\partial Det(k, M_1, M_2)}{\partial M_2} = 0$ we obtain:

$$(a_{11}a_{22} - ca_{11})\left(\phi_{M_2}(k) + \frac{ca_{11} - a_{11}D_T k^2}{a_{11}a_{22} - ca_{11}}\right)\frac{\partial\overline{\phi_{M_1}}}{\partial M_1} = 0,$$

and

$$(a_{11}a_{22} - ca_{11})\left(\phi_{M_1}(k) - \frac{D_G(a_{22} - c)k^2 + a_{12}a_{21}}{a_{11}a_{22} - ca_{11}}\right)\frac{\partial\overline{\phi_{M_2}}}{\partial M_2} = 0.$$

Then we have:

$$\left\{ \begin{array}{l} \overline{\phi_{M_2}}(k) = \dfrac{a_{11}D_T k^2 - ca_{11}}{a_{11}a_{22} - ca_{11}} = \dfrac{D_T k^2 - c}{a_{22} - c} \;\; \text{or} \;\; \dfrac{\partial\overline{\phi_{M_1}}}{\partial M_1} = 0, \\ \\ \overline{\phi_{M_1}}(k) = \dfrac{D_G(a_{22} - c)k^2 + a_{12}a_{21}}{a_{11}a_{22} - ca_{11}} \;\; \text{or} \;\; \dfrac{\partial\overline{\phi_{M_2}}}{\partial M_2} = 0. \end{array} \right. \qquad (3.125)$$

First, if:

$$\overline{\phi_{M_2}}(k) = \frac{D_T k^2 - c}{a_{22} - c} \ \text{ and } \ \overline{\phi_{M_1}}(k) = \frac{D_G(a_{22} - c)k^2 + a_{12}a_{21}}{a_{11}a_{22} - ca_{11}},$$

then $Det(k, M_1, M_2) = \dfrac{a_{12}a_{21}}{c - a_{22}} D_T k^2 + c\dfrac{a_{12}a_{21}}{a_{22} - c}$. Using the fact that $\dfrac{\partial Det(k, M_1, M_2)}{\partial k} = 0$, we obtain $k = 0$ and then we return to the temporal case.

Second, if:

$$\overline{\phi_{M_2}}(k) = \frac{D_T k^2 - c}{a_{22} - c} \ \text{ and } \ \frac{\partial \overline{\phi_{M_2}}}{\partial M_2} = 0,$$

then as previously: $Det(k, M_1, M_2) = \dfrac{a_{12}a_{21}}{c - a_{22}} D_T k^2 + c\dfrac{a_{12}a_{21}}{a_{22} - c}$ and we can not have Turing bifurcation there.

Third, if:

$$\overline{\phi_{M_1}}(k) = \frac{D_G(a_{22} - c)k^2 + a_{12}a_{21}}{a_{11}a_{22} - ca_{11}} \ \text{ and } \ \frac{\partial \overline{\phi_{M_2}}}{\partial M_2} = 0,$$

then we have the same results as before. Finally, if:

$$\frac{\partial \overline{\phi_{M_1}}}{\partial M_1} = 0 \ \text{ and } \ \frac{\partial \overline{\phi_{M_2}}}{\partial M_2} = 0,$$

we obtain

$$\tan(kM_1) = kM_1 \ \text{ and } \ \tan(kM_2) = kM_2.$$

Set $z_1 = kM_1$ and $z_2 = kM_2$, then z_1 and z_2 are solutions of :

$$\tan(z) = z. \tag{3.126}$$

Set:

$$\mu_1 = \frac{\sin(z_1)}{z_1} \ \text{ and } \ \mu_2 = \frac{\sin(z_2)}{z_2}.$$

$Det(k, M_1, M_2) = 0$ gives that:

$$(k^T)^2 = \frac{D_G a_{22}\mu_2 + a_{11}D_T\mu_1 + cD_G(1 - \mu_2) + \sqrt{\Sigma}}{2D_G D_T}, \tag{3.127}$$

with:

$$\Sigma = (D_G a_{22}\mu_2 + a_{11}D_T\mu_1 + cD_G(1 - \mu_2))^2 - 4D_G D_T\left((a_{11}a_{22} - ca_{11})\mu_1\mu_2 - a_{12}a_{21}\mu_2 + ca_{11}\mu_1\right),$$

and using the fact that $\dfrac{\partial Det(k, M_1, M_2)}{\partial k} = 0$ we obtain:

$$(D_G a_{22}\mu_2 + a_{11}D_T\mu_1 + cD_G(1 - \mu_2))^2 = 4D_G D_T\left((a_{11}a_{22} - ca_{11})\mu_1\mu_2 - a_{12}a_{21}\mu_2 + ca_{11}\mu_1\right). \tag{3.128}$$

Note that $(a_{11}a_{22} - ca_{11})\mu_1\mu_2 - a_{12}a_{21}\mu_2 + ca_{11}\mu_1 > 0$ thanks to the second assumption in (3.121).

Thus, the relation in (3.128) is well defined and therefore:

$$(k^T)^2 = \sqrt{\frac{(a_{11}a_{22} - ca_{11})\mu_1\mu_2 - a_{12}a_{21}\mu_2 + ca_{11}\mu_1}{D_G D_T}}. \tag{3.129}$$

The associated values of M_1 and M_2 are

$$M_1^T = z_1 \left(\frac{D_G D_T}{(a_{11}a_{22} - ca_{11})\,\mu_1\mu_2 + ca_{11}\mu_1 - a_{12}a_{21}\mu_2} \right)^{1/4}, \tag{3.130}$$

and

$$M_2^T = z_2 \left(\frac{D_G D_T}{(a_{11}a_{22} - ca_{11})\,\mu_1\mu_2 + ca_{11}\mu_1 - a_{12}a_{21}\mu_2} \right)^{1/4}. \tag{3.131}$$

☐

Because of the difficulty of interpretation of the second condition in (3.121), we find a sufficient condition to the previous one. Set:

$$\mathcal{R}_3^* = \frac{\gamma_T(1 + \Omega T^*)}{p\lambda_{fT} f \omega(G^*) \exp(-pT^*)}.$$

It is straightforward to observe that $\mathcal{R}_1^* - \mathcal{R}_2^* < \mathcal{R}_3^*$ and that $\mathcal{R}_3^* > 1$ implies that the second condition of (3.121) is valid. Therefore, the following result holds true:

Theorem 3.8. (Sufficient condition)
If $\mathcal{R}_1^* - \mathcal{R}_2^* > 1$, then the conclusion of Theorem 3.7 is valid.

Remark 3.8. $\mathcal{R}_3^*$ is the primary production of tree biomass and additional production of tree biomass due to tree-tree facilitation relative to fire induced tree biomass loss.

2. Condition (3.122) gives the range beyond which the nonlocal interactions are sufficient for the coexistence of both tree and grass inhomogeneous solutions in the same space domain.
3. Due to the implicit nature of the equation (3.117), it is difficult to provide explicit expression of Turing bifurcation threshold analytically and hence we have describe one way in previous theorem to determine a triplet (k^T, M_1^T, M_2^T) as a suitable choice of M_1 and M_2 to obtain stationary Turing Pattern (see also Banerjee and Volpert [8]). However the space period of spatial structure is $\sigma = \dfrac{2\pi}{k_{max}}$ where k_{max} is the most unstable mode, that could be computed numerically.

3.4 Numerical illustration

Our model is designed for humid savannas. Then, carrying capacities considered, before the normalization of biomasses are $K_G = 17t.ha^{-1}$ and $K_T = 340t.ha^{-1}$. These values were obtained from Yatat Djeumen et al. [124] considering that the mean annual rainfall **W** is equal to $1500\ \ mm.yr^{-1}$.

We also assume that $D_G = 0.1$ and $D_T = 1$ (see e.g. Yatat Djeumen et al. [122]). The finite difference method was used to discretize the spatial part and on the other hand, the non standard finite difference method (Anguelov et al. [4]) was used to discretize the temporal part of the problem given by the system (3.5).

The numerical scheme for the problem given by system (3.5) is obtained by using non standard finite method for the discretization of the temporal part of the system and difference finite method for the spatial part. We subdivided the space domain $(0, l)$ in $n + 1$ intervals such that:

$$x_0 = 0 < x_1 < x_2 < ... < x_n < x_{n+1} = l,$$

where

$$\forall\ j = 1, ..., n\ \ \Delta x = x_{j+1} - x_j = \frac{l}{n+1}\ \text{ and }\ x_j = j\Delta x.$$

In the same way, we subdivided the time interval such that:

$$t_0 < t_1 < t_2 < ... < t_i < t_{j+1} < ...\ \text{ and }\ t_i = i\Delta t.$$

We denote by G_j^i and T_j^i respectively the value of G and T at the time t_i and at the space point x_j. Remark first that in non standard method, non linear terms are substituted by a non local approximation. Second, the standard denominator Δt in each discrete derivative is replaced by a time-step function $0 < \varphi(\Delta t) < 1$ such that $\varphi(\Delta t) = \Delta t + \mathcal{O}(\Delta t)$. The non-standard approximation for the system (3.5) are given by:

$$\begin{cases} \dfrac{G_j^{i+1} - G_j^i}{\varphi_1(\Delta t)} &= D_G \dfrac{G_{j+1}^i + G_{j-1}^i - 2G_j^i}{\Delta x^2} + \left(\gamma_G - \delta_G - \lambda_{fG} f\right) G_j^i - \left(\gamma_G \phi_{M_1} * G_j^i + \gamma_{TG} \phi_{M_2} * T_j^i\right) G_j^{i+1}, \\ \\ \dfrac{T_j^{i+1} - T_j^i}{\varphi_2(\Delta t)} &= D_T \dfrac{T_{j+1}^i + T_{j-1}^i - 2T_j^i}{\Delta x^2} + \left(\gamma_T - \delta_T + \Omega\gamma_T T_j^i\right) T_j^i - \left(\gamma_T \phi_{M_2} * T_j^i + \Omega\gamma_T T_j^i \phi_{M_2} * T_j^i\right) T_j^{i+1} \\ & \quad -\lambda_{fT} f \omega(G_j^i) \exp\left(-p\phi_{M_2} * T_j^i\right) T_j^{i+1}, \end{cases} \tag{3.132}$$

with

$$\begin{cases} \varphi_1(\Delta t) &= \dfrac{e^{(\gamma_G - \delta_G - \lambda_{fG} f)\Delta t} - 1}{\gamma_G - \delta_G - \lambda_{fG} f}, \\ \\ \varphi_2(\Delta t) &= \dfrac{e^{(\gamma_T - \delta_T)\Delta t} - 1}{\gamma_T - \delta_T}, \end{cases} \tag{3.133}$$

and

$$\left\{\begin{array}{rcl}\Delta x & \leq & \min\left(\sqrt{\dfrac{2D_G}{\gamma_G-\delta_G-\lambda_{fG}f}};\sqrt{\dfrac{2D_T}{\gamma_T-\delta_T}}\right),\\[4ex] \Delta t & \leq & \min\left(\dfrac{\ln\left[1+\dfrac{\gamma_G-\delta_G-\lambda_{fG}f}{\frac{2D_G}{\Delta x^2}-(\gamma_G-\delta_G-\lambda_{fG}f)}\right]}{\gamma_G-\delta_G-\lambda_{fG}f};\dfrac{\ln\left[1+\dfrac{\gamma_T-\delta_T}{\frac{2D_T}{\Delta x^2}-\gamma_T-\delta_T}\right]}{\gamma_T-\delta_T}\right).\end{array}\right. \tag{3.134}$$

Recall that $\mathcal{R}_G > 1$ implies that $\gamma_G - \delta_G - \lambda_{fG}f > 0$ and $\mathcal{R}_{T,0} > 1$ implies that $\gamma_T - \delta_T > 0$. Second, in system (3.132) $\phi_{M_2} * T_j^i$ is an approximation of the convolution term $\int_{-\infty}^{+\infty} \phi_{M_2}(x-y)T(y,t)dy$, done by the Matlab function “trapz". It is the same for $\phi_{M_1} * G_j^i$.

Our numerical illustrations in this paper are suitable for a 9 hectare (ha) savanna square domain (for instance, Martinez-Garcia et al. [68] considered for example a square patch of savanna of 1 ha). Due to the fact that we have restricted the mathematical analysis to a domain of dimension 1, numerical illustrations are carried out in the space interval $[0; 300]$. The unit of space considered is meter (m) and unit of time is year (yr).

Parameters	D_G	D_T	γ_G	δ_G	γ_T	δ_T	λ_{fG}	λ_{fT}	p	g_0	Ω
Values	0.1	1	2.7	0.1	1	0.3	0.7	0.8	3.4	0.14	5

Table 3.4: Parameter values for simulation.

Parameter values (see Table 4.2, page 163) used for model (3.5) are based on (Yatat Djeumen et al. [123, 124], Accatino et al. [2]). Only Ω, D_G and D_T are assumed.

We first illustrate a bifurcation diagram, for the space-implicit model related to system (3.5), with respect to variations of the fire frequency f and γ_{TG}, the parameter that shapes the competition of tree on grass.

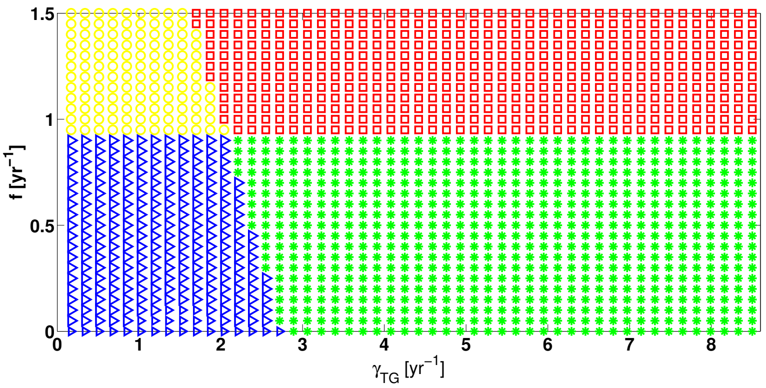


Figure 3.2: Bifurcation diagram according to variations of γ_{TG} and f. The blue triangle corresponds to the savanna monostability, the red square stands for the forest-grassland bistability, the green star denotes the forest monostability and the yellow circle represents the grassland-savanna bistability.

From figure 4.2, one deduces that, at least for parameters considered in table 4.2, stable forest state is easily found, but, for low values of tree-grass competition, savannas are present. We also notice that when we approach the annual fire regime and proceed beyond we manage to recover the grassland state as part of a bistability situation. Then, the increase of the tree-grass competition parameter γ_{TG}, leads to the transition from savanna to forest or grassland-savanna to forest-grassland. In fact, in humid zone, the vegetation is intrinsically dominated by trees, that exert competition pressure on grass biomass, such that grass may be easily suppressed.

The increase of fire frequency leads to the reduction of tree biomass but thanks to tree-tree cooperation, trees can perpetuate. Therefore depending on the tree-grass competition parameter, the system switches are either savanna to grassland-savanna or forest to forest-grassland.

Now we want to illustrate the spatial structuring of trees and grasses in the various cases displayed on the previous bifurcation diagram (see figure 4.2 in page 163).

3.4.1 Case of forest monostability $(f = 0.9 \quad \text{and} \quad \gamma_{TG} = 5.1)$

With the choice of parameters in table 4.2, the homogeneous forest steady state $E_{T_2} = (0, 0.9477)'$ is locally asymptotic stable in absence of nonlocal interactions. Based on theorem 3.6, figure 3.3 depicts that the forest homogeneous steady state is unstable for those M_2 such that the curve of $S_2(z)$ intersects with the line $\dfrac{1}{(M_2)^2}$. For those values, we see that the minimum M_2 required for the emergence of the Turing instability verified approximately $\dfrac{1}{(M_2^T)^2} = 0.0798$ (see the red dashed line in figure 3.3), then $M_2^T = 3.54m$. Therefore, we choose $M_1 = 0.5m$ and $M_2 = 20m$ and we consider the initial data as a random perturbation of the forest homogeneous steady state $(0, T_2)'$:

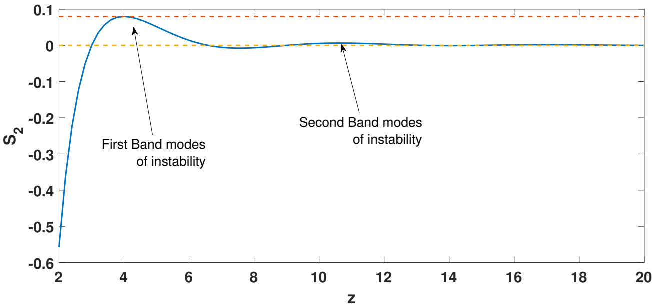
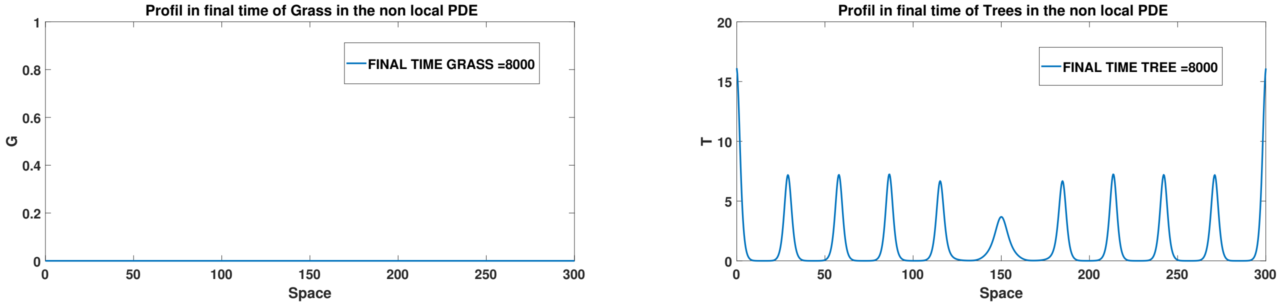


Figure 3.3: Graph of S_2 as a function of z. The parameter values are given in table 4.2. The red dashed line stands for $\dfrac{1}{(M_2)^2}$.

$$G(x,0) = 0 + \epsilon_1, \quad T(x,0) = T_2 + \epsilon_2 \ \text{ with } \ 0 \le \epsilon_i \le 10^{-3}$$



(a) Grass distribution in space at $t = 8000$.

(b) Tree distribution in space at $t = 8000$.

Figure 3.4: Illustration of Grass and Tree profiles in space.

We observe from Figure 3.4 that the solution of system (3.5) converges toward a space inhomogeneous forest solution, thanks to a Turing bifurcation.

The key thresholds in that situation are $\mathcal{R}_F$ (the primary production of grass biomass, relative to grass biomass loss induced by fire, herbivory (grazing) or human action and additional grass suppression due to tree competition, at the closed forest equilibrium) and M_2 the range of nonlocal competition of trees on grasses. Using a periodogram, we can numerically determine the number of patches in our inhomogeneous solution and we can therefore compute the associated spatial wavelength.

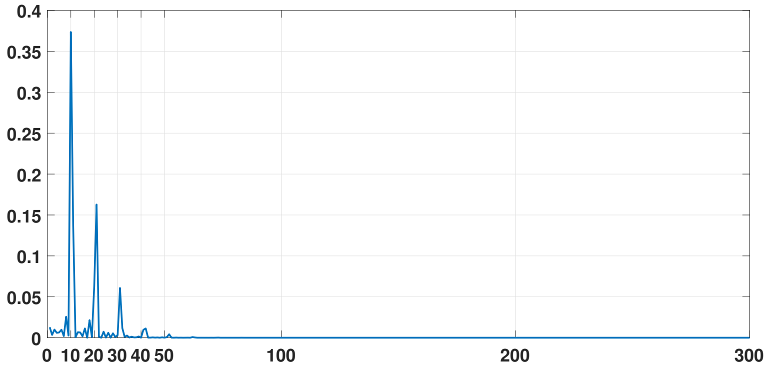
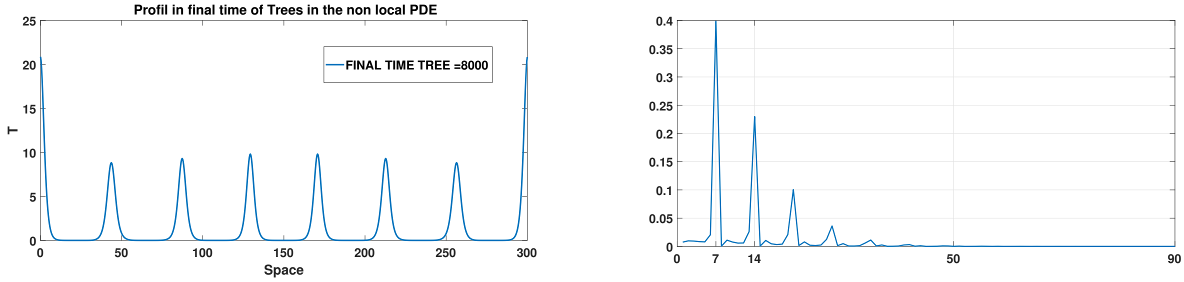
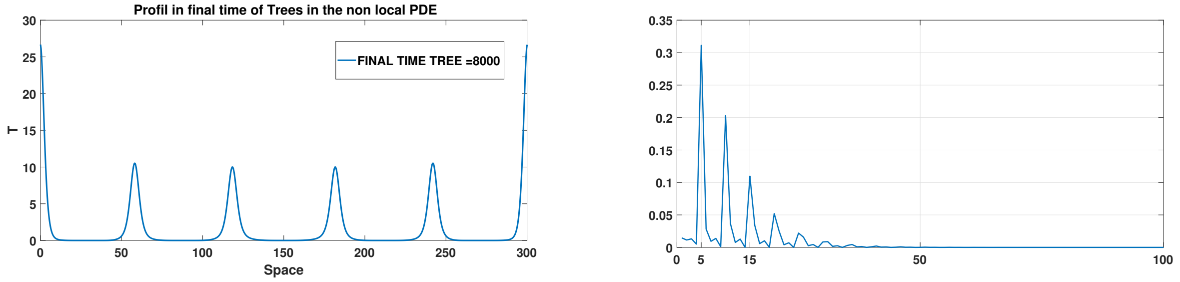


Figure 3.5: Graph of a periodogram of forest inhomogeneous solution

From figure 3.5, we have 10 patches in the spatial profile of forest distribution (see also panel (d) in figure 3.4). Therefore, the numerical wavelength is $\sigma_T = \frac{300}{10} = 30m$. However, from the linear stability analysis and the parameter values considered in this case, the theoretical wavelength is $\sigma_T = 31.4m$ which is quite close of the numerical space period. We also found that for increasing values of M_2, the space period (wavelength) σ_T increases.



(a) Trees distribution at $t = 8000$ with $M_2 = 30m$. (b) Periodogram of trees distribution with $M_2 = 30m$



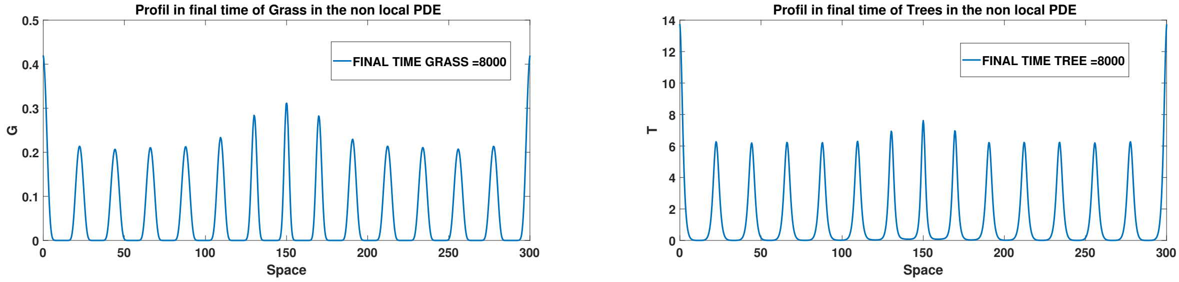
(c) Trees distribution at $t = 8000$ with $M_2 = 40m$. (d) Periodogram of trees distribution with $M_2 = 40m$

Figure 3.6: Illustration of Trees distribution profiles in final time and the corresponding periodogram.

Figure 3.6 shows that the numerical wavelength of tree distribution is $\sigma_T = \frac{300}{7} = 42.8m$ with $M_2 = 30m$ and $\sigma_T = \frac{300}{5} = 60m$ with $M_2 = 40m$. By linear stability analysis, the space period is $\sigma_T = 47.1m$ for $M_2 = 30m$, $\sigma_T = 62.8m$ for $M_2 = 40m$

3.4.2 Case of savanna monostability $(f = 0.9$ and $\gamma_{TG} = 1.7)$

We find that the savanna steady state $E^* = (0.1345, 0.9453)'$ is locally asymptotically stable in the absence of nonlocal interactions. Moreover, the minimal positive solution of the equation $\tan(z) = z$ is $z_1 = 4.49$. We take $z_2 = 10.9$ which is also solution of $\tan(z) = z$.
From these two values, we find $\mu_1 = -0.22, \ \ \mu_2 = -0.09$ and we get the Turing bifurcation condition: $M_1 > 5.07m$ and $M_2 > 12.32m$. For illustration we choose $M_1 = 5.5m$, $M_2 = 15m$ and we consider the initial data as a random perturbation of the savanna equilibrium (G^*, T^*).



(a) grass distribution at $t = 8000$. (b) Tree distribution at $t = 8000$.

Figure 3.7: Illustration of Grass and Tree profiles in space at final times.

We observe from figure 3.7 that, solutions of system (3.5) converge toward a space inhomogeneous tree-grass coexistence solution thanks, to a Turing bifurcation. In the same way as before, we are interested in the wavelength resulting from this inhomogeneous solution.

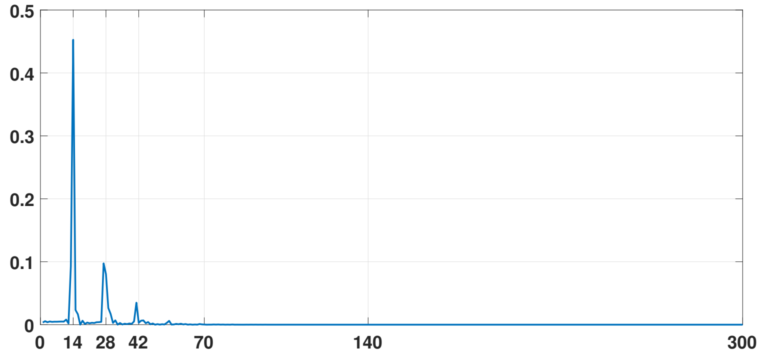


Figure 3.8: Periodogram of savana inhomogeneous solution

Figure 3.8 depicts that the savanna inhomogenous solution illustrated in figure 3.7 has 14 cells. Then, the numerical wavelength in this case is $\sigma = \frac{300}{14} = 21.43m$. Theoretically, it is necessary to determine the most growing mode k_{max} for wich $Det(k_{max}, M_1, M_2) < 0$ and the wavelength will be $\sigma = \dfrac{2\pi}{k_{max}}$.

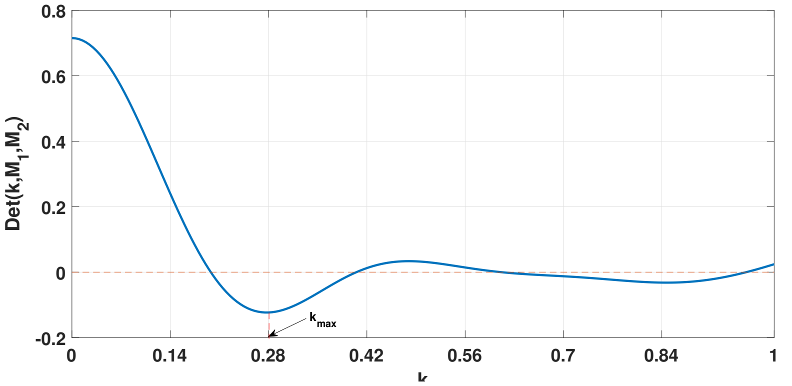


Figure 3.9: Graph of $Det(k, M_1, M_2)$ with $M_1 = 5.5m$ and $M_2 = 15m$

From figure 3.9, $k_{max} = 0.28$ and then $\sigma = 22.43m$.

The value of M_1 used previously could be questioned and seen too large. However, note that the Turing condition that we obtained is only a sufficient condition. Therefore, it may be possible that outside of these values, we can have a change of sign of $Det(k, M_1, M_2)$ which leads to a Turing bifurcation. To illustrate that point, we consider $M_1 = 0.5m$ and $M_2 = 25m$ and we draw $Det(k, M_1, M_2)$.

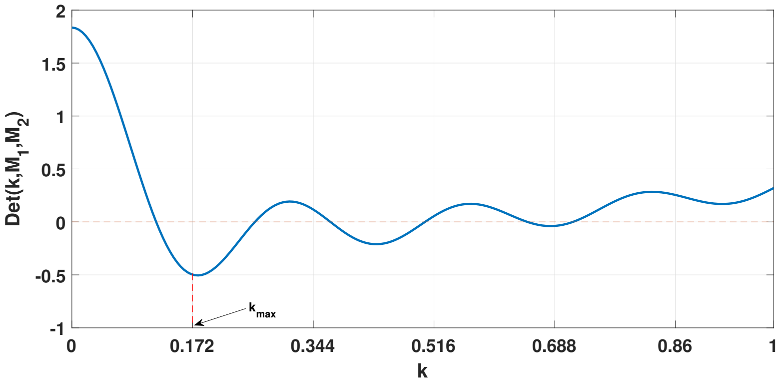
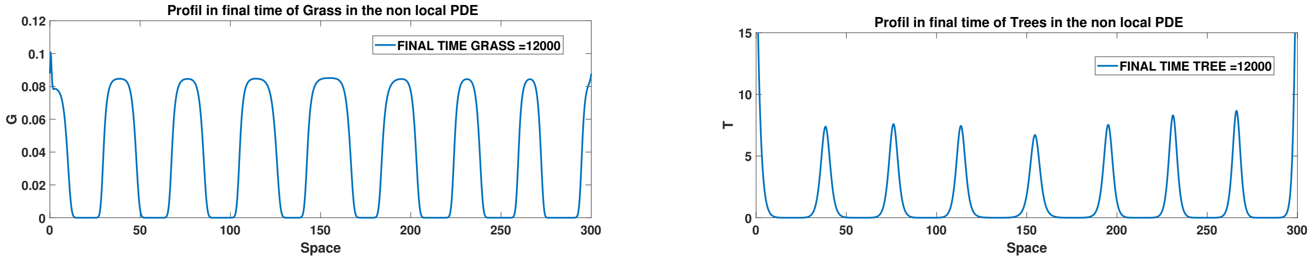


Figure 3.10: Graph of $Det(k, M_1; M_2)$ with $M_1 = 0.5m$ and $M_2 = 25m$.

From figure 3.10, we observe that it is possible to have a Turing bifurcation with $M_1 = 0.5m$ and $M_2 = 25m$ due to the change of sign of $Det(k, M_1, M_2)$. For these values of M_1 and M_2 we can thus illustrate the inhomogeneous solution obtained (see figure 3.11).



(a) Grass distribution at $t = 12000$.

(b) Tree distribution at $t = 12000$.

Figure 3.11: Illustration of Grass and Tree profiles in space at final times.

The graph of periodogram is illustrated in figure 3.12.

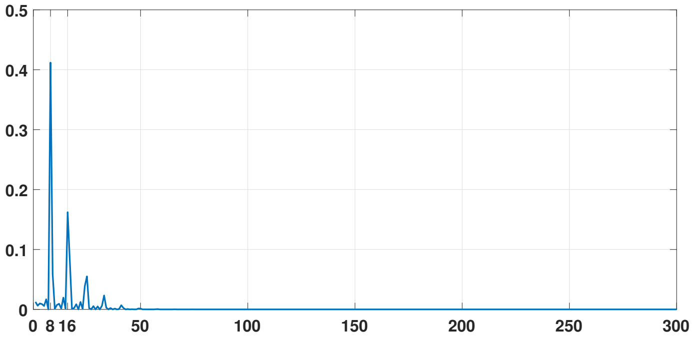


Figure 3.12: Periodogram of savana inhomogeneous solution

From figure 3.12, the numerical space period is $\sigma_T = 37.5m$ and the theoretical wavelength is $\sigma_T = \dfrac{2\pi}{0.172} = 36.5m$.

3.4.3 Case of bistability forest-grassland $(f = 0.98$ and $\gamma_{TG} = 5.1)$

In this case, in absence of nonlocal interactions we have a bistability situation with two homogeneous steady states: a grassland steady state E_G and a forest steady state E_{T_2}. We may observe the spatial structuring of the two state variables in two cases: first around the grassland homogeneous steady state and second around the forest homogeneous steady state.

3.4.3.1 Around the grassland homogeneous steady state

In this section we will consider $D_G = 0.01$ and, for an easy display of figures, we reduce the size of the domain to $100m$, with $E_G = (0.7089; 0)'$.

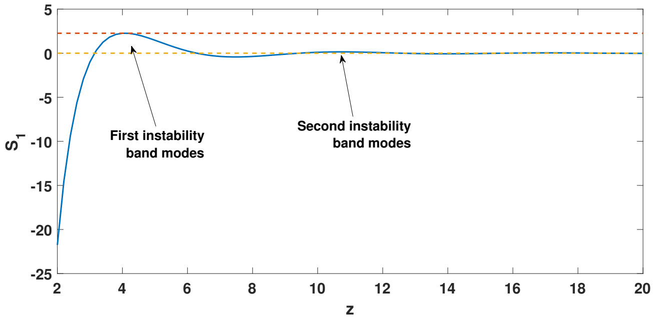
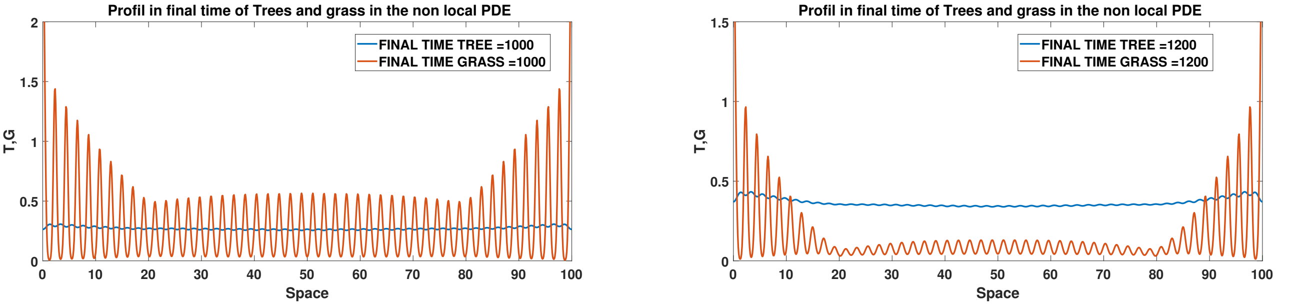


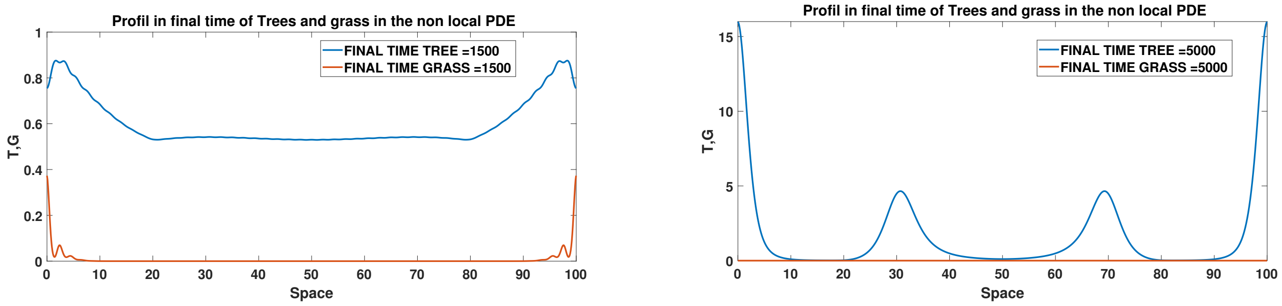
Figure 3.13: Graph of S_1 as a function of z with the parameter values given in table 4.2. The red dashed line stands for $\dfrac{1}{(M_1)^2}$.

Based on theorem 3.4, figure 3.13 illustrates that grassland homogeneous steady state is unstable for values of M_1 such that the curve of $S_1(z)$ intersects with the line $\dfrac{1}{(M_1)^2}$. The minimal value of M_1 such that the grassland equilibrium is unstable verified $\dfrac{1}{(M_1^T)^2} = 2.26$ (then $M_1^T = 0.6647m$) and we choose for illustration around the grassland equilibrium $M_1 = 1.5m$ and $M_2 = 20m$.



(a) Tree-grass distribution at $t = 1000$.

(b) Tree-grass distribution at $t = 1200$.



(c) Tree-grass distribution at $t = 1500$.

(d) Tree-grass distribution at $t = 5000$.

Figure 3.14: Illustration of Grass and Tree distributions.

Figure 3.14 suggests the existence of a metastable tree-grass pattern. In fact, from panel (a) one could believe that we have an inhomogeneous solution of coexistence of the two species; but when we increase the simulation time, we observe that we are moving rather towards the inhomogeneous forest solution. So in this case we have the coexistence of unstable grassland inhomogenous solution and stable tree inhomogeneous solution. This type of solution is called a metastable state (see also Eigentler and Sheratt [35]). However, if we stop at a final time equal to 1000, we observe that the grassy biomass benefits from the space freed by the trees. We can further illustrate it with figure 3.15 for $M_1 = 3m$ and $M_2 = 20m$.

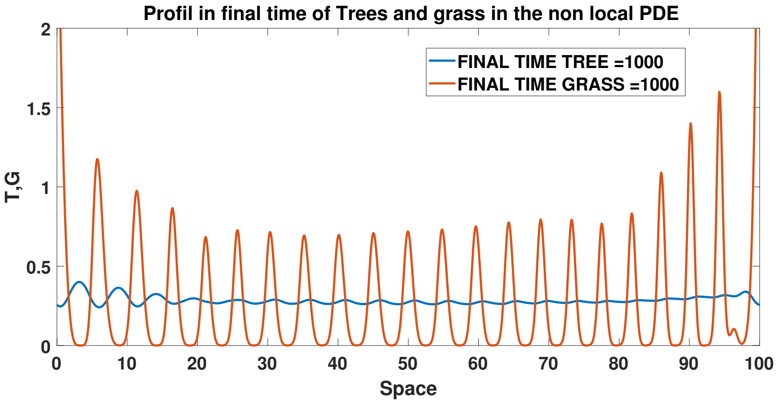


Figure 3.15: Tree-grass distribution at $t = 1000$

3.4.3.2 Around the forest homogeneous steady state

The forest homogeneous steady state is $E_{T_2} = (0, 0.9477)'$. As previously, to find the Turing bifurcation threshold M_2^T, we need to draw the curve of $S_2(z)$.

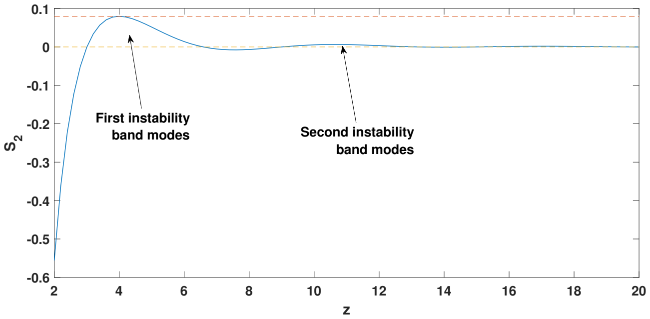
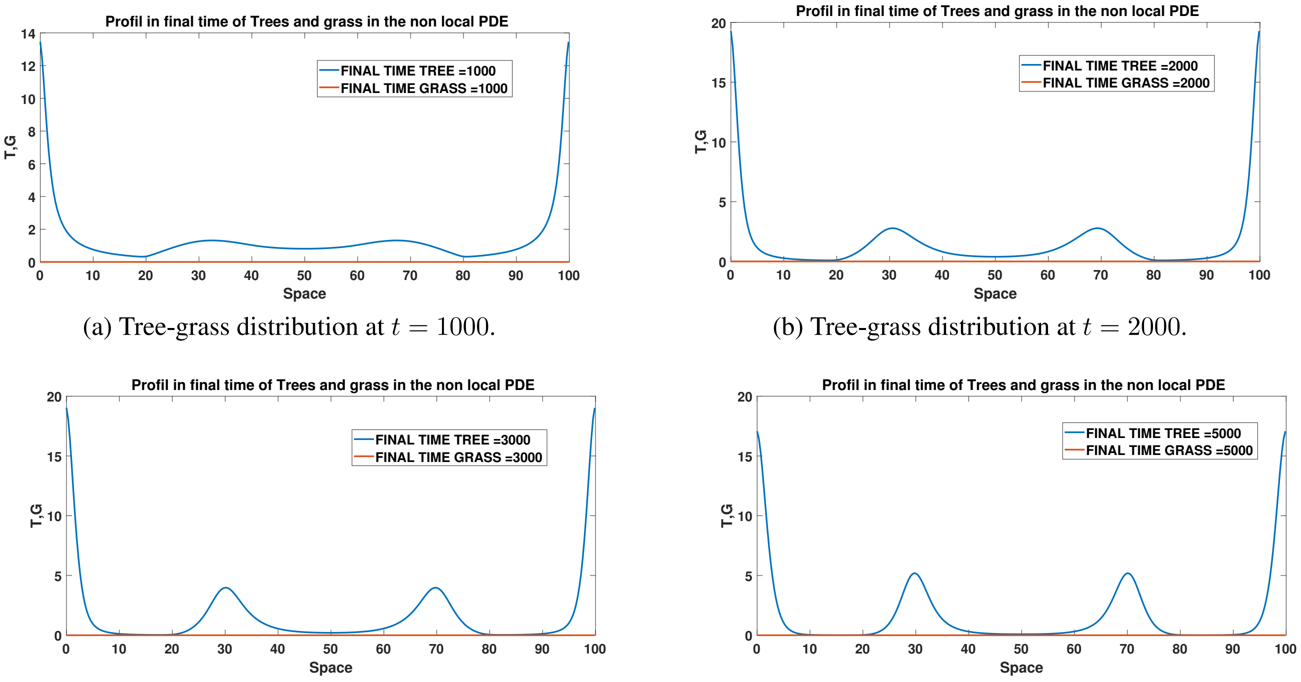


Figure 3.16: Graph of S_2 as a function of z with the parameter values given in table (4.2)

Figure 3.16 shows that the minimal value of M_2 such that the forest homogeneous steady state is unstable verified $\dfrac{1}{(M_2^T)^2} = 0.0798$ (then $M_2^T = 3.54m$). Hence, for illustration, we choose $M_1 = 0.5m$ and $M_2 = 20m$.



(a) Tree-grass distribution at $t = 1000$.

(b) Tree-grass distribution at $t = 2000$.

(c) Tree-grass distribution at $t = 3000$.

(d) Tree-grass distribution at $t = 5000$.

Figure 3.17: Illustration of Grass and Tree distributions.

Then by figure 3.17 the nonlocal system (3.5) converges toward a forest inhomogeneous stable solution, and numerical space period is $\sigma_T = 33.33m$ while theoretically, the space period is $\sigma_T = 31.42m$.

3.4.4 Case of bistability savanna-grassland $(f = 0.98$ and $\gamma_{TG} = 1.7)$

Considering parameter values in table 4.2, the savanna homogeneous steady state E^* and the grassland homogeneous steady state E_G are both locally asymptotically stable for the space implicit model related to system (3.5). In this section, the space domain is $[0, 100]$ and $D_G = 0.01$

3.4.4.1 Around the savanna homogeneous steady state

Around the savanna homogeneous steady state $E^* = (0.1136, 0.9455)'$ the Turing bifurcation condition are $M_1 > 2.97m$ and $M_2 > 7.21m$. To illustrate the appearance of inhomogeneous solution, we choose $M_1 = 3m$ and $M_2 = 20m$. Therefore, we have figure 3.18.

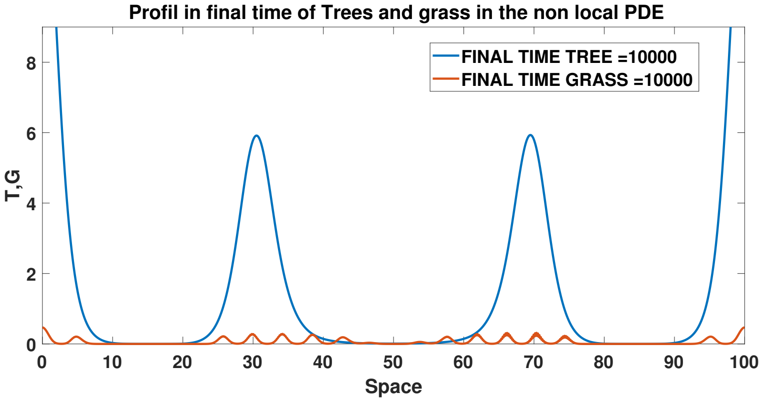


Figure 3.18: Tree-grass distribution at $t = 10000$.

The system converges toward a space inhomogeneous tree-grass coexistence solution (savanna) thanks, to a Turing bifurcation. We also observe that in figure 3.18, we have grass localized solution in space and regular tree spots.

3.4.4.2 Around the grassland homogeneous steady state

The grassland homogeneous steady state is $E_G = (0.7089, 0)'$ and is the same as before (see section 3.4.3.1). The Turing bifurcation threshold is the same as before. We choose $M_1 = 1.5m$, $M_2 = 20m$ for illustration. Figure 3.19 illustrates the spatial distribution of the inhomogeneous tree-grass (i.e. savanna) solution.

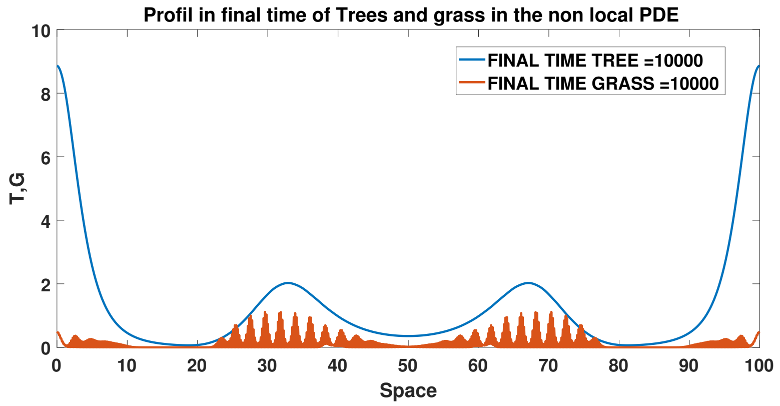
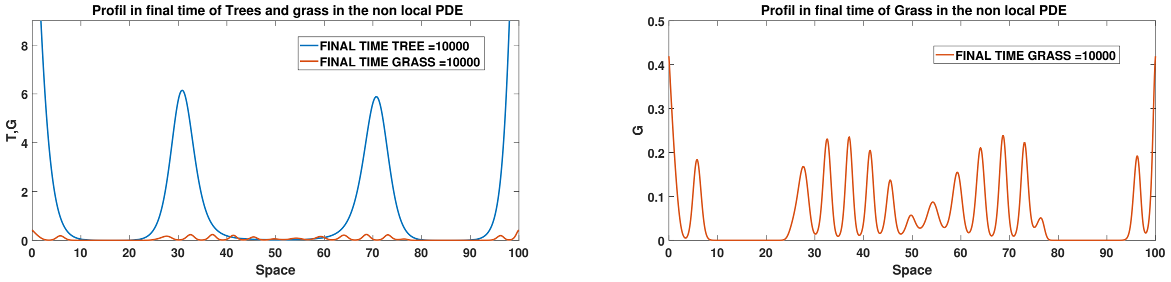


Figure 3.19: Tree-grass distributions at $t = 10000$.

Figure 3.19 shows a high density under the trees which is due to the range of interactions between the grasses which is quite low. However, if we push this range to $M_1 = 3m$, we obtain the following figure 3.20 which is similar to the structure obtained around the savanna homogeneous steady state.



(a) Tree-grass distribution at $t = 10000$.

(b) zooming of grass distribution at $t = 10000$.

Figure 3.20: Illustration of Grass and Tree profiles in space at final times.

3.5 Discussion

We analyzed an integro-differential reaction-diffusion fire-mediated tree-grass interactions model, allowing to reach spatial patterns (namely, regular spotted pattern) sometimes observed in humid savannas. Starting from the parsimonious 2-dimensional ODE-based model of grassy and woody biomasses fire-mediated interactions studied in Yatat Djeumen et al. [124], we introduced local biomass propagation through Laplace operators, like in Yatat Djeumen et al. [122], as well as nonlocal interaction terms. Hence, our model improves and extends previous ODE models (e.g. Yatat Djeumen et al. [122, 124]) by explicitly taking into account spatial components and nonlocal terms of tree-grass interactions. We showed that the combination of the nonlocal tree-tree facilitation and the nonlocal tree-tree, grass-grass and tree-grass competition, may induce spatial patterns. In fact, nonlocal interactions break up the homogeneous distribution of tree and grass biomass resulting in the emergence of a regular spotted pattern (see for instance Tian et al. [104]). Then, novelties in this chapter include the consideration of nonlocal interaction terms (both facilitation and competition) on both trees and grasses dynamics. Indeed, in the absence of nonlocal terms, our model is unable to produce spatial patterns since the associated reaction-diffusion model is monotone decreasing (see also Yatat Djeumen et al. [122]). A key technical point is the requirement on kernels that must be constant functions with compact supports. Indeed, we show that Gaussian or Laplace-like kernels are not able to produce patterns in our model. According to Martinez-Garcia et al. [68], kernels whose Fourier transforms take negative values for some wavenumber values, will lead to clustering in some specific models with short range facilitation.

On the other hand, it is now acknowledged that fire is one of the key factors that shape the physiognomy of savanna vegetation, in general, and particularly, in humid savannas where rainfall is sufficient to promote very high grass biomass production which in turn constitutes the principal fuel for fires. However, as a response to the negative impact of fires, trees have developed 'defence' or resilience mechanisms in order to limit or to reduce the fire-induced tree mortality. Indeed, tree-tree facilitation or cooperation promotes germination of tree's seeds, the recruitment of new trees by improving the conditions under canopy (shading, litter and nutriments, enhanced water infiltration). We modelled this effect thanks to the Ω parameter that was added to the reference ODE model of Yatat Djeumen et al. [124] to make the unsaturated logistic growth a non linear function of trees biomass (T). By enhancing woody biomass growth, tree-tree facilitation indirectly reduces the grass layer or favours an heterogeneous spatial distribution of the grass layer which reduces fire intensity along with the potential of fire to spread all over the landscape.

Based on parameter values used for the bifurcation diagram (see figure 4.2, page 163), we explore and illustrate in the different regions of the bifurcation diagram, the spatial structuring of trees and grasses resulting from nonlocal interaction terms. We obtained broadly four types of inhomogeneous solutions: first, what we call forest inhomogeneous solution (obtained around the monostable forest space-homogeneous equilibrium) which are characterized by an absence of grass biomass and regular tree spots in the space domain. Second, the savanna inhomogeneous solutions which featured both tree and grass spots. Third, the coexistence of “localized" grass pattern and regular tree spots and, finally, the presence of metastable patterns obtained in the conditions of the forest-grassland bistable state. In each of these cases we were able to characterize a minimal range of nonlocal interactions for the appearance of spatial structures. In the case of the forest inhomogeneous state, we note that the grass biomass does not take advantage of the space between the ligneous plants, where it is absent. This may result from the fact that grassland space-homogeneous equilibrium is unstable and also from the strong pressure (competition) led by trees on grasses. We also observe the presence of extinction zones were none of the two life forms establish (see for example figure 3.4, page 116). In the case of the savanna inhomogeneous solution, we consider an initial distribution of the vegetation around the monostable savanna equilibrium. We find that the ligneous plants are in phase with the grass biomass. Likewise in this case, grasses do not take advantage of the space between the trees and exclusion zones are also created (see figure 3.7, page 117). On the other hand, we notice that the savanna inhomogeneous state is favored by the high level of woody biomass due to the fact that $\mathcal{R}_1^* - \mathcal{R}_2^*$ is an increasing function of T^*. In fact, one of the necessary conditions for the existence of savanna inhomogeneous solution is $\mathcal{R}_1^* - \mathcal{R}_2^* > 1$. We also notice the appearance of metastable structures when the initial setting is the forest-grassland bistability. Precisely, we considered vegetation initial distribution around the stable grassland homogeneous steady state while parameter values ensured that the forest homogeneous state is also stable. Therefore, for a substantial time of simulation up to an order of 10^3 years, we can see that the grass biomass takes advantage of the space between the trees (see figure 3.15, page 121).

Here, nonlocal competition between the grass tuft is responsible for this configuration. However, when the final simulation time is high ($> 10^3$ years), the previous tree and grass spots configuration is no longer observed. Instead, we find a regular structure of tree spots (see panel (d) figure 3.14, page 121). In this case, coexistence of tree and grass spots appears as a long transient phase to a tree spots pattern, which seems qualitatively compatible with the type of pattern illustrated in Fig.1-a). According to Eigentler and Sheratt [35], metatastable pattern is an unstable pattern whose instability is caused by a very small unstable eigenvalue. In case of savanna-grassland bistability, we numerically observe another type of structure that we assimilate to a coexistence of localized grass inhomogeneous solution and regular tree spots (see figure 3.20, page 124). In fact, Vanag and Epstein [114] suggested that if the system is in the spatial bistability domain, then we must apply a perturbation of appropriate shape and sufficient amplitude in order to cause a transition to possibly localized inhomogeneous patterns. The necessary and sufficient condition for localized patches is the coexistence of homogeneous cover and periodic pattern (Tlidi et al. [106], Koga and Kuramoto [59]). In this case, localized inhomogeneous solutions can be interpreted as a nonlinear front between spatially periodic tree distribution and aperiodic grass distribution.

Another line of discussion relies on the size of tree patches observed numerically (i.e. σ_T) and its comparison with the size (width) of the tree nonlocal interaction kernel (M_2) and the value of the cooperation factor Ω. Our illustration around, the forest and savanna homogeneous steady states showed that $\sigma_T \approx 1.5M_2$. Note that, M_2 is to be related to the lateral extend of tree roots or tree canopy. In all cases, where we obtained regular spots, we find that the size of vegetation patches goes above $20m$. In Lejeune et al. [65] for example, the size of vegetation patches in Marahoué National Park in Ivory Coast, ranges from $10m$ to $20m$. The value of Ω used in our work was chosen for illustrative purposes. Nevertheless, within the framework of this paper, we noticed that Ω plays a role on the kinetics of our structures. In fact, for low values of Ω ($\Omega < 1$), the structures take longer time to set up, while the reverse occurs with Ω at large values. Lefever et al. [64] gave a range of value for Ω in the case of arid vegetation. Finally in this chapter, first, we choose to work in first approximation with local operator for spatial propagation (Laplace operators). This choice allows us, from a mathematical point of view, to find a good characterization of the ranges of nonlocal interactions enabling the appearance of structures. Without these local operators it would become difficult to find a mathematical characterization of spatial ranges of nonlocal interactions that can be easily manipulated numerically. Secondly and as a perspective of this work, it is necessary to improve our numerical schemes, where for which during the simulations the densities sometimes exceed the carrying capacities. This ambiguity has also been observed in other models with similar structures of equations, notably in Banerjee and Volpert [8] and Genieys et al. [42]. It is also necessary to emphasize on the mathematical conditions allowing this model to exhibit localized structures and metastable patterns, that we observed numerically, and that may be of substantial relevance to account for field observations.

3.6 Conclusion

In this work, we developed and studied a spatio-temporal tree-grass fire-mediated interactions model allowing to illustrate the spatial structuring of vegetation in the wet savanna zone, where regular spotted patterns (tree groves) have been casually reported in presence of high grass production and frequent fires. To achieve this aim, we extended previous temporal models studied in Yatat Djeumen et al. [122, 124] into integro-differential reaction-diffusion systems. We explore in this model, the combination of nonlocal facilitation and nonlocal competition for the emergence of inhomogeneous solutions. In this context, we integrated kernel functions describing the area of influence of tree and grass roots and the extent of tree canopy-induced shadow effect. Both are modeled like in Martinez-Garcia et al. [68], Banerjee and Volpert [7, 8], Banerjee and Zhang [9] by a constant function of finite range. Accordingly, one of the major key in this chapter is the simultaneous presence of nonlocal tree-tree facilitation along with nonlocal tree-tree, tree-grass and grass-grass competition. In fact, the associated model that results from the present contribution, takes into account the tree-tree cooperation mechanisms modelled by the parameter Ω which is not considered in most of the works dedicated to tree-grass interactions in fire-prone savannas, specifically in Yatat Djeumen et al. [122, 124]. Thanks to the stability analysis, we found conditions of existence of patterned inhomogeneous solutions around space-homogeneous steady states of our system. From a mathematical point of view, our work summarizes all the methods generally used to capture inhomogeneous solutions in nonlocal reaction-diffusion systems, and it appeared necessary to include nonlocal terms as to induce the symmetry breaking instability leading to the patterns. The sequences of patterns observed in this paper consist of regular spot vegetation (tree and grass spots noticed around the forest and grassland homogeneous steady state), “localized" grass structures and metastable pattern. In all cases where we obtained regular spotted patterns, wavelength is an increasing function of the range of tree competitive or tree canopy influence, M_2. As a first approximation, we assumed that both grass and tree biomasses have local propagation through Laplace operators which is in line with rendering clonal propagation. But in reality, wind or even animals may also favor plant propagation through propagule dipersion. Thus Pueyo et al. [79] suggested that it is more reasonable to use nonlocal terms to describe plant dispersal, than diffusion terms. Hence, a line of improvement of the current work could rely on the consideration of nonlocal dispersion terms. Another important objective is to consider the same problem in a two-dimensional spatial domain as to reach more realistic prospects on the patterning processes addressed in the present chapter. In so doing, we may expect to obtain very interesting multi-scale vegetation patterns.

TRAVELING WAVE SOLUTION IN A NONLOCAL DISPERSAL TREE-GRASS SYSTEM

Abstract[1]

PRediction and characterisation of the possible transitions of the various components of vegetation mosaics, are two main points for the sustainable management of resources in humid tropical savannas. These could consequently allow us to anticipate the effect of climate change in ecological system. This work attempts to propose the key parameters of processes that induce or possibly counteract the forest encroachment in grassland/savanna landscape mosaics. For this purpose, we build a reaction-dispersion model with nonlocal dispersion and we study the long-term dynamics of forest-grassland mosaic by the mean of a traveling wave solution setting. Precisely, we characterise the minimal wave speed of the aforementioned forest-grassland traveling wave. We also provide numerical simulations that depict how some parameters of the model may shape variations of the minimal speed. Our results suggest that, the increase of tree seeds dispersal range can accelerate forest encroachment and that, the increase of fire frequency can slow down the progression of the forest.

4.1 Introduction

Savannas are widely defined as the durable co-existence of a continuous grass-cover with a discontinuous tree-cover which can vary in density depending on the location and context (Whitecross et al. [117], Scholes and Archer [91]). They represent one of the largest part of the African landscape and thus play an important role in the livelihoods of millions of people (Devine et al. [25]). Indeed, under humid climate, environmental conditions characterised by sufficient rainfall may lead in absence of fire, to a stable forest state (i.e. a dominance of tree layer that almost excludes grasses). It is there-

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[1] This chapter is an edited version of *Minimal wave speed of a traveling wave for prediction and control of forest encroachment in humid tropical savannas*, preprint submited to the Journal of Differential Equations and Dynamical Systems (DEDS), Springer.

fore acknowledged that fire and herbivory are the major disturbance factors required in order to avoid canopy closure and maintain savannas under sufficient rainfall (Scholes and Archer [91], Higgins and Bond [50], Bond et al. [14], Van Langevelde et al. [109], Govender et al. [46], Bond [13], Staver et al. [98, 99], Jeffery et al. [55], Yatat Djeumen et al. [122] and references therein).

In various locations, the vegetation may display several physiognomies that include one of the four archetypes, namely desert (or absence of vegetation), grassland, forest and savanna. Grassland is characterized by an absence or a total suppression of trees and shrubs. Moreover, many observations of the spatial structuring of vegetation around the world, in relation to precipitation gradient, reveal the presence of particular physiognomies also termed as mosaics of vegetation where several of the four archetypes are present at the landscape scale. Specifically iconic vegetation mosaics are characterised by the simultaneous presence of bare soil and vegetation (grasses or shrubs) in arid and semi-arid regions (Lefever and Lejeune [62], HilleRisLambers et al. [52], Couteron and Lejeune [19], Pueyo et al. [79], Lefever et al. [64], Deblauwe et al. [24], Lefever and Turner [63], Couteron et al. [20], Yatat Djeumen et al. [122]). Also highly tricking yet less investigated are grasslands/savannas and forest in the humid regions (Youta Happi [126], Hirota et al. [53], Jeffery et al. [55], Xu et al. [120], Stall et al. [95], Tega II et al. [103] and references therein). Triggered by these observations, theoretical studies that aimed to understand the mosaics transition amid vegetation states (Couteron [18]) and substantiated the theory of abrupt shifts or disastrous transitions in ecology were carried out (see e.g. Favier et al. [39], Scheffer et al. [89], Zelnik and Meron [129], Yatat Djeumen et al. [122], Tega II et al. [103]).

According to Devine et al. [25], growing evidences suggest a continent-wide phenomenon of savanna and grassland alteration by “woody encroachment" (see e.g figure 4.1, in page 131). For example, Jeffery et al. [55] reported in “Lopé National Park" in Gabon that, without fires, forest invades the savanna. In fact, main drivers for woody encroachment in classical theory of vegetation systems are climate, soil moisture, herbivory and fire regime. However although promoting increasing carbon storage, woody encroachment may have adverse effects on pastoral and game resources, and even on biodiversity (Bond [13]). Specifically, it reduces the landscape diversity and the variety of animal species presents. From another point of view, biotic interactions such as facilitation (understood as the mechanism that either enhance or limit the suppression of plant biomass density) and competition between plants may influence durably the structure of the landscape (Tega II et al. [103] and references therein). Precisely, water availability and limited resources are the two factors that induce facilitation and competition and then, structure the physiognomies of landscapes in arid and semi arid savannas (Lefever and Lejeune [62], Couteron and Lejeune [19], Gilad et al. [43], Pueyo et al. [79], Lefever et al. [64], Pueyo et al. [80], Lefever and Turner [63], Couteron et al. [20]). Conversely, in humid savannas, water is not limited due to frequent and abundant rainfall. Then, competition and facilitation in wet savannas should be based on external factors such as light availability, herbivory and the resistance of trees in face of recurrent and intense fires. Therefore, facilitation against fire impacts, inter-specific competition for light or nutriments, intra-specific competition and ranges of seeds dispersal could be the main mechanisms that promote and maintain such mosaics of vegetation in wet ecosystems. Being able to characterise or predict the different transitions in vegetation mosaics is consequently one of the main objectives in the study of vegetation dynamics.

The mathematical modelling of transition mechanisms in savannas has been the subject of several works related in particular, to the modelling of the spatial mechanisms of tree-grass interactions taking into account external disturbances (fire, herbivory). Such work have used discrete kernel models or cellular automaton (CA) and/or systems of partial differential equations. Notably, Accatino et al. [3] developed a CA-model able to mimic the invasion of trees in the grass stratum in humid savannas despite repeated fires (see also Yatat Djeumen et al. [123], Tega II et al. [103]). One may note that, CA-model are principally treated or analysed numerically because they present a lack of mathematical tractability. In fact, it is not easy to use mathematical analysis to infer or to understand the behaviour and properties of CA-Models. Therefore to avoid extensive, yet no always comprehensive simulations, mathematically tractable models like Partial Differentials Equations (PDE models) could be advocated. Indeed, tractability is a desirable feature of model since it could help to systematically identify the possible outcomes of the model in relation to parameters variations.

Yatat Djeumen et al. [122], developed and studied the long term dynamics of mosaics of forest and grassland by considering local biomass diffusion and local tree-grass interactions. They proposed a reaction-diffusion model which is an extension of a temporal model that unified Tchuinte Tamen et al. [102] and Yatat Djeumen et al. [125] works. Specifically, they characterized how model parameters (like the fire frequency) shape forest to grassland invasion landscape in a context where sharp boundaries are observed between forest and grassland. However, Yatat Djeumen et al. [122] did not consider non-local interactions which are empirically evidenced as mechanisms of plant dispersal, plant competition for light and nutrients as well as plant protection from fires (Pueyo et al. [79], Eigentler and Sheratt [35]). In a previous paper, Tega II et al. [103] extended the work of Yatat Djeumen et al. [122], by taking into account nonlocal tree-grass interactions (facilitation/competition, thanks to kernels of intra and inter specific interactions) as to identify the condition of the emergence of spatial structuring. However, they assumed as in HilleRisLambers et al. [52], Rietkerk et al. [81], Gilad et al. [43], Pueyo et al. [80], Yatat Djeumen et al. [122] that, plant biomasses experience a local isotropic diffusion modelled by Laplace operators which is questionable since plants "do not walk” (see for instance Lefever et al. [64], Lefever and Turner [63]). Diffusion is widely believed to be inadequate for modelling plant dispersal due to frequent long range dispersal events like water, wind, animals (see e.g. figure 4.1).

In this vein, Eigentler and Sheratt [36] proposed and analysed an extension of the reaction-diffusion-advection model of Klausmeier [58] (which studied the spatio-temporal dynamic of plant and water density for arid and semi-arid environments) by taking into account nonlocal convolution terms in order to describe plant seed dispersal. Precisely, Eigentler and Sheratt [36] replaced the diffusion term in Klausmeier [58] by an integral operator (convolution) in the prospect to signify that, a seed located at a space point x can actually come from a plant situated at a space point y possibly far from x. This convolution term in their model, thus describe the probability for a seed to be dispersed. The analysis of Eigentler and Sheratt [36] model deals with a mathematical characterisation of a threshold (depending of rainfall parameter) value for transition between uniform solution and non-uniform periodic solution by the means of symmetric breaking instability. Thus their model end with the comparison in term of model prediction between the local dispersal case and the non-local one. However, for humid environments, we didn't find a mathematically tractable model that deals with PDE formalism and focus on the role of long range seed-dispersal for possible vegetation physiognomic transitions.

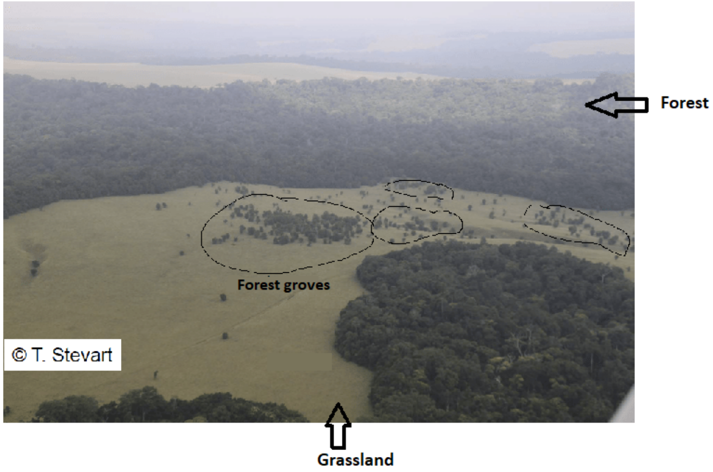


Figure 4.1: Nonlocal forest dispersal/encroachment in a forest-grassland vegetation mosaic in Central Africa. (Gabon, Wonga Wongue Reserve)

The aims in this paper are first, to identify the key parameters of the process that induces transitions in vegetation mosaics and second, to predict and control the speed of some observed transitions. Following figure 4.1, we hypothesized the nonlocal tree seed (or propagules) dispersal to be one of the main process that induce forest encroachment. Therefore, we proposed a reaction-dispersion model where nonlocal tree and grass seeds dispersal are taken into account and we aim to characterise the minimal wave speed of the traveling wave front between forest and grassland homogeneous steady states of the system. This minimal wave speed would latter help us to identify the key parameters of the model able to control the forest encroachment in grasslands. The rest of the paper is organized as follows, section 4.2 presents the model construction, section 4.3 is dedicated to the local stability analysis of homogeneous stationary solutions of the system, section 4.4 deals with the characterisation of the minimal wave speed of the traveling wave solutions between the forest and the grassland homogeneous steady states of the system. Section 4.5 deals with numerical illustrations of our theoretical results.

4.2 Model formulation

We consider in this work, the following nonlocal dispersal and fire mediated tree-grass interaction system adapted for humid tropical ecosystems

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial G(x,t)}{\partial t} & = & \overbrace{D_G\Big(J_{L_G} * G(x,t) - G(x,t)\Big)}^{\text{G dispersion}} + \overbrace{\gamma_G G(x,t)\,(1-G(x,t))}^{\text{G logistic growth}} - \overbrace{\delta_G G(x,t)}^{\text{G loss (herbivorie)}} \\[2ex]
 & & - \overbrace{\gamma_{TG} G(x,t)(\phi_M * T(x,t))}^{\text{G loss (interspecific competition)}} - \overbrace{\lambda_{fG} f G(x,t)}^{\text{G loss (fire)}}, \\[2ex]
\dfrac{\partial T(x,t)}{\partial t} & = & \underbrace{D_T\Big(J_{L_T} * T(x,t) - T(x,t)\Big)}_{\text{T dispersion}} + \underbrace{\gamma_T T(x,t)(1+\Omega T(x,t))(1-T(x,t))}_{\text{T facilitative logistic growth}} - \underbrace{\delta_T T(x,t)}_{\text{T loss (human activities)}} \\[2ex]
 & & - \underbrace{\lambda_{fT} f \omega(G) \exp(-p\phi_M * T(x,t)) T(x,t)}_{\text{T loss (facilitation against fire)}},
\end{array}
\right.
\tag{4.1}
$$

where:

- $G(x,t)$ and $T(x,t)$ stands respectively for the normalised grass and tree biomasses at the location $x \in \mathbb{R}$ and time $t > 0$.
- The dispersal coefficients D_G and D_T, that scale the convolution terms in (4.1), describe the lifeforms (trees and grasses) dispersal rate by taking into account the seed production, seed mortality and germination rate and seed establishment ability. They therefore embody particular aspects of facilitation relating to production and fate of propagule biomass, which is complementary to tree-tree facilitation expressed by Ω; These two dispersal coefficients are both dimensionless.
- $J_{L_h} * h(x,t) = \displaystyle\int_{\mathbb{R}} J_{L_h}(x-y)h(y,t)dy, \;\; h = G, T;$ and $\phi_M * T(x,t) = \displaystyle\int_{\mathbb{R}} \phi_M(x-y)T(y,t)dy.$
- The kernel function ϕ_M describes the strength of nonlocal interaction due to the roots system of trees or to the canopy-induced shadow effect. Specifically, it describes an area of influence of the root system or of the shadow effect. We modeled this kernel function by constant, precisely:

$$
\phi_M(x) = \begin{cases} \dfrac{1}{2M} & , \;\; |x| \leq M, \\ 0 & , \;\; |x| > M, \end{cases}
$$

  with ϕ_0 the δ-function and $\displaystyle\int_{-\infty}^{+\infty} \phi_M(y)dy = 1.$

- The function $\omega(G)$ is on the form of a Holling type III function, describing the potential impact of fires on tree biomass ("fire momentum”).
- $\exp\left(-p\phi_M * T(x,t)\right)$ describe the decrease of tree mortality with cumulative tree biomass.

System (4.1) is a reformulation of a previous integro-differential reaction-diffusion system studied in Tega II et al. [103] which however substantially differs in two mains points:

(a) As we pointed out in the introduction, the description of plant species propagation using the Laplace operator (like it was in Tega II et al. [103]) is not sufficiently relevant in the sense that this latter is essentially a local behavior. Precisely, it described movements of species between adjacent spatial locations and does not take into account all processes of long-range dispersal. In fact, nonlocal processes such as seeds dispersal by e.g. wind or animals is often involved (see figure 4.1, page 131). Then, referring to Pan [73], Zhang and Li [130], Pan and Lin [75], Yu and Yuan [128], Zhao et al. [132], Ducrot et al. [33], Dong et al. [32], we consider the following form of nonlocal dispersal models by assuming isotropic dispersion of seeds:

$$\frac{\partial \mathbf{U}}{\partial t}(x,t) = D\Big(J * \mathbf{U} - \mathbf{U}\Big)(x,t) + \mathbf{f}(\mathbf{U}), \tag{4.2}$$

where $J * \mathbf{U} - \mathbf{U}$ models the fact that the diffusion of density $\mathbf{U}$ at a point x and time t depends on all the values of $\mathbf{U}$ in a neighborhood of x through the convolution term $J * \mathbf{U}$:

$$J * \mathbf{U}(x,t) = \int_{\mathbb{R}} J(x-y)\mathbf{U}(y,t)dy; \tag{4.3}$$

The kernel function $J(x-y)$ describes the probability per unit of seed generated at any point y to being dispersed to the space point x.

Therefore, based on Lefever et al. [64], Lefever and Turner [63], we consider for seed dispersal in system (4.1) Gaussian kernels:

$$J_{L_h}(x) = \frac{1}{\sqrt{2\pi}L_h} e^{-\dfrac{x^2}{2L_h^2}}, \ \ h = G, T, \tag{4.4}$$

where $L_h \quad (h = G, T)$ represents the range of tree and grass seed propagation or dispersion along the space domain. The kernel J_{L_h} defined in (4.4) has the following properties:

(A_1) $J_{L_h} \in C^1(\mathbb{R}), \ \ h = G, T.$

(A_2) $J_{L_h} \in L^1(\mathbb{R})$ and satisfies :

$$\int_{\mathbb{R}} J_{L_h}(x)dx = 1, \ \ J_{L_h}(x) = J_{L_h}(-x), \ \ h = G, T. \tag{4.5}$$

(A_3) J_{L_h} is a δ-Dirac function as L_h tends to zero, $h = G, T$.

(A_4) For $\kappa \in [0, +\infty)$,

$$\int_{\mathbb{R}} J_{L_h}(x)e^{-\kappa x}dx < \infty, \tag{4.6}$$

and $\int_{\mathbb{R}} J_{L_h}(x)e^{-\kappa x}dx \longrightarrow +\infty$ as $\kappa \to +\infty$.

(b) We neglected the nonlocal tree-tree and grass-grass competition present in Tega II et al. [103]. This simplification is done for a mathematical duty. In fact, the presence of nonlocal intraspecefic competition terms make the system to be non-monotone and this latter cannot be turn into a cooperative system. Consequently the different well established methods (theory of monotone semiflows (Fang and Zhao [37]), upper and lower solution solution method with Schauder fixed point theorem (Ducrot et al. [33], Hao et al. [49])) for the existence and spreading speed of traveling wave solution cannot be applied. Note that in humid environment, inter-species competition is not expected to be a driving force but if approaching steric close packing, ie at a very local scale.

We consider the following thresholds:

$$\begin{cases} \mathcal{R}_G = \dfrac{\gamma_G}{\delta_G + f\lambda_{fG}}, \\ \mathcal{R}_{G,0} = \dfrac{\gamma_G}{\delta_G}, \\ \mathcal{R}_{T,0} = \dfrac{\gamma_T}{\delta_T}, \\ \mathcal{R}_{T_2} = \dfrac{\gamma_T T_2\left[(1-\Omega)+\Omega T_2\right]}{\lambda_{fT} f\omega(G_e)}, \\ \mathcal{R}_T = \dfrac{\gamma_T}{\delta_T + \lambda_{fT} f\omega(G_e)}, \\ \mathcal{R}_F = \dfrac{\gamma_G}{\delta_G + \lambda_{fG} f + \gamma_{TG} T_2}, \end{cases} \tag{4.7}$$

and we assume that $\mathcal{R}_G > 1$, $\mathcal{R}_{G,0} > 1$ and $\mathcal{R}_{T,0} > 1$ in order to avoid systematic extinction of biomasses.

System (4.1) has, assuming tree-tree cooperation (i.e $\Omega > 0$), at most four homogeneous steady states (see also Tega II et al. [103]):

- The desert steady state: $E_0 = (0,0)'$.
- The Grassland steady state: $E_{G_e} = (G_e, 0)' = \left(1 - \dfrac{1}{\mathcal{R}_G}, 0\right)'$.
- The forest steady state $E_{T_2} = (0, T_2)' = \left(0, \dfrac{\sqrt{(1-\Omega)^2 + 4\Omega\left(1 - \dfrac{1}{\mathcal{R}_{T,0}}\right)} - (1-\Omega)}{2\Omega}\right)'$.
- The Savanna steady state $E^* = (G^*, T^*)'$.

Firstly, we will focus on local stability of homogeneous steady state of system (4.1). Secondly, we will prove the existence of traveling wave solutions of system (4.1) connecting E_{G_e} with E_{T_2}, through the construction of a truncated problem combining the upper and lower solutions method and the Schauder's fixed-point theorem. Finally, we will give some illustrations of traveling wave solutions of system (4.1).

Symbols	Description	Units
γ_G	Intrinsic growth rate of grasses	yr^{-1}
δ_G	Portion of grass biomass loss due to human activities and herbivory	yr^{-1}
λ_{fG}	Portion of grass biomass loss due to fire	
γ_{TG}	Tree grass interaction parameter	yr^{-1}
γ_T	Intrinsic growth rate of trees	yr^{-1}
δ_T	Portion of tree biomass loss due to human activities	yr^{-1}
λ_{fT}	Portion of tree biomass loss due to fire	
p	Proportional to the inverse of biomass destroyed at intermediate level of mortality	
Ω	Cooperation factor	
f	fire frequency	yr^{-1}
D_G	Grass biomass dispersal rate	
D_T	Tree biomass dispersal rate	
L_G	Range of grass-seed dispersion	m
L_T	Range of tree-seed dispersion	m
M	Range of tree spatial nonlocal interaction	m

Table 4.1: Definition of parameters used in the model.

4.3 Local stability analysis

In this section we are interested by the local asymptotic stability of homogeneous steady state of system (4.1).

Let us fix $(G_s, T_s)' \in \{(G_e, 0)'; (0; T_2)'; (G^*; T^*)'\}$, and consider $u(x,t) = G(x,t) - G_s$ and $v(x,t) = T(x,t) - T_s$ two spatial perturbations, then linearization of system (4.1) around a non trivial homogeneous steady state $(G_s, T_s)'$ is given by:

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} &=& D_G\left(J_{L_G} * u - u\right) + \left[\gamma_G(1-G_s) - \delta_G - \gamma_{TG}T_s - \lambda_{fG}f - \gamma_G G_s\right]u - \gamma_{TG}G_s \displaystyle\int_{-\infty}^{+\infty} \phi_M(x-y)v(y,t)dy, \\ \dfrac{\partial v}{\partial t} &=& D_T\left(J_{L_T} * v - v\right) + \left[(\gamma_T(1+2\Omega T_s)(1-T_s) - \gamma_T T_s(1+\Omega T_s) - \delta_T - \lambda_{fT}f\omega(G_s)\exp(-pT_s))\right]v \\ && + \left(p\lambda_{fT}f\omega(G_s)\exp(-pT_s)T_s\right)\displaystyle\int_{\mathbb{R}} \phi_M(x-y)v(y,t)dy - \lambda_{fT}f\omega'(G_s)\exp(-pT_s)T_s u. \end{array}\right. \tag{4.8}$$

We refer the reader to Tega II et al. [103] and Tian et al. [104] for the process of linearization. Let us set:

$$\begin{array}{rcl} a_{11} &=& \gamma_G G_e, \\ a_{12} &=& \gamma_{TG} G_e, \\ a_{22} &=& \gamma_T - \delta_T - \lambda_{fT} f\omega(G_e), \\ b_{11} &=& \gamma_G - (\delta_G + \lambda_{fG} f) + \gamma_{TG} T_2, \\ b_{22} &=& \gamma_T T_2 \sqrt{(1-\Omega)^2 + 4\Omega\left(1 - \dfrac{1}{\mathcal{R}_{T,0}}\right)}. \end{array}$$

4.3.1 Stability property of the grassland homogeneous steady state

By replacing in system (4.8) G_s and T_s respectively with G_e and 0, we get the following linearised system

$$\left\{\begin{array}{rcl}\dfrac{\partial u}{\partial t} & = & D_G\left(J_{L_G} * u - u\right) - a_{11}u - a_{12}\displaystyle\int_{-\infty}^{+\infty}\phi_M(x-y)v(y,t)dy,\\ \dfrac{\partial v}{\partial t} & = & D_T\left(J_{L_T} * v - v\right) + a_{22}v.\end{array}\right. \tag{4.9}$$

Then, by the Fourier analysis, we get the following eigenvalue problem for the system (4.9) by setting $u(x,t) = q_1\exp(\lambda t)\exp(ikx)$ and $v(x,t) = q_2\exp(\lambda t)\exp(ikx)$, where λ is the eigenvalue associated with the wavenumber $k \geq 0$, q_1 and q_2 are two positive constants:

$$\left\{\begin{array}{rcl}\lambda q_1 & = & D_G\left(\overline{J_{L_G}}(k)q_1 - q_1\right) - a_{11}q_1 - a_{12}\overline{\phi_M}(k)q_2,\\ \lambda q_2 & = & D_T\left(\overline{J_{L_T}}(k)q_2 - q_2\right) + a_{22}q_2,\end{array}\right. \tag{4.10}$$

where $\overline{J_{L_h}}(k) = \exp\left(-\dfrac{k^2L_h^2}{2}\right)$, $h = G, T$, $\overline{\phi_M}(k) = \dfrac{\sin kM}{kM}$ are the Fourier transforms of the functions J_{L_h}, ϕ_M respectively.

Now we are able to write the system in (4.10) in the matrix form:

$$\lambda\begin{pmatrix} q_1 \\ \\ q_2\end{pmatrix} = \begin{bmatrix} D_G\left(\overline{J_{L_G}}(k) - 1\right) - a_{11} & -a_{12}\overline{\phi_M}(k) \\ \\ 0 & D_T\left(\overline{J_{L_T}}(k) - 1\right) + a_{22}\end{bmatrix}\begin{pmatrix} q_1 \\ \\ q_2\end{pmatrix}, \tag{4.11}$$

and we denote

$$\mathbf{M}_k = \begin{bmatrix} D_G\left(\overline{J_{L_G}}(k) - 1\right) - a_{11} & -a_{12}\overline{\phi_M}(k) \\ \\ 0 & D_T\left(\overline{J_{L_T}}(k) - 1\right) + a_{22}\end{bmatrix}. \tag{4.12}$$

The set of eigenvalues of $\mathbf{M}_k$, is $sp(\mathbf{M}_k) = \left\{D_G\left(\overline{J_{L_G}}(k) - 1\right) - a_{11}; D_T\left(\overline{J_{L_T}}(k) - 1\right) + a_{22}\right\}$, and we have the following results:

Theorem 4.1. Assume that $\overline{J_{L_T}}(k) < 1 - \dfrac{a_{22}}{D_T}$ for all $k \in \mathbb{R}$, then the grassland homogeneous steady state is locally asymptotically stable (LAS).

Proof. If $\overline{J_{L_T}}(k) < 1 - \dfrac{a_{22}}{D_T}$, then the two eigenvalues values associated to $\mathbf{M}_k$ are always negatives, due to the fact that $\overline{J_{L_i}}(k) < 1$, $i = G, T$. □

Remark 4.1. If $\mathcal{R}_T < 1$, the grassland homogeneous steady state is LAS. In fact, in this case we have $a_{22} < 0$ and the condition in Theorem 4.1 is verified.

- If $\mathcal{R}_T > 1$, the grassland homogeneous steady state is unstable, because we can find $k_0 > 0$ such that $\overline{J_{L_T}}(k_0) > 1 - \dfrac{a_{22}}{D_T}$.

4.3.2 Stability property of the forest and savanna homogeneous steady states

Now we consider $u(x,t) = G(x,t)$ and $v(x,t) = T(x,t) - T_2$ be two perturbations around the forest homogeneous steady state. The system obtained after linearization is:

$$\left\{ \begin{array}{lcl} \dfrac{\partial u}{\partial t} & = & D_G \left(J_{L_G} * u - u \right) + b_{11} u, \\ \dfrac{\partial v}{\partial t} & = & D_T \left(J_{L_T} * v - v \right) - b_{22} v. \end{array} \right. \tag{4.13}$$

In the same manner like in the previous subsection, if we consider the eigenvalue problem of the system (4.13) then we obtain the following theorem:

Theorem 4.2. Assume that $\overline{J_{L_G}}(k) < 1 - \dfrac{b_{11}}{D_G}$ for all $k \in \mathbb{R}$, then the forest homogeneous steady state is LAS.

Proof. The proof is done like for Theorem 4.1, page 136. ☐

Remark 4.2. If $\mathcal{R}_F < 1$, the forest homogeneous steady state is LAS. In fact, in this case we have $b_{11} < 0$ and the condition in Theorem 4.2 is verified.

- If $\mathcal{R}_F > 1$, the grassland homogeneous steady state is unstable, because we can find $k_1 > 0$ such that $\overline{J_{L_G}}(k_1) > 1 - \dfrac{b_{11}}{D_G}$.

Proceeding in a similar way, we deduce conditions for the stability of the savanna homogeneous steady sate $(G^*, T^*)'$. Let us set:

$$\begin{array}{lcl} \mathcal{R}_1^* & = & \dfrac{\gamma_T \left[(1-\Omega) + 2\Omega T^* \right]}{p\lambda_{fT} f \omega(G^*) \exp(-pT^*)}, \\ \mathcal{R}_2^* & = & \dfrac{\gamma_{TG} \omega'(G^*)}{p\gamma_G \omega(G^*)}, \end{array} \tag{4.14}$$

then we have the following proposition:

Proposition 4.1. (Stability condition of a savanna steady state).
Assume $\mathcal{R}_1^* - \mathcal{R}_2^* > 1$, then for all $M \in \mathbb{R}_+$ the savanna homogeneous steady state $(G^*, T^*)'$ is LAS.

Proof. Let us set:

$$\begin{array}{lcl} g_{11} & = & -\gamma_G G^*, \\ g_{12} & = & -\gamma_{TG} G^*, \\ g_{21} & = & -\lambda_{fT} f \omega'(G^*) \exp(-pT^*) T^*, \\ g_{22} & = & -\gamma_T \left[(1-\Omega) T^* + 2\Omega (T^*)^2 \right] + p\lambda_{fT} f \omega(G^*) \exp(-pT^*) T^*, \\ c & = & p\lambda_{fT} f \omega(G^*) \exp(-pT^*) T^*, \\ \mathcal{R}_1^* & = & \dfrac{\gamma_T \left[(1-\Omega) + 2\Omega T^* \right]}{p\lambda_{fT} f \omega(G^*) \exp(-pT^*)}, \\ \mathcal{R}_2^* & = & \dfrac{\gamma_{TG} \omega'(G^*)}{p\gamma_G \omega(G^*)}. \end{array} \tag{4.15}$$

Recall that if $\mathcal{R}_1^* - \mathcal{R}_2^* > 1$ then $g_{11}g_{22} - g_{12}g_{21} > 0$ and $g_{22} < 0$ (see also Tega II et al. [103]).
We therefore define the following two functions:

$$\begin{aligned}
Tr(k,M) &= \Big(D_G\left(\overline{J_{L_G}}(k)-1\right)+g_{11}\Big)+\Big(D_T\left(\overline{J_{L_T}}(k)-1\right)+g_{22}+c\left(\overline{\phi_M}(k)-1\right)\Big),\\
Det(k,M) &= \Big(D_G\left(\overline{J_{L_G}}(k)-1\right)+g_{11}\Big)\times\Big(D_T\left(\overline{J_{L_T}}(k)-1\right)+g_{22}+c\left(\overline{\phi_M}(k)-1\right)\Big)\\
&\quad -g_{12}g_{21}\overline{\phi_M}(k),
\end{aligned} \tag{4.16}$$

where $k, M \in \mathbb{R}_+$, $\overline{J_{L_h}}(k)$ and $\overline{\phi_M}(k)$ are respectively the Fourier transform of J_{L_h} and ϕ_M, $h = G, T$. Moreover for a given value of M, if for all $k \in \mathbb{R}_+$, $Tr(k,M) < 0$ and $Det(k,M) > 0$ then the savanna homogeneous steady state $(G^*, T^*)'$ is stable.
Suppose that $\mathcal{R}_1^* - \mathcal{R}_2^* > 1$. Remark that $\overline{J_{L_h}}(k) \leq 1,\ \ h = G, T$ and $\overline{\phi_M}(k) \leq 1$ for all $k, M \in \mathbb{R}_+$.
Therefore:

$$Tr(k,M) = \overbrace{\Big(D_G\left(\overline{J_{L_G}}(k)-1\right)+g_{11}\Big)}^{<0}+\overbrace{\Big(D_T\left(\overline{J_{L_T}}(k)-1\right)+g_{22}+c\left(\overline{\phi_M}(k)-1\right)\Big)}^{<0} < 0,$$

and

$$\begin{aligned}
Det(k,M) &= \Big(D_G\left(\overline{J_{L_G}}(k)-1\right)+g_{11}\Big)\times\Big(D_T\left(\overline{J_{L_T}}(k)-1\right)+g_{22}+c\left(\overline{\phi_M}(k)-1\right)\Big)\\
&\quad -g_{12}g_{21}\overline{\phi_M}(k),\\
&= \Big(D_G\left(\overline{J_{L_G}}(k)-1\right)+g_{11}\Big)\times\Big(D_T\left(\overline{J_{L_T}}(k)-1\right)+c\left(\overline{\phi_M}(k)-1\right)\Big)\\
&\quad +g_{22}\Big(D_G\left(\overline{J_{L_G}}(k)-1\right)\Big)+g_{11}g_{22}-g_{12}g_{21}\overline{\phi_M}(k),\\
&\geq \overbrace{\Big(D_G\left(\overline{J_{L_G}}(k)-1\right)+g_{11}\Big)}^{<0}\times\overbrace{\Big(D_T\left(\overline{J_{L_T}}(k)-1\right)+c\left(\overline{\phi_M}(k)-1\right)\Big)}^{<0}\\
&\quad \underbrace{+\,g_{22}\Big(D_G\left(\overline{J_{L_G}}(k)-1\right)\Big)}_{\geq 0}+\underbrace{g_{11}g_{22}-g_{12}g_{21}}_{>0},\\
&> 0
\end{aligned}$$

Then for all values of $M \in \mathbb{R}_+$ $(G^*, T^*)'$ is LAS. □

In the next section we now address the issue of minimal wave speed by studying the existence of traveling waves solution for system (4.1).

4.4 Minimal wave speed

Our aim in this section is to derive by the study of the existence and non existence of a traveling wave solution for system (4.1) connecting the grassland homogeneous steady state $(G_e, 0)$ to the forest homogeneous steady state $(0, T_2)$, the minimal wave speed of the aforementioned traveling wave.

4.4.1 Preliminaries

Definition 4.1. A solution (G, T) of system (4.1) is called traveling wave solution if there exists a constant $c \in \mathbb{R}_+$ called the wave speed and a pair (φ_1, φ_2) of positive functions defined on $\mathbb{R}$ called the wave profiles such that:

$$G(x,t) = \varphi_1(z), \;\; T(x,t) = \varphi_2(z) \;\text{ where }\; z = x + ct. \tag{4.17}$$

By the Definition 4.1, we rewrite the system (4.1) as follows:

$$\left\{\begin{array}{lcl} c\varphi_1'(z) & = & D_G\left(J_{L_G} * \varphi_1(z) - \varphi_1(z)\right) + \gamma_G\varphi_1(z)\left(1 - \varphi_1(z)\right) - \delta_G\varphi_1(z) - \gamma_{TG}\varphi_1(z)\phi_M * \varphi_2(z) \\ & & -\lambda_{fG} f \varphi_1(z), \\ & & \\ c\varphi_2'(z) & = & D_T\left(J_{L_T} * \varphi_2(z) - \varphi_2(z)\right) + \gamma_T\varphi_2(z)\left(1 + \Omega\varphi_2(z)\right)\left(1 - \varphi_2(z)\right) - \delta_T\varphi_2(z) \\ & & -\lambda_{fT} f\omega(\varphi_1(z))\exp(-p\phi_M * \varphi_2(z))\varphi_2(z), \end{array}\right. \tag{4.18}$$

where the prime denotes differentiation with respect to z.

We are interested by the existence of $c > 0$ and (φ_1, φ_2) solution of the system (4.18) with the following asymptotic boundary conditions:

$$\lim_{z\to-\infty}(\varphi_1(z), \varphi_2(z)) = (G_e, 0) \;\text{ and }\; \lim_{z\to+\infty}(\varphi_1(z), \varphi_2(z)) = (0, T_2). \tag{4.19}$$

By the change of variables:

$$\tilde{\varphi}_1 = G_e - \varphi_1 \;\text{ and }\; \tilde{\varphi}_2 = \varphi_2, \tag{4.20}$$

system (4.18) and asymptotic boundary assumptions in (4.19) become:

$$\left\{\begin{array}{lcl} c\varphi_1'(z) & = & D_G\left(J_{L_G} * \varphi_1(z) - \varphi_1(z)\right) - \gamma_G\left(G_e - \varphi_1(z)\right)\varphi_1(z) + \gamma_{TG}\left(G_e - \varphi_1(z)\right)\phi_M * \varphi_2(z), \\ & & \\ c\varphi_2'(z) & = & D_T\left(J_{L_T} * \varphi_2(z) - \varphi_2(z)\right) + \gamma_T\varphi_2(z)\left(1 + \Omega\varphi_2(z)\right)\left(1 - \varphi_2(z)\right) - \delta_T\varphi_2(z) \\ & & -\lambda_{fT} f\omega(G_e - \varphi_1(z))\exp(-p\phi_M * \varphi_2(z))\varphi_2(z), \end{array}\right. \tag{4.21}$$

and

$$\lim_{z\to-\infty}(\varphi_1(z), \varphi_2(z)) = (0, 0) \;\text{ and }\; \lim_{z\to+\infty}(\varphi_1(z), \varphi_2(z)) = (G_e, T_2), \tag{4.22}$$

all of these by dropping the tilde in (4.21) and (4.22) for convenience.
We will get the existence of (φ_1, φ_2) by using a standard method combining the upper and lower solution with Schauder fixed point theorem. We refer the reader to Zhang and Li [130], Yu and Yuan [128], Zhao et al. [132], Ducrot et al. [33], Dong et al. [32], Hao et al. [49] for papers using similar method. Then we find in a first step the traveling wave solution connecting $(0,0)$ at $z = -\infty$, to a nontrivial state at $z = +\infty$, meaning that this traveling wave stays away from zero at $z = +\infty$. Precisely,

$$\liminf_{z\to+\infty} \varphi_1(z) > 0 \ \text{ and } \ \liminf_{z\to+\infty} \varphi_2(z) > 0, \tag{4.23}$$

and

$$\limsup_{z\to+\infty} \varphi_1(z) \leq G_e, \ \text{ and } \ \limsup_{z\to+\infty} \varphi_2(z) \leq T_2. \tag{4.24}$$

In a second step, for the asymptotic condition at $+\infty$, we shall use the condition of local stability of the forest homogeneous steady state $(0, T_2)$ for system (4.1).

Definition 4.2. The pairs of functions $(\overline{\varphi_1}, \overline{\varphi_2})$ and $(\underline{\varphi}_1, \underline{\varphi}_2)$ are respectively called upper and lower solutions of system (4.21) if they are bounded and satisfy the following inequalities:

$$\begin{cases} D_G\left(J_{L_G} * \overline{\varphi_1}(z) - \overline{\varphi_1}(z)\right) - \gamma_G(G_e - \overline{\varphi_1}(z))\overline{\varphi_1}(z) + \gamma_{TG}(G_e - \overline{\varphi_1}(z))\phi_M * \overline{\varphi_2}(z) - c\overline{\varphi_1}'(z) \leq 0, \\ \\ D_T\left(J_{L_T} * \overline{\varphi_2}(z) - \overline{\varphi_2}(z)\right) + \gamma_T\overline{\varphi_2}(z)(1 + \Omega\overline{\phi_2}(z))(1 - \overline{\varphi_2}(z)) - \delta_T\overline{\varphi_2}(z) \\ -\lambda_{fT} f\omega(G_e - \overline{\varphi_1}(z))\exp(-p\phi_M * \overline{\varphi_2}(z))\overline{\varphi_2}(z) - c\overline{\varphi_2}'(z) \leq 0, \\ \\ D_G\left(J_{L_G} * \underline{\varphi_1}(z) - \underline{\varphi_1}(z)\right) - \gamma_G(G_e - \underline{\varphi_1}(z))\underline{\varphi_1}(z) + \gamma_{TG}(G_e - \underline{\varphi_1}(z))\phi_M * \underline{\varphi_2}(z) - c\underline{\varphi_1}'(z) \geq 0, \\ \\ D_T\left(J_{L_T} * \underline{\varphi_2}(z) - \underline{\varphi_2}(z)\right) + \gamma_T\underline{\varphi_2}(z)(1 + \Omega\underline{\phi_2}(z))(1 - \underline{\varphi_2}(z)) - \delta_T\underline{\varphi_2}(z) \\ -\lambda_{fT} f\omega(G_e - \underline{\varphi_1}(z))\exp(-p\phi_M * \underline{\varphi_2}(z))\underline{\varphi_2}(z) - c\underline{\varphi_2}'(z) \geq 0, \end{cases} \tag{4.25}$$

for $z \in \mathbb{R}\backslash D$, with some finite set $D = \{z_1, ..., z_m\}$ where $\underline{\varphi_j}$ and $\overline{\varphi_j}$, $j = 1, 2$ are differentiable in $\mathbb{R}\backslash D$.

We also assume that a pair of upper and lower solutions of system (4.21) satisfies:

(C_1) $(0,0) \leq (\underline{\varphi_1}(z), \underline{\varphi_2}(z)) \leq (\overline{\varphi_1}(z), \overline{\varphi_2}(z)) \leq (G_e, T_2)$.

(C_2) $\sup_{z<s} \underline{\varphi_i}(z) \leq \inf_{z>s} \overline{\varphi_i}(z)$, $\forall i = 1, 2$ and $\inf_{z\in\mathbb{R}} \overline{\varphi_2}(z) > 0$.

We define :

$$\Delta_1(\lambda, c) = D_T\left(\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda y}dy - 1\right) - c\lambda + (\delta_T + \lambda_{fT} f\omega(G_e))(\mathcal{R}_T - 1). \tag{4.26}$$

We assume that $\mathcal{R}_T > 1$ and by a simple calculation for $\lambda \geq 0$, $c > 0$, we have:

(a) $\Delta_1(0,c) = (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)$ and $\lim_{\lambda \longrightarrow +\infty} \Delta_1(\lambda, c) = +\infty$.

(b) $\dfrac{\partial \Delta_1(\lambda,c)}{\partial c} = -\lambda < 0$ and $\dfrac{\partial \Delta_1(0,c)}{\partial \lambda} = -c < 0$.

(c) $\dfrac{\partial^2 \Delta_1(\lambda,c)}{\partial \lambda^2} > 0$.

(a), (b) and (c) help us to get the following Lemma 4.1.

Lemma 4.1. There exists a unique positive pair of (λ_T^*, c_1^*) such that:

$$\Delta_1(\lambda_T^*, c_1^*) = 0 \ \text{ and } \ \frac{\partial \Delta_1(\lambda_T^*, c_1^*)}{\partial \lambda} = 0.$$

Furthermore,

(i) if $c > c_1^*$, then there exists two positive real numbers $\lambda_{11} = \lambda_{11}(c)$ and $\lambda_{12} = \lambda_{12}(c)$, with $0 < \lambda_{11} < \lambda_T^* < \lambda_{12}$, such that:

$$\Delta_1(\lambda, c) \begin{cases} = 0 & \text{for} \quad \lambda = \lambda_{11} = \lambda_{12}, \\ < 0 & \text{for} \quad \lambda \in (\lambda_{11}, \lambda_{12}), \\ > 0 & \text{for} \quad \lambda \in (0, \lambda_{11}) \cup (\lambda_{12}, +\infty). \end{cases}$$

(ii) If $0 < c < c_1^*$, then $\Delta_1(\lambda, c) > 0$ for all $\lambda \in (0, +\infty)$.

(iii)

$$c_1^* = \min_{\lambda > 0} \frac{D_T\left(\displaystyle\int_{\mathbb{R}} J_{L_T}(y) e^{-\lambda y} dy - 1\right) + (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)}{\lambda}. \tag{4.27}$$

Proof. At first, remark that:

$$\Delta_1(\lambda, c) = 0 \Longleftrightarrow c = \frac{1}{\lambda}\left(D_T\left(\int_{\mathbb{R}} J_{L_T}(y) e^{-\lambda y} dy - 1\right) + (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)\right), \tag{4.28}$$

and

$$\frac{\partial \Delta_1(\lambda, c)}{\partial \lambda} = D_T\left(\int_{\mathbb{R}} (-y) J_{L_T}(y) e^{-\lambda y} dy\right) - c. \tag{4.29}$$

Using the fact that:

$$\lim_{\lambda \to +\infty} \frac{\partial \Delta_1(\lambda, c)}{\partial \lambda} = +\infty, \ \ \frac{\partial \Delta_1(0, c)}{\partial \lambda} < 0, \ \text{ and } \ \frac{\partial^2 \Delta_1(\lambda, c)}{\partial \lambda^2} > 0,$$

by the intermediate values theorem there exist an unique $\lambda_T^* \in (0, +\infty)$ such that:

$$\frac{\partial \Delta_1(\lambda_T^*, c)}{\partial \lambda} = 0, \ \ \forall c > 0. \tag{4.30}$$

Inserting λ_T^* in (4.28) and by (4.30), we will get:

$$c = c_1^* = \frac{1}{\lambda_T^*}\left(D_T\left(\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda_T^* y}dy - 1\right) + (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)\right), \text{ and } \begin{cases} \Delta_1(\lambda_T^*, c_1^*) & = 0, \\ \dfrac{\partial\Delta(\lambda_T^*, c_1^*)}{\partial\lambda} & = 0. \end{cases}$$

At second, by (4.30) and knowing that $\dfrac{\partial^2\Delta_1(\lambda, c)}{\partial\lambda^2} > 0$ (this by the condition (c) in page 141), we reach to the conclusion that $\Delta_1(\lambda_T^*, c) = \min_{\lambda>0}\Delta_1(\lambda, c)$.
Furthermore due to the fact that $\dfrac{\partial\Delta_1(\lambda_T^*, c)}{\partial c} < 0$, we get the following cases :

- for $c > c_1^*$, $\Delta_1(\lambda_T^*, c) < \Delta_1(\lambda_T^*, c_1^*) = 0$. Then, with:

$$\Delta_1(0, c) > 0 \text{ and } \lim_{\lambda\to+\infty}\Delta_1(\lambda, c) = +\infty,$$

and using the intermediate values theorem in $(0, \lambda_T^*]$ and $[\lambda_T^*, +\infty)$, there exist λ_{11} and λ_{12}, $(\lambda_{11} < \lambda_T^* < \lambda_{12})$ such that $\Delta_1(\lambda_{11}, c) = \Delta_1(\lambda_{12}, c) = 0$. On the above, we get :

$$\Delta_1(\lambda, c)\begin{cases} < 0 & \text{for} \quad \lambda \in (\lambda_{11}, \lambda_{12}), \\ > 0 & \text{for} \quad \lambda \in (0, \lambda_{11}) \cup (\lambda_{12}, +\infty). \end{cases}$$

- for $c < c_1^*$, $\Delta_1(\lambda_T^*, c) > \Delta_1(\lambda_T^*, c_1^*) = 0$, and due to the fact that $\Delta_1(\lambda_T^*, c) = \min_{\lambda>0}\Delta_1(\lambda, c)$, we get $\Delta_1(\lambda, c) > 0, \ \forall\lambda \in (0; +\infty)$.

At third, we consider the following function of λ defines on $(0; +\infty)$ by:

$$\Phi(\lambda) = \frac{1}{\lambda}\left[D_T\left(\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda y}dy - 1\right) + (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)\right]. \tag{4.31}$$

Then $\Delta_1(\lambda, c) = \lambda\,[\Phi(\lambda) - c]$. We have: $\lim_{\lambda\to 0}\Phi(\lambda) = +\infty$ and $\lim_{\lambda\to+\infty}\Phi(\lambda) = +\infty$. Moreover,

$$\frac{\partial\Delta_1(\lambda, c)}{\partial\lambda} = (\Phi(\lambda) - c) + \lambda\Phi'(\lambda). \tag{4.32}$$

then:

$$\begin{cases} \Delta_1(\lambda, c) & = 0, \\ \\ \dfrac{\partial\Delta_1(\lambda, c)}{\partial\lambda} & = 0, \end{cases} \Leftrightarrow \begin{cases} c = \Phi(\lambda), \\ \\ \Phi'(\lambda) = 0. \end{cases} \tag{4.33}$$

λ_T^* is therefore the unique critic point of Φ on $(0; +\infty)$. Moreover, due to the fact that J_{L_T} is symmetric, compactly supported and $\int_{\mathbb{R}} J_{L_T}(x)dx = 1$ we have:

$$\begin{aligned}
\Phi(\lambda) &= \tfrac{1}{\lambda}\left[D_T\Big(\int_{\mathbb{R}} J_{L_T}(y)e^{\lambda y}dy - 1\Big) + (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)\right], \\
&= \frac{D_T\left(\displaystyle\int_{\mathbb{R}} J_{L_T}(y)\sum_{k=0}^{+\infty}\frac{(\lambda y)^k}{k!}dy - 1\right) + (\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)}{\lambda}, \\
&= D_T\textstyle\sum_{k=1}^{+\infty}\left(\dfrac{\lambda^{2k-1}\displaystyle\int_{\mathbb{R}} J_{L_T}(y)y^{2k}dy}{(2k)!}\right) + \dfrac{(\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1)}{\lambda}.
\end{aligned}$$

Then the second derivative of $\Phi(\lambda)$ is:

$$\Phi''(\lambda) = D_T\sum_{k=1}^{+\infty}\left((2k-1)(2k-2)\lambda^{2k-3}\frac{\displaystyle\int_{\mathbb{R}} J_{L_T}(y)y^{2k}dy}{(2k)!}\right) + \frac{2}{\lambda^3}\,(\delta_T + \lambda_{fT} f\omega(G_e))\,(\mathcal{R}_T - 1) > 0. \tag{4.34}$$

Consequently $\Phi''(\lambda_T^*) > 0$, thus λ_T^* is the unique point where of the minimum of $\Phi(\lambda)$ occurs in $(0; +\infty)$. Then:

$$c_1^* = \Phi(\lambda_T^*) = \min_{\lambda>0}\Phi(\lambda).$$

□

For any $\lambda \geq 0$, we define :

$$A(\lambda) = D_G\Big(\int_{\mathbb{R}} J_{L_G}(y)e^{-\lambda y}dy - 1\Big) - c\lambda. \tag{4.35}$$

Proposition 4.2. There exist $\overline{\lambda} \in (0; \lambda_{11})$ such that $\mathcal{A}(\overline{\lambda}) < 0$.

Proof. By a simple calculation, we have: $\mathcal{A}(0) = 0$ and

$$\mathcal{A}'(0) = \lim_{\lambda\to 0} D_G\Big(\int_{\mathbb{R}}(-y)J_{L_T}(y)e^{-\lambda y}dy - 1\Big) - c = -c < 0.$$

Then, there exist $\overline{\lambda} \in (0; \lambda_{11})$ such that $\mathcal{A}(\overline{\lambda}) = D_G\Big(\int_{\mathbb{R}} J_{L_T}(y)e^{-\overline{\lambda} y}dy - 1\Big) - c\overline{\lambda} < 0.$ □

4.4.2 Upper and Lower solution of system (4.21)

Assume that $c > c_1^*$ and let defines the followings continuous functions:

$$\overline{\varphi_1}(z) = \min(G_e e^{\overline{\lambda}(z-z_0)}, G_e) \quad \text{and} \quad \overline{\varphi_2}(z) = \min(T_2 e^{\lambda_{11} z}, T_2),$$

$$\underline{\varphi_1}(z) = 0 \quad \text{and} \quad \underline{\varphi_2}(z) = \begin{cases} 0 \ \text{ if } \ z \geq z_2, \\ \\ T_2\biggl(e^{\lambda_{11} z} - q e^{\eta \lambda_{11} z}\biggr) \ \text{ if } \ z < z_2, \end{cases}$$

with $z_2 = \dfrac{\log(q)}{\lambda_{11}(1-\eta)}$ and:

$$\eta \in \left(1, \min\left(2; \frac{\lambda_{12}}{\lambda_{11}}\right)\right) \ \text{ and } \ q = 1 + \frac{\gamma_T}{-\Delta(\eta\lambda_{11}, c)}. \tag{4.36}$$

We choose $z_0 < 0$ such that:

$$\left(\frac{\gamma_{TG} T_2}{\gamma_G G_e} \int_{\mathbb{R}} \phi_{M_2}(y) e^{-\lambda_{11} y} dy\right) e^{(\lambda_{11} - \bar{\lambda}) z_0} < 1. \tag{4.37}$$

Therefore, we can observe that:

$$\left(\gamma_{TG} T_2 \int_{\mathbb{R}} \phi_{M_2}(y) e^{-\lambda y} dy\right) e^{\lambda_{11} z} < \gamma_G G_e e^{\bar{\lambda} z} \ \text{ for all } \ z < z_0. \tag{4.38}$$

Lemma 4.2. The pair $(\underline{\varphi_1}(z), \underline{\varphi_2}(z))$ is a lower solution for system (4.21).

Proof. Recall that: $\underline{\varphi_1}(z) = 0$ and $\underline{\varphi_2}(z) = \begin{cases} 0 \ \text{ if } \ z \geq z_2, \\ \\ T_2\biggl(e^{\lambda_{11} z} - q e^{\eta \lambda_{11} z}\biggr) \ \text{ if } \ z < z_2 \end{cases}$.

Set:

$$\begin{aligned} A_1 &= D_G\biggl(J_{L_G} * \underline{\varphi_1}(z) - \underline{\varphi_1}(z)\biggr) - \gamma_G\left(G_e - \underline{\varphi_1}(z)\right)\underline{\varphi_1}(z) + \gamma_{TG}\left(G_e - \underline{\varphi_1}(z)\right)\phi_M * \underline{\varphi_2}(z) - c\underline{\varphi_1}'(z), \\ A_2 &= D_T\biggl(J_{L_T} * \underline{\varphi_2}(z) - \underline{\varphi_2}(z)\biggr) + \gamma_T \underline{\varphi_2}(z)(1 + \Omega\underline{\varphi_2}(z))(1 - \underline{\varphi_2}(z)) - \delta_T \underline{\varphi_2}(z) \\ &\quad -\lambda_{fT} f \omega(G_e)\exp(-p\phi_M * \underline{\varphi_2}(z))\underline{\varphi_2}(z) - c\underline{\varphi_2}'(z). \end{aligned}$$

We have to prove that $A_1 \geq 0$ and $A_2 \geq 0$.
If $z \geq z_2$, then $\underline{\varphi_1}(z) = \underline{\varphi_2}(z) = 0$, and it is easy to show that $A_1 \geq 0$ and $A_2 \geq 0$.
If $z \leq z_2$, then $\underline{\varphi_1}(z) = 0$ and $\underline{\varphi_2}(z) = T_2\biggl(e^{\lambda_{11} z} - q e^{\eta\lambda_{11} z}\biggr)$.
Therefore we have, knowing that $\underline{\varphi_2}(z) \leq T_2 e^{\lambda_{11} z}$:

$A_1 = \gamma_{TG} G_e \phi_M * \underline{\varphi_2}(z) \geq 0.$

$$\begin{aligned}
A_2 &\geq D_T\Bigg(J_{L_T} * \underline{\varphi_2}(z) - \underline{\varphi_2}(z)\Bigg) + \gamma_T \underline{\varphi_2}(z)(1 - \underline{\varphi_2}(z)) - \delta_T \underline{\varphi_2}(z) - \lambda_{fT} f\omega(G_e)\underline{\varphi_2}(z) - c\underline{\varphi_2}'(z), \\
&= D_T\Bigg(J_{L_T} * \underline{\varphi_2}(z) - \underline{\varphi_2}(z)\Bigg) + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)\underline{\varphi_2}(z) - \gamma_T \left(\underline{\varphi_2}(z)\right)^2 - c\underline{\varphi_2}'(z), \\
&\geq T_2 e^{\lambda_{11} z}\left[D_T\left(\int_{\mathbb{R}} J_{L_T}(y) e^{-\lambda_{11} y} dy - 1\right) - c\lambda_{11} + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)\right] - \\
&\quad qT_2 e^{\eta\lambda_{11} z}\left[D_T\left(\int_{\mathbb{R}} J_{L_T}(y) e^{-\eta\lambda_{11} y} dy - 1\right) - c\lambda_{11} + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)\right] - \gamma_T T_2 e^{2\lambda_{11} z}, \\
&= T_2 e^{\lambda_{11} z}\Delta_1(\lambda_{21}, c) - qT_2 e^{\eta\lambda_{11} z}\Delta_1(\eta\lambda_{21}, c) - \gamma_T T_2 e^{2\lambda_{21} z}, \\
&= T_2 e^{\eta\lambda_{11} z}\left(-q\Delta_1(\eta\lambda_{11}, c) - \gamma_T e^{(2-\eta)\lambda_{11} z}\right), \\
&\geq T_2 e^{\eta\lambda_{11} z}\left(-q\Delta(\eta\lambda_{11}, c) - \gamma_T\right).
\end{aligned}$$

By the choice of q in (4.36) we get:
$A_2 \geq -T_2 e^{\eta\lambda_{21} z}\Delta(\eta\lambda_{21}, c) \geq 0,$ □

For a given $\lambda > 0$ and $a \leq 0$ we define the following functions:

$$I_a(\lambda) = \Omega\gamma_T T_2 - \gamma_T T_2(1 + \Omega T_2)e^{a\lambda} + \lambda_{fT} f\omega(G_e)e^{-a\lambda}. \tag{4.39}$$

Proposition **4.3.** Assume that $\mathcal{R}_{T_2} > 1$. There exist $\tilde{\lambda_2} > 0$ such that for all $\lambda \in (0, \tilde{\lambda_2})$, $I_a(\lambda) < 0$.

Proof. Assuming that $\mathcal{R}_{T_2} > 1$, it suffice to remark that: $I_a(0) = \lambda_{fT} f\omega(G_e)\left(1 - \mathcal{R}_{T_2}\right) \leq 0$. □

Lemma **4.3.** Assume that $\mathcal{R}_{T_2} > 1$ and $\lambda_{11} \in \left(0, \tilde{\lambda_2}\right)$, then the pair $(\overline{\varphi_1}(z), \overline{\varphi_2}(z))$ is an upper solution for system (4.21).

Proof. Set:

$$B_1 = D_G\left(J_{L_G} * \overline{\varphi_1}(z) - \overline{\varphi_1}(z)\right) - \gamma_G(G_e - \overline{\varphi_1}(z))\overline{\varphi_1}(z) + \gamma_{TG}(G_e - \overline{\varphi_1}(z))\phi_M * \overline{\varphi_2}(z) - c\overline{\varphi_1}'(z),$$

$$\begin{aligned}
B_2 = &\ D_T\left(J_{L_T} * \overline{\varphi}_2(z) - \overline{\varphi}_2(z)\right) + \gamma_T \overline{\varphi_2}(z)(1 + \Omega\overline{\varphi_2}(z)(1 - \overline{\varphi_2}) - \delta_T \overline{\varphi_2}(z) \\
&-\lambda_{fT} f\omega(G_e - \overline{\varphi_1}(z))\exp(-p\phi_M * \overline{\varphi_2}(z))\overline{\varphi_2}(z) - c\overline{\varphi_2}'(z).
\end{aligned}$$

We have to prove that $B_1 \leq 0$ and $B_2 \leq 0$.
If $z > 0$, $\overline{\varphi_1}(z) = G_e$ and $\overline{\varphi_2}(z) = T_2$ and it is easy to show that $B_1 \leq 0$ and $B_2 \leq 0$.

Similarly, if $z < 0$ and $z > z_0$, $\overline{\varphi_1}(z) = G_e$ and $\overline{\varphi_2}(z) = T_2 e^{\lambda_{11} z}$ and it is easy to show that

$B_1 \leq 0.$

$$\begin{aligned}
B_2 &= D_T\left(J_{L_T} * \overline{\varphi}_2(z) - \overline{\varphi}_2(z)\right) + \gamma_T\overline{\varphi_2}(z)(1+\Omega\overline{\varphi_2}(z))(1-\overline{\varphi_2}) - \delta_T\overline{\varphi_2}(z),\\
&= D_T\left(J_{L_T} * \overline{\varphi}_2(z) - \overline{\varphi}_2(z)\right) + \gamma_T\overline{\varphi_2}(z)(1+\Omega\overline{\varphi_2}(z))(1-\overline{\varphi_2}) - \delta_T\overline{\varphi_2}(z) - \lambda_{fT}f\omega(G_e)\overline{\varphi_2}(z)\\
&\quad +\lambda_{fT}f\omega(G_e)\overline{\varphi_2}(z) - c\overline{\varphi_2}'(z),\\
&= T_2e^{\lambda_{11}z}\left(D_T\left(\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda_{11}y}dy - 1\right) - c\lambda_{11} + \gamma_T - \delta_T - \lambda_{fT}f\omega(G_e) + \Omega\gamma_T T_2e^{\lambda_{11}z}\right)\\
&\quad +T_2e^{\lambda_{11}z}\left(-\gamma_T T_2e^{\lambda_{11}z}(1+\Omega T_2e^{\lambda_{11}z}) + \lambda_{fT}f\omega(G_e)\right),
\end{aligned}$$

and then,

$$\begin{aligned}
B_2 &= T_2e^{\lambda_{11}z}\left[\Delta_1(\lambda_{11},c) + \Omega\gamma_T T_2e^{\lambda_{11}z} - \gamma_T T_2e^{\lambda_{11}z}\left(1+\Omega T_2e^{\lambda_{11}z}\right) + \lambda_{fT}f\omega(G_e)\right],\\
&= T_2e^{\lambda_{11}z}\left(\Delta_1(\lambda_{11},c) + e^{\lambda_{11}z}I_z(\lambda_{11})\right),\\
&\leq 0 \text{ due to the fact that } \Delta_1(\lambda_{11},c) = 0 \text{ and } I_z(\lambda_{11}) \leq 0.
\end{aligned}$$

If $z < 0$ and $z < z_0$, $\overline{\varphi_1}(z) = G_e e^{\bar{\lambda}(z-z_0)}$ and $\overline{\varphi_2}(z) = T_2e^{\lambda_{11}z}$; Then:

$$\begin{aligned}
D_G\left(J_{L_G} * \overline{\varphi_1}(z) - \overline{\varphi_1}(z)\right) - c\overline{\varphi_1}'(z) &= D_G\left(\int_{\mathbb{R}} J_{L_G}(y)\overline{\varphi_1}(z-y)dy - G_e e^{\bar{\lambda}(z-z_0)}\right),\\
&= G_e e^{\bar{\lambda}(z-z_0)}\left[D_G\left(\int_{\mathbb{R}} J_{L_G}(y)e^{-\bar{\lambda}y}dy - 1\right) - c\bar{\lambda}\right],\\
&= G_e e^{\bar{\lambda}(z-z_0)}A(\bar{\lambda}) < 0.
\end{aligned}$$

We also have for $z < z_0$,

$$\begin{aligned}
(G_e - \overline{\varphi_1}(z))\left(\gamma_{TG}\phi_{M_2} * \overline{\varphi_2}(z) - \gamma_G\overline{\varphi_1}(z)\right) &= (G_e - \overline{\varphi_1}(z))\Bigg(\gamma_{TG}T_2e^{\lambda_{11}z}\int_{\mathbb{R}}\phi_{M_2}(y)e^{-\lambda y}dy\\
&\quad -\gamma_G G_e e^{\bar{\lambda}(z-z_0)}\Bigg),\\
&\leq (G_e - \overline{\varphi_1}(z))\left(\gamma_G G_e e^{\bar{\lambda}z} - \gamma_G G_e e^{\bar{\lambda}(z-z_0)}\right),\\
&= \gamma_G G_e e^{\bar{\lambda}z}(G_e - \overline{\varphi_1}(z))\left(1 - e^{-\bar{\lambda}z_0}\right),\\
&\leq 0.
\end{aligned}$$

Therefore $B_1 \leq 0$. In the same vain,

$$
\begin{aligned}
B_2 \leq\ & D_T \left(J_{L_T} * \overline{\varphi}_2(z) - \overline{\varphi}_2(z)\right) + \gamma_T \overline{\varphi_2}(z)(1+\Omega\overline{\varphi_2}(z))(1-\overline{\varphi_2}) - \delta_T \overline{\varphi_2}(z) - \lambda_{fT} f\omega(G_e)\overline{\varphi_2}(z) \\
& + \lambda_{fT} f\omega(G_e)\overline{\varphi_2}(z) - c\overline{\varphi_2}'(z), \\
=\ & T_2 e^{\lambda_{11} z}\left(D_T\left(\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda_{11} y}dy - 1\right) - c\lambda_{11} + \gamma_T - \delta_T - \lambda_{fT} f\omega(G_e) + \Omega\gamma_T T_2 e^{\lambda_{11} z}\right) \\
& + T_2 e^{\lambda_{11} z}\left(\gamma_T T_2 e^{\lambda_{11} z}(1+\Omega T_2 e^{\lambda_{11} z}) + \lambda_{fT} f\omega(G_e)\right), \\
=\ & T_2 e^{\lambda_{11} z}\left[\Delta_1(\lambda_{11}, c) + \Omega\gamma_T T_2 e^{\lambda_{11} z} - \gamma_T T_2 e^{\lambda_{11} z}\left(1+\Omega T_2 e^{\lambda_{11} z}\right) + \lambda_{fT} f\omega(G_e)\right], \\
=\ & T_2 e^{\lambda_{11} z}\left(\Delta_1(\lambda_{11}, c) + e^{\lambda_{11} z} I_z(\lambda_{11})\right), \\
\leq\ & 0 \ \text{ due to the fact that } \ \Delta_1(\lambda_{11}, c) = 0 \ \text{ and } \ I_z(\lambda_{11}) \leq 0.
\end{aligned}
$$

□

4.4.3 Existence of traveling wave for the model

4.4.3.1 The case $c > c_1^*$

Let denote by

$$X = \left\{\Phi = (\varphi_1, \varphi_2) : \Phi \ \text{ is bounded and uniformly continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\right\}. \qquad (4.40)$$

Note that X is a Banach space equipped with the standard sup norm (Pan et al. [76], Pan [73]). Let $||\cdot||$ denotes the norm in $\mathbb{R}^2$, defined by $||(u,v)|| = \max\left(|u|, |v|\right), \ \text{ for all } \ (u,v) \in \mathbb{R}^2$.
Set

$$X_{T_2}^{G_e} = \left\{(\varphi_1, \varphi_2) \in X : 0 \leq \varphi_1(z) \leq G_e \ \text{ and } \ 0 \leq \varphi_2(z) \leq T_2 \ \text{ for all } z \in \mathbb{R}\right\}. \qquad (4.41)$$

Consider $(\overline{\varphi}_1, \overline{\varphi}_2)$, $(\underline{\varphi}_1, \underline{\varphi}_2)$ the upper and lower solutions of system (4.21). We have $\underline{\varphi}_1 \leq \overline{\varphi}_1, \underline{\varphi}_2 \leq \overline{\varphi}_2$ and

$$\text{for } \ z \in \mathbb{R} \ \ \underline{\varphi}_i(z), \ \ \overline{\varphi}_i(z) \in X_{T_2}^{G_e}, \ \ i = 1, 2. \qquad (4.42)$$

Let us define on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, the two following continuous functions F_1 and F_2:

$$
\left\{
\begin{aligned}
F_1(y_1, y_2) &= -\gamma_G\left(G_e - y_1\right)y_1 + \gamma_{TG}\left(G_e - y_1\right)y_2, \\
F_2(y_1, y_2, y_3) &= \gamma_T y_2(1+\Omega y_2)(1-y_2) - \delta_T y_2 - \lambda_{fT}\omega(G_e - y_1)\exp(-p y_3)y_2.
\end{aligned}
\right. \qquad (4.43)
$$

We also defines the two following operators $\mathbf{H} = (H_1, H_2)$ and $\mathbf{P} = (P_1, P_2)$, defines on $X_{T_2}^{G_e}$ by:

$$\begin{cases} H_1(\varphi_1, \varphi_2)(z) &= D_G(J_{L_G} * \varphi_1(z) - \varphi_1(z)) + F_1(\varphi_1, \phi_M * \varphi_2)(z) + \beta\varphi_1(z), \\ \\ H_2(\varphi_1, \varphi_2)(z) &= D_T(J_{L_T} * \varphi_2(z) - \varphi_2(z)) + F_2(\varphi_1, \varphi_2, \phi_M * \varphi_2)(z) + \beta\varphi_2(z), \end{cases} \tag{4.44}$$

where

$$\beta > \max\left(D_G + \gamma_G G_e + \gamma_{TG} T_2; D_T + \gamma_T T_2(1 + \Omega T_2) + \lambda_{fT} f\right), \tag{4.45}$$

and

$$\begin{cases} P_1(\varphi_1, \varphi_2)(z) = \dfrac{1}{c}\displaystyle\int_{-\infty}^{z} e^{-\dfrac{\beta(z-y)}{c}} H_1(\varphi_1, \varphi_2)(y)dy, \\ \\ P_2(\varphi_1, \varphi_2)(z) = \dfrac{1}{c}\displaystyle\int_{-\infty}^{z} e^{-\dfrac{\beta(z-y)}{c}} H_2(\varphi_1, \varphi_2)(y)dy. \end{cases} \tag{4.46}$$

Proposition 4.4. A fixed point of $\mathbf{P}$ corresponds to a solution of system (4.21).

Proof. Let $(\varphi_1, \varphi_2) \in X_{T_2}^{G_e}$ such that $\mathbf{P}(\varphi_1, \varphi_2) = (\varphi_1, \varphi_2)$. Then for all $z \in \mathbb{R}$ we have

$$P_1(\varphi_1, \varphi_2)(z) = \varphi_1(z) \text{ and } P_2(\varphi_1, \varphi_2)(z) = \varphi_2(z). \tag{4.47}$$

With the first equality in (4.47) we have:

$$\begin{aligned} P_1(\varphi_1, \varphi_2)(z) = \varphi_1(z) \iff & \tfrac{1}{c}\int_{-\infty}^{z} e^{\frac{-\beta(z-y)}{c}} H_1(\varphi_1, \varphi_2)(y)dy = \varphi_1(z), \\ \iff & \int_{-\infty}^{z} e^{\frac{\beta y}{c}} H_1(\varphi_1, \varphi_2)(y)dy = e^{\frac{\beta z}{c}} c\varphi_1(z), \\ \iff & e^{\frac{\beta z}{c}} H_1(\varphi_1, \varphi_2)(z) = e^{\frac{\beta z}{c}} \left(\beta\varphi_1(z) + c\varphi_1'(z)\right), \\ \iff & H_1(\varphi_1, \varphi_2)(z) = \beta\varphi_1(z) + c\varphi_1'(z), \\ \iff & D_G(J_{L_G} * \varphi_1(z) - \varphi_1(z)) + F_1(\varphi_1, \phi_{M_1} * \varphi_1, \phi_M * \varphi_2)(z) = c\varphi_1'(z). \end{aligned}$$

In the same way, for the second equality in (4.47) we get:

$$P_2(\varphi_1, \varphi_2)(z) = \varphi_2(z) \iff D_T(J_{L_T} * \varphi_2(z) - \varphi_2(z)) + F_2(\varphi_1, \varphi_2, \phi_M * \varphi_2)(z) = c\varphi_1'(z),$$

and then (φ_1, φ_2) is a solution of system (4.21). □

Let α be a small constant such that $0 < \alpha < \dfrac{\beta}{c}$, and

$$B_\alpha(\mathbb{R},\mathbb{R}^2) = \left\{ (\varphi_1,\varphi_2) \in X_{T_2}^{G_e} \ : \sup_{z\in\mathbb{R}} \max\left(|\varphi_1(z)|, |\varphi_2(z)|\right) e^{-\alpha|z|} < \infty \right\}. \tag{4.48}$$

Then, the following result is valid.

Proposition 4.5. (see Ducrot et al. [33]) $B_\alpha(\mathbb{R},\mathbb{R}^2)$ is a Banach space equipped with the norm $|\cdot|_\alpha$, given by:

$$|(\varphi_1,\varphi_2)|_\alpha := \sup_{z\in\mathbb{R}} \max\left(|\varphi_1(z)|, |\varphi_2(z)|\right) e^{-\alpha|z|}. \tag{4.49}$$

Let us consider the $\Gamma \subset B_\alpha(\mathbb{R},\mathbb{R}^2)$ and defined by:

$$\Gamma = \left\{ (\varphi_1,\varphi_2) \in X, \underline{\varphi}_i(z) \leq \varphi_i(z) \leq \overline{\varphi}(z),\ \ \forall z\in\mathbb{R}, i=1,2 \right\}. \tag{4.50}$$

Lemma 4.4. Γ is non empty, bounded, convex and closed with respect to the norm $|\cdot|_\alpha$.

Proof.• Γ is non empty because $\left(\dfrac{G_e}{1+e^{-\bar{\lambda}(z-z_0)}}; \dfrac{T_2}{1+e^{-\lambda_{11}z}} \right) \in \Gamma$.

- Γ is bounded. In fact for any $\varphi = (\varphi_1,\varphi_2) \in \Gamma$, we have: $|\varphi|_\alpha = \sup_{z\in\mathbb{R}} \max\left(|\varphi_1(z)|, |\varphi_2(z)|\right) e^{-\alpha|z|}$ and then:

$$\begin{aligned} |\varphi|_\alpha &\leq \sup\nolimits_{z\in\mathbb{R}} \left(\overline{\varphi}_1(z) + \overline{\varphi}_2(z)\right) e^{-\alpha|z|}, \\ &\leq \sup\nolimits_{z\in\mathbb{R}} \left(\overline{\varphi}_1(z) + \overline{\varphi}_2(z)\right), \\ &\leq G_e + T_2, \\ &\leq 2. \end{aligned}$$

- Clearly Γ is convex.
- Γ is closed. In fact, assume that $(\varphi_n,\phi_n) \in \Gamma$ such that $\lim_{n\to+\infty}(\varphi_n,\phi_n) = (\varphi,\phi)$, then: for all $n\in\mathbb{N}$,we have:

$$\underline{\varphi}_1(z)e^{-\alpha|z|} \leq \varphi_n(z)e^{-\alpha|z|} \leq \overline{\varphi}_1(z)e^{-\alpha|z|} \text{ and } \underline{\varphi}_2(z)e^{-\alpha|z|} \leq \phi_n(z)e^{-\alpha|z|} \leq \overline{\varphi}_2(z)e^{-\alpha|z|} \ \ z\in\mathbb{R}. \tag{4.51}$$

Moreover, for $\varepsilon > 0$, there exits $N_\varepsilon \in \mathbb{N}$ such that:

$$\begin{aligned} n \geq N_\varepsilon \ &\Longrightarrow\ |(\varphi_n,\phi_n) - (\varphi,\phi)|_\alpha < \varepsilon, \\ &\Longrightarrow\ \sup\nolimits_{z\in\mathbb{R}} \left\{ \max\left(|\varphi_n(z) - \varphi(z)|e^{-\alpha|z|}; |\phi_n(z) - \phi(z)|e^{-\alpha|z|} \right) \right\} < \varepsilon, \\ &\Longrightarrow\ \forall z\in\mathbb{R},\ |\varphi_n(z)e^{-\alpha|z|} - \varphi(z)e^{-\alpha|z|}| < \varepsilon \text{ and } |\phi_n(z)e^{-\alpha|z|} - \phi(z)e^{-\alpha|z|}| < \varepsilon. \end{aligned}$$

Consequently the sequence of functions $\varphi_n(z)e^{-\alpha|z|}$, converge uniformly on $\mathbb{R}$ to $\varphi(z)e^{-\alpha|z|}$. Similarly, $\phi_n(z)e^{-\alpha|z|}$, converge uniformly on $\mathbb{R}$ to $\phi(z)e^{-\alpha|z|}$. Taking the limit $n\to+\infty$ in (4.51) we have:

$$\underline{\varphi}_1(z)e^{-\alpha|z|} \leq \varphi(z)e^{-\alpha|z|} \leq \overline{\varphi}_1(z)e^{-\alpha|z|} \text{ and } \underline{\varphi}_2(z)e^{-\alpha|z|} \leq \phi(z)e^{-\alpha|z|} \leq \overline{\varphi}_2(z)e^{-\alpha|z|}, \tag{4.52}$$

therefore:

$$\underline{\varphi}_1(z) \leq \varphi(z) \leq \overline{\varphi}_1(z) \text{ and } \underline{\varphi}_2(z) \leq \phi(z) \leq \overline{\varphi}_2(z), \tag{4.53}$$

and $(\varphi, \phi) \in \Gamma$.

□

Lemma 4.5. If $\overline{\Phi} = (\overline{\varphi}_1, \overline{\varphi}_2)$ and $\underline{\Phi} = (\underline{\varphi}_1, \underline{\varphi}_2)$ are upper and lower solutions of (4.21) satisfying (P_1) and (P_2) then we have:

$$\begin{array}{l} \underline{\varphi}_1 \leq P_1(\underline{\varphi}_1, \underline{\varphi}_2) \leq P_1(\overline{\varphi}_1, \overline{\varphi}_2) \leq \overline{\varphi}_1, \\ \underline{\varphi}_2 \leq P_2(\underline{\varphi}_1, \underline{\varphi}_2) \leq P_2(\overline{\varphi}_1, \overline{\varphi}_2) \leq \overline{\varphi}_2. \end{array} \tag{4.54}$$

Consequently, $\mathbf{P}(\Gamma) \subset \Gamma$.

Proof. By the definition of lower solution and the operator $\mathbf{P}$, it follows that:

$$H_1(\underline{\varphi_1}, \underline{\varphi}_2)(z) \geq c\underline{\varphi_1}'(z) + \beta\underline{\varphi_1}(z) \;\; z \in \mathbb{R} \setminus \{z_i, i = 1, ..., m\}\,. \tag{4.55}$$

Letting $z_0 = -\infty$ and $z_{m+1} = +\infty$, then

$$\begin{aligned} P_1(\underline{\varphi}_1, \underline{\varphi}_2)(z) &= \tfrac{1}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}} H_1(\underline{\varphi_1}, \underline{\varphi_2})(y)dy, \\ &\geq \tfrac{1}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}} \left(c\underline{\varphi_1}'(y) + \beta\underline{\varphi_1}(y)\right) dy, \\ &= \tfrac{1}{c}e^{-\frac{\beta z}{c}} \left\{\textstyle\sum_{i=1}^{m-1} \displaystyle\int_{z_{i-1}}^{z_i} + \int_{z_{m-1}}^{z}\right\} e^{\frac{\beta y}{c}} \left(c\underline{\varphi_1}'(y) + \beta\underline{\varphi_1}(y)\right) dy, \\ &= \underline{\varphi_1}(z), \end{aligned}$$

for $z_{i-1} < z < z_i$ with $i = 1, 2, ..., m-1$. By a similar argument, we can prove that:

$$P_1(\overline{\varphi_1}, \overline{\varphi_2})(z) \leq \overline{\varphi_1}(z), \;\; P_2(\underline{\varphi_1}, \underline{\varphi_2})(z) \geq \underline{\varphi_2}(z), \;\; P_2(\overline{\varphi_1}, \overline{\varphi_2})(z) \leq \overline{\varphi_2}(z). \tag{4.56}$$

Moreover for any (φ_1, φ_2) in Γ, we have:

$$\begin{aligned}
H_1(\varphi_1,\varphi_2)(z) &= D_G\left[J_{L_G} * \varphi_1(z) - \varphi_1(z)\right] + \beta\varphi_1(z) - \gamma_G(G_e - \varphi_1(z))\varphi_1(z) \\
&\quad +\gamma_{TG}(G_e - \varphi_1(z))\phi_M * \varphi_2(z), \\
&\geq D_G\left[J_{L_G} * \underline{\varphi_1}(z)\right] + \left(\beta - D_G - \gamma_G G_e\right)\varphi_1(z) + \gamma_G\left(\underline{\varphi_1}(z)\right)^2 + \gamma_{TG}\left(G_e - \varphi_1(z)\right) \times \\
&\quad \phi_M * \underline{\varphi_2}(z), \\
&= D_G\left[J_{L_G} * \underline{\varphi_1}(z)\right] + \Big(\beta - \Big(D_G + \gamma_G G_e + \gamma_{TG}\phi_M * \underline{\varphi}_2(z)\Big)\Big)\varphi_1(z) + \gamma_G\left(\underline{\varphi_1}(z)\right)^2 + \\
&\quad \gamma_G G_e \phi_M * \underline{\varphi_2}(z).
\end{aligned}$$

By the definition of β in (4.45), $\beta > D_G + \gamma_G G_e + \gamma_{TG}T_2$, and then:

$$H_1(\varphi_1,\varphi_2)(z) \geq H_1(\underline{\varphi_1},\underline{\varphi_2})(z) \text{ for all } z \in \mathbb{R}.$$

Thus,

$$\begin{aligned}
P_1(\varphi_1,\varphi_2)(z) &= \tfrac{1}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}} H_1(\varphi_1,\varphi_2)(y)dy, \\
&\geq \tfrac{1}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}} H_1(\underline{\varphi_1},\underline{\varphi_2})(y)dy, \\
&= P_1(\underline{\varphi_1},\underline{\varphi_2})(z).
\end{aligned}$$

In the same manner, we get $P_1(\varphi_1,\varphi_2)(z) \leq P_1(\overline{\varphi}_1,\overline{\varphi}_2)(z)$. Similarly:

$$\begin{aligned}
H_2(\varphi_1,\varphi_2)(z) &= D_T\left[J_T * \varphi_2(z) - \varphi_2(z)\right] + \beta\varphi_2(z) + \gamma_T\varphi_2(z)\left(1 + \Omega\varphi_2(z)\right)\left(1 - \varphi_2(z)\right) - \\
&\quad \lambda_{fT} f \exp(-p\phi_M * \varphi_2(z))\varphi_2(z), \\
&\geq D_T\left[J_T * \underline{\varphi_2}(z)\right] + (\beta - D_T)\varphi_2(z) + \gamma_T\underline{\varphi_2}(z)\left(1 + \Omega\underline{\varphi_2}(z)\right)\left(1 - \varphi_2(z)\right) \\
&\quad -\lambda_{fT} f \exp(-p\phi_M * \underline{\varphi_2}(z))\varphi_2(z), \\
&= D_T\left[J_T * \underline{\varphi_2}(z)\right] + \left(\beta - \left(D_T + \gamma_T\underline{\varphi_2}(z)\left(1 + \Omega\underline{\varphi_2}(z)\right) + \lambda_{fT} f \exp(-p\phi_M * \underline{\varphi_2}(z))\right)\right) \times \\
&\quad \varphi_2(z) + \gamma_T\underline{\varphi_2}(z)\left(1 + \Omega\underline{\varphi_2}(z)\right),
\end{aligned}$$

Moreover, $\beta > D_T + \gamma_T T_2(1 + \Omega T_2) + \lambda_{fT} f$, then:

$$H_2(\varphi_1,\varphi_2)(z) \geq H_2(\underline{\varphi_1},\underline{\varphi_2})(z) \text{ for all } z \in \mathbb{R}.$$

thus,

$$\begin{aligned} P_2(\varphi_1,\varphi_2)(z) &= \tfrac{1}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}} H_2(\varphi_1,\varphi_2)(y)dy, \\ &\geq \tfrac{1}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}} H_2(\underline{\varphi_1},\underline{\varphi_2})(y)dy, \\ &= P_2(\underline{\varphi_1},\underline{\varphi_2})(z). \end{aligned}$$

In the same manner we have $P_2(\varphi_1,\varphi_2)(z) \leq P_2(\overline{\varphi}_1,\overline{\varphi}_2)(z)$.
Therefore, with $\beta > \max(D_G + \gamma_G G_e + \gamma_{TG} T_2; D_T + \gamma_T T_2(1+\Omega T_2) + \lambda_{fT} f)$, we have:

$$\begin{array}{l} \underline{\varphi}_1 \leq P_1(\underline{\varphi}_1,\underline{\varphi}_2) \leq P_1(\varphi_1,\varphi_2) \leq P_1(\overline{\varphi}_1,\overline{\varphi}_2) \leq \overline{\varphi}_1, \\ \underline{\varphi}_2 \leq P_2(\underline{\varphi}_1,\underline{\varphi}_2) \leq P_2(\varphi_1,\varphi_2) \leq P_2(\overline{\varphi}_1,\overline{\varphi}_2) \leq \overline{\varphi}_2. \end{array} \tag{4.57}$$

Therefore, (4.57) leads to $\mathbf{P}(\Gamma) \subset \Gamma$. □

Lemma 4.6. The operator $\mathbf{P} : \Gamma \longrightarrow \Gamma$ is continuous in the sense of $|\cdot|_\alpha$.

Proof. We consider $\varphi = (\varphi_1,\varphi_2)$ and $\psi = (\psi_1,\psi_2)$ two pairs of function belong to Γ. We have :

$$\begin{aligned} |H_1(\varphi_1,\varphi_2)(z) - H_1(\psi_1,\psi_2)(z)| \leq\ & D_G \int_{\mathbb{R}} J_{LG}(z-y)|\varphi_1(y)-\psi_1(y)|dy + (\beta - D_G)|\varphi_1(z)-\psi_1(z)| \\ & + \gamma_G G_e|\varphi_1(z)-\psi_1(z)| + \gamma_{TG} G_e|\phi_M * \varphi_2(z) - \phi_M * \psi_2(z)| \\ & + \gamma_G|(\varphi_1(z))^2 - (\psi_1(z))^2| \\ & + \gamma_{TG}|\varphi_1(z)\phi_M * \varphi_2(z) - \psi_1(z)\phi_M * \psi_2(z)|, \end{aligned}$$

Due to the definition of $X_{T_2}^{G_e}$ and due to the fact that $\Gamma \subset X_{T_2}^{G_e}$, we have:

$$|\varphi_1(z)| \leq G_e,\ \ |\psi_1(z)| \leq G_e \ \text{ and } \ |\varphi_2(z)| \leq T_2,\ \ |\psi_2(z)| \leq T_2.$$

Consequently,

$$\begin{aligned} |H_1(\varphi_1,\varphi_2)(z) - H_1(\psi_1,\psi_2)(z)| \leq\ & D_G \int_{\mathbb{R}} J_{LG}(z-y)|\varphi_1(y)-\psi_1(y)|dy + (\beta - D_G + 3\gamma_G G_e)|\varphi_1(z) - \psi_1 \\ & + \gamma_{TG} T_2|\varphi_1(z)-\psi_1(z)| + 2\gamma_{TG} G_e|\phi_M * \varphi_2(z) - \phi_M * \psi_2(z)|. \end{aligned}$$

Therefore, we deduce that:

$$|P_1(\varphi_1,\varphi_2)(z) - P_1(\psi_1,\psi_2(z)|e^{-\alpha|z|} \leq Z_1 + Z_2 + Z_3,$$

where

$$\begin{aligned}
Z_1 &= D_G \frac{e^{-\alpha|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}}\left[\int_{\mathbb{R}} J_{LG}(y-s)|\varphi_1(s)-\psi_1(s)|ds\right]dy,\\
Z_2 &= (\beta - D_G + 3\gamma_G G_e + \gamma_{TG}T_2)\,\frac{e^{-\alpha|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}}|\varphi_1(y)-\psi_1(y)|dy,\\
Z_3 &= 2\gamma_{TG}G_e\frac{e^{-\alpha|z|}}{c}\int_{-\infty}^{z} e^{\frac{-\beta(z-y)}{c}}|\phi_M * \varphi_2(y) - \phi_M * \psi_2(y)|dy.
\end{aligned}$$

For Z_1 we have:

$$\begin{aligned}
Z_1 &= D_G \frac{e^{-\alpha|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}}\left[\int_{\mathbb{R}} J_{LG}(s)|\varphi_1(y-s)-\psi_1(y-s)|ds\right]dy,\\
&= D_G \frac{e^{-\alpha|z|}}{c}\int_{-\infty}^{z} e^{-\frac{\beta(z-y)}{c}}\left[\int_{\mathbb{R}} J_{LG}(s)e^{\alpha|y-s|}|\varphi_1(y-s)-\psi_1(y-s)|e^{-\alpha|y-s|}ds\right]dy,\\
&\leq \frac{D_G|\varphi-\psi|_\alpha}{c}\int_{\mathbb{R}} J_{LG}(s)e^{\alpha|s|}ds\int_{-\infty}^{z} e^{\frac{-\beta(z-y)}{c}}e^{-\alpha|z|}e^{\alpha|y|}dy,\\
&\leq D_G\int_{\mathbb{R}} J_{LG}(s)e^{\alpha|s|}ds\left(\frac{1}{\beta-c\alpha}+\frac{1}{\beta+c\alpha}\right)|\varphi-\psi|_\alpha.
\end{aligned}$$

Similarly, for Z_2 and Z_3, we get the following inequalities:

$$\begin{aligned}
Z_2 &\leq (\beta - D_G + 3\gamma_G G_e + \gamma_{TG}T_2)\times\left(\frac{1}{\beta-c\alpha}+\frac{1}{\beta+c\alpha}\right)|\varphi-\psi|_\alpha,\\
Z_3 &\leq \left(2\gamma_{TG}G_e\int_{\mathbb{R}}\phi_M(s)e^{\alpha|s|}ds\right)\times\left(\frac{1}{\beta-c\alpha}+\frac{1}{\beta+c\alpha}\right)|\varphi-\psi|_\alpha.
\end{aligned}$$

Let :

$$K_1^* = \left(D_G\int_{\mathbb{R}} J_{LG}(s)e^{\alpha|s|}ds + \beta - D_G + 3\gamma_G G_e + \gamma_{TG}T_2 + 2\gamma_{TG}G_e\int_{\mathbb{R}}\phi_M(s)e^{\alpha|s|}ds\right)\times\left(\frac{1}{\beta-c\alpha}+\frac{1}{\beta+c\alpha}\right)$$

then,

$$|P_1(\varphi_1,\varphi_2)(z) - P_1(\psi_1,\psi_2)(z)|e^{-\alpha z} \leq K_1^*|\varphi-\psi|_\alpha. \tag{4.58}$$

Proceeding in a analogous way for P_2, we get:

$$|P_2(\varphi_1,\varphi_2)(z) - P_2(\psi_1,\psi_2)(z)|e^{-\alpha z} \leq K_2^*|\varphi-\psi|_\alpha, \tag{4.59}$$

where

$$\begin{aligned} K_2^* \;\; = \;\; & \left(D_T \int_{\mathbb{R}} J_{L_T}(s) e^{\alpha|s|} ds + \beta - D_T + \gamma_T - \delta_T + 3\Omega\gamma_T T_2^2 + 2\gamma_T T_2 |1-\Omega| + \lambda_{fT} f + \lambda_{fT} f T_2 k_1 \right. \\ & \left. + \lambda_{fT} f p T_2 k_2 \int_{\mathbb{R}} \phi_M(s) e^{\alpha|s|} ds \right) \times \left(\frac{1}{\beta - c\alpha} + \frac{1}{\beta + c\alpha} \right). \end{aligned}$$

and $k_i, i = 1, 2$ are Lipschitz constant of functions of $\omega(G)$ and $\exp(-pT)$. Due to the two inequalities in (4.58) and (4.59), we deduce the uniform continuity of $\mathbf{P}$ with respect to the norm $|.|_\alpha$. Then $\mathbf{P}$ is continuous with respect to the norm $|.|_\alpha$. ☐

For any given $n \in \mathbb{N}$, let $\mathbb{R}_{n,-} = (-\infty, n]$. Consider the domain of functions in the space B_α

$$B_\alpha(\mathbb{R}_{n,-}, \mathbb{R}^2) = \left\{ \Phi = (\varphi_1, \varphi_2) \in X_{T_2}^{G_e} | \mathbb{R}_{n,-} \sup_{z \in \mathbb{R}_{n,-}} \max\left(|\varphi_1(z)|, |\varphi_2(z)|\right) e^{-\alpha|z|} < \infty \right\}. \quad (4.60)$$

Proposition 4.6. (see Zhao et al. [132]) $B_\alpha(\mathbb{R}_{n,-}, \mathbb{R}^2)$ is a Banach space equipped with the norm $|\cdot|_\alpha^n$, defined by:

$$|(\varphi_1, \varphi_2)|_\alpha^n = \sup_{z \in \mathbb{R}_{n,-}} \max\left(|\varphi_1(z)|, |\varphi_2(z)|\right) e^{-\alpha z}. \quad (4.61)$$

Lemma 4.7. (Corduneanu [17])(see also Zhao et al. [132])
Let $\mathbf{E} \subset B_\alpha(\mathbb{R}_{n,-}, \mathbb{R}^2)$ be a set satisfying the following conditions:

(i) $\mathbf{E}$ is bounded in $B_\alpha(\mathbb{R}_{n,-}, \mathbb{R}^2)$.

(ii) the functions belonging to $\mathbf{E}$ are equicontinuous on any compact interval of $\mathbb{R}_{n,-}$.

(iii) The functions in $\mathbf{E}$ are equiconvergent.

Then, $\mathbf{E}$ is compact in $B_\alpha(\mathbb{R}_{n,-}, \mathbb{R}^2)$.

Lemma 4.8. $\mathbf{P}(\Gamma)$ is compact in $B_\alpha(\mathbb{R}_{n,-}, \mathbb{R}^2)$..

Proof. We define on Γ the operator $\mathbf{P}^n$ by:

$$\mathbf{P}^n(\Phi)(z) = \begin{cases} \mathbf{P}(\Phi)(n), & z > n, \\ \mathbf{P}(\Phi)(z), & z \in (-\infty, n], \end{cases} \quad (4.62)$$

for any $\Phi = (\varphi_1, \varphi_2) \in \Gamma$, and $n \in \mathbb{N}$. It is easy to remark that, $\mathbf{P}^n(\Phi)(z)$ is compact if $\mathbf{P}(\Phi)(z)_{|(-\infty,n]}$ is compact. Consequently, we have to show that $\mathbf{P}(\Gamma)(z)_{|(-\infty,n]}$ satisfies all the conditions of Corduneanu [17] theorem gives in Lemma 4.7.
In a first step, by Lemma 4.4, $\mathbf{P}(\Gamma)_{|(-\infty,n]}$ is bounded, because $\mathbf{P}(\Gamma) \subset \Gamma$.
In the second step, we aims to prove that, all functions in $\mathbf{P}(\Gamma)(z)_{|(-\infty,n]}$ are equicontinous on any compact intervall of $(-\infty; n]$. Then for any $\Phi = (\varphi_1, \varphi_2) \in \Gamma$ and for any $z_1 < z_2 \in (-\infty; n]$, we

have:

$$|P_1(\varphi_1,\varphi_2)(z_1)e^{-\alpha|z_1|} - P_1(\varphi_1,\varphi_2)(z_2)e^{-\alpha|z_2|}|$$

$$= \frac{1}{c}\left|e^{-\alpha|z_1|}\int_{-\infty}^{z_1} e^{-\frac{\beta(z_1-y)}{c}} H_1(\varphi_1,\varphi_2)(y)dy - e^{-\alpha|z_2|}\int_{-\infty}^{z_2} e^{-\frac{\beta(z_2-y)}{c}} H_1(\varphi_1,\varphi_2)(y)dy\right|,$$

$$\leq \frac{1}{c}\left|e^{-(\alpha|z_1|+\frac{\beta}{c}z_1)}\int_{-\infty}^{z_1} e^{\frac{\beta y}{c}} H_1(\varphi_1,\varphi_2)(y)dy - e^{-(\alpha|z_2|+\frac{\beta}{c}z_2)}\int_{-\infty}^{z_2} e^{-\frac{\beta y}{c}} H_1(\varphi_1,\varphi_2)(y)dy\right|,$$

$$\leq \tfrac{1}{c}|e^{-(\alpha|z_1|+\frac{\beta}{c}z_1)} - e^{-(\alpha|z_2|+\frac{\beta}{c}z_2)}| \times \left|\int_{-\infty}^{z_1} e^{\frac{\beta y}{c}} H_1(\varphi_1,\varphi_2)(y)dy\right| + \frac{1}{c}e^{-(\alpha|z_2|+\frac{\beta}{c}z_2)}\left|\int_{z_1}^{z_2} e^{\frac{\beta y}{c}} H_1(\varphi_1,\varphi_2)(y)dy\right|,$$

Due to the fact that $0 \leq \varphi_1(z) \leq G_e$ and $0 \leq \varphi_2(z) \leq T_2$, we have:

$$|P_1(\varphi_1,\varphi_2)(z_1)e^{-\alpha z_1} - P_1(\varphi_1,\varphi_2)(z_2)e^{-\alpha z_2}| \leq G_e K^*|z_1 - z_2|, \tag{4.63}$$

where $K^* = \theta_1 e^{\frac{\beta z_1}{c}} + \theta_2 e^{-(\alpha|z_2|+\frac{\beta}{c}z_2)}$ and θ_i, $i = 1,2$ are respectively the Lipschitz constant of $e^{-(\alpha|z|+\frac{\beta}{c}z)}$ and e^z. In fact it suffice to remark by setting $\tilde{\varphi_1}(z) = G_e$ and $\tilde{\varphi_2}(z) = T_2$ for all $z \in \mathbb{R}$, that:

$$D_G\left(J_{L_G} * \tilde{\varphi_1}(z) - \tilde{\varphi_1}(z)\right) - \gamma_G\left(G_e - \tilde{\varphi_1}(z)\right)\tilde{\varphi_1}(z) + \gamma_{TG}\left(G_e - \tilde{\varphi_1}(z)\right)\phi_M * \tilde{\varphi_2}(z) - c\tilde{\varphi_1}'(z) = 0,$$

and then:

- $$\int_{-\infty}^{z_1} e^{\frac{\beta y}{c}} H_1(\varphi_1,\varphi_2)(y)dy \leq \int_{-\infty}^{z_1} e^{\frac{\beta y}{c}} H_1(\tilde{\varphi_1},\tilde{\varphi_2})(y)dy = cG_e e^{\frac{\beta z_1}{c}}, \tag{4.64}$$
- $$\begin{aligned}\int_{z_1}^{z_2} e^{\frac{\beta y}{c}} H_1(\varphi_1,\varphi_2)(y)dy &\leq \int_{z_1}^{z_2} e^{\frac{\beta y}{c}} H_1(\tilde{\varphi_1},\tilde{\varphi_2})(y)dy,\\ &= cG_e\left(e^{\frac{\beta z_2}{c}} - e^{\frac{\beta z_1}{c}}\right).\end{aligned} \tag{4.65}$$

Therefore, by (4.64) and (4.65) it is easy to get (4.63). In the same way we have

$$|P_2(\varphi_1,\varphi_2)(z_1)e^{-\alpha|z_1|} - P_2(\varphi_1,\varphi_2)(z_1)e^{-\alpha|z_1|}| \leq T_2 K^*|z_1 - z_2|. \tag{4.66}$$

Then, the relation in (4.63) and (4.66) implies that, the condition (ii) in Lemma 4.7 is satisfied.
In the third step, we prove that $\mathbf{P}(\Gamma)(z)_{|(-\infty,n]}$ is equiconvergent. Moreover, using $\lim_{z\to-\infty}\Phi(z) = 0$, then for a given $\varepsilon > 0$, there exist a $N(\varepsilon) < 0$ such that for any $z < N(\varepsilon)$ we have: $|\varphi_1(z)e^{-\alpha|z|}| < \frac{\varepsilon}{2}$ and $|\varphi_2(z)e^{-\alpha|z|}| < \frac{\varepsilon}{2}$. Consequently:

$$||\Phi(z) - \Phi(-\infty)||e^{-\alpha|z|} < \varepsilon,$$

whenever $z \leq N(\varepsilon)$. Then in this case, the condition (iii) in Lemma 4.7 is satisfied.
Lemma 4.7 indicates that $\mathbf{P}(\Gamma)(z)_{|(-\infty,n]}$ is compact in the sens of $|.|_{\alpha}^{n}$, so $\mathbf{P}^{n}(\Phi)(z)$ is precompact subset of $B_{\alpha}(\mathbb{R}_{n,-},\mathbb{R}^{2})$ with respect to the norm $|.|_{\alpha}$. Furthermore,

$$|\mathbf{P}^{n}(\Phi)(z)-\mathbf{P}(\Phi)(z)|_{\alpha} \leq 2(G_{e}+T_{2})e^{-\alpha n} \longrightarrow 0 \text{ as } n \to +\infty. \tag{4.67}$$

Thus, $\mathbf{P}^{n}(\Phi)(z)$ converges to $\mathbf{P}(\Phi)(z)$ in the sense of the norm $|.|_{\alpha}$, and compactness of $\mathbf{P}^{n}(\Phi)(z)$ implies that $\mathbf{P}(\Phi)(z)$ is compact. □

We recall some results on Schauder's fixed point theorem.

Theorem 4.3. (Schauder Fixed Point theorem)[see Gilbarg and Trudinger [44], page 279] Let Θ be a compact convex set in a Banach space $\mathcal{B}$ and let T be a continuous mapping of Θ into itself. Then T has a fixed point, that is, $Tx = x$ for some $x \in \Theta$.

Corollary 4.1 (see Gilbarg and Trudinger [44], page 280)**.** Let Θ be a closed convex set in a Banach space $\mathcal{B}$ and let T be a continuous mapping of Θ into itself such that the image $T\Theta$ is precompact. Then T has a fixed point.

Theorem 4.4. Assume that:

- $(A_1)-(A_4)$ in page 133 holds,
- $\mathcal{R}_T > 1$ and $\mathcal{R}_{T_2} > 1$.

For any $c > c_1^*$ (where c_1^* is given in (4.85),page 161), system (4.21) has a positive solution $\varphi_i(z) \in X_{T_2}^{G_e}$ satisfying $\underline{\varphi_i}(z) \leq \varphi_i(z) \leq \overline{\varphi}_i(z)$

Proof. It suffices to prove that the operator $\mathbf{P}$ defined in (4.46) has a fixed point. We first note by Lemma 4.4 page 149 that Γ is closed, bounded, convex and nonempty. Moreover, operator $\mathbf{P}$ is continuous (see for instance Lemma 4.6, page 152). With Lemma 4.8 page 154, $\mathbf{P}(\Gamma)$ is relatively compact on the space B_{α} since Γ is bounded. Then, according to the corollary 4.1, the operator $\mathbf{P}$ has a fixed point $\mathbf{\Phi}^* = (\varphi_1^*, \varphi_2^*) \in \Gamma$. Due to the Proposition 4.4 page 148, $\mathbf{\Phi}^* = (\varphi_1^*, \varphi_2^*)$ is a positive solution of system (4.21). □

Now we are interested by the asymptotic discussion. Thanks to Theorem 4.4, the case $z \longrightarrow -\infty$ is quite easy. So we consider the asymptotic boundary condition of traveling wave solution described in Theorem 4.4 as $z \longrightarrow +\infty$. Let $(\varphi_1^*, \varphi_2^*)$ be a traveling wave connecting $(0,0)$ and a nontrivial state.

Proposition 4.7. If $\mathcal{R}_F < 1$, then $\lim_{z\to+\infty}(\varphi_1^*, \varphi_2^*) = (G_e, T_2)$.

Proof. Suppose $\mathcal{R}_F < 1$, then the forest homogeneous steady state $(0, T_2)$ for the system (4.1) is LAS and therefore by the Definition 4.1 and the change of variables in (4.20), we have the result. □

For the case where $\mathcal{R}_F \geq 1$ we use the following proposition:

Proposition 4.8. Assume that the limit:

$$(g,h) = \lim_{z \longrightarrow +\infty} (\varphi_1^*, \varphi_2^*)(z)$$

exist. Then $(g,h) = (G_e, T_2)$.

Proof. We remark first by the assumption in (4.23) that $0 < g < 1$ and $0 < h < 1$. The proposition 4.8 follows if we can show that

$$\left(J_{L_G} * \varphi_1^*\right)(z) \longrightarrow g \quad \text{and} \quad \left(J_{L_T} * \varphi_2^*\right)(z) \longrightarrow h \text{ as } z \longrightarrow +\infty, \tag{4.68}$$

since the only nontrivial solution of

$$\begin{cases} -\gamma_G(G_e - \varphi_1^*)\varphi_1^* + \gamma_{TG}(G_e - \varphi_1^*)\phi_M * \varphi_2^* = 0, \\ \gamma_T(1 + \Omega\varphi_2^*)(1 - \varphi_2^*) - \delta_T - \lambda_{fT} f\omega(G_e - \varphi_1^*)\exp(-p\phi_M * \varphi_2^*) = 0, \end{cases}$$

is (G_e, T_2).

Then, we need only to treat the case of the first limit in (4.68) since the proof of the second is similar to the first one.

For $\varepsilon \in (0, g)$, since $\varphi_1^*(+\infty) = g$ and $\int_{\mathbb{R}} J_{L_G}(y)dy = 1$, there exist $a > 0$ such that:

$$\int_a^{+\infty} J_{L_G}(y)dy < \frac{\varepsilon}{2} \text{ and } g - \frac{\varepsilon}{2} < \varphi_1^*(z) < g + \frac{\varepsilon}{2}, \ \forall z > a.$$

Therefore:

$$\int_{\mathbb{R}} J_{L_G}(z')\varphi_1^*(z-z')dz' = \int_{-\infty}^{a} J_{L_G}(z')\varphi_1^*(z-z')dz' + \int_a^{+\infty} J_{L_G}(z')\varphi_1^*(z-z')dz' := C_1(z) + C_2(z).$$

Evaluating C_1 and C_2, we yields in a first step to:

$$C_1(z) + C_2(z) \leq \left(g + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = g + \varepsilon, \tag{4.69}$$

for all $z \geq 2a$, by using $z - z' \geq a$ for $z' \leq a$ and $z \geq 2a$, $\int_{\mathbb{R}} J_{L_G}(y)dy = 1$ and $\varphi_1^* \leq 1$. In a second step, we have:

$$C_1(z) + C_2(z) \geq C_1(z) \geq (g - \frac{\varepsilon}{2})(1 - \frac{\varepsilon}{2}) \geq g - \varepsilon. \tag{4.70}$$

for all $z \geq 2a$ by using $g \leq 1$. Combining (4.69) and (4.70) we have shown that $\left(J_{L_G} * \varphi_1^*\right)(z) \longrightarrow g$ as $z \longrightarrow +\infty$ and so (4.68) follows. ☐

4.4.3.2 The case $c = c_1^*$

We consider, in this subsection the existence of a positive solution of system (4.21) if $c = c_1^*$, by referring to Lin [67] and Dong et al. [32].

Lemma 4.9. Let (φ_1, φ_2) be the positive solution of system 4.21 with the speed $c > c_1^*$ obtained in theorem 4.4. We have, $\varphi_1, \varphi_2, \varphi_1', \varphi_2'$ are uniformly bounded and equicontinuous.

Proof. Since $(\varphi_1, \varphi_2) \in \Gamma$, we get φ_1, $\varphi_2 \in C(\mathbb{R})$ and $0 \leq \varphi_1(z) \leq G_e$ and $0 \leq \varphi_2(z) \leq T_2$, for all $z \in \mathbb{R}$. Then by system (4.21), we get φ_1' $\varphi_2' \in \mathbb{R}$ and $0 \leq \varphi_1'(z) \leq \frac{G_e}{c}\left(D_G + \gamma_{TG} T_2\right)$, $0 \leq \varphi_2'(z) \leq \frac{T_2}{c} D_T$. Then, by the virtue of the derivative of variable z on both side of the system (4.21) we have φ_1', φ_2' are uniformly bounded . ☐

Lemma 4.10. Let (φ_1, φ_2) be the nonnegative solution of system 4.21 with the speed $c > c_1^*$, we have:

$$\lim_{z \to -\infty} \frac{\varphi_2'(z)}{\varphi_2(z)} = \lambda_T^* \tag{4.71}$$

Proof. For $c > c_1^*$, we have $\varphi_2(z) > 0$ for $z \in \mathbb{R}$. In fact, if there exist $z_0 \in \mathbb{R}$ such that $\varphi_2(z_0) = 0$, then $\varphi_2'(z_0) = 0$ and by the second equation of system (4.21), we obtain $J_{L_T} * \varphi_2(z_0) =$, which implies that $\varphi_2(z) = 0$ for all $z \in \mathbb{R}$. This leads to a contradiction, hence $\varphi_2(z) > 0$ for all $z \in \mathbb{R}$. Let:

$$\theta(z) = \frac{\varphi_2'(z)}{\varphi_2(z)} \text{ and } B(z) = \gamma_T \left(1 + \Omega\varphi_2(z)\right)\left(1 - \varphi_2(z)\right) - \delta_T - \lambda_{fT} f\omega(G_e - \varphi_1(z)) \exp(-p\phi_{M_2} * \varphi_2(z)) \tag{4.72}$$

It follows from the second equation in system (4.21), that :

$$c\theta(z) = D_T\left(\int_{\mathbb{R}} J_{L_T}(y) e^{\int_z^{z-y} \theta(s)ds} dy - 1\right) + B(z). \tag{4.73}$$

and

$$\lim_{z \to -\infty} B(z) = \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right), \tag{4.74}$$

Referring to Zhang et al. [131] (proposition 3.7), $\lambda = \lim_{z \to -\infty} \theta(z)$ exist and satisfies:

$$c\lambda = D_T\left(\int_{\mathbb{R}} J_{L_T}(y) e^{-\lambda y} dy - 1\right) + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right). \tag{4.75}$$

By lemma 4.1, we have $\lambda = \lambda_T^*$. ☐

Lemma 4.11. (Arzelà-Ascoli theorem)
Let $K \subseteq \mathbb{R}^N$ be compact and let f_n a sequence in $\mathcal{C}(K)$ which is uniformly bounded and equicontinuous. Then f_n has a subsequnce which converge uniformly on K to a function f in $\mathcal{C}(K)$.

Remark 4.3. The proof of lemma 4.11, involve sequences of sequences of functions and what is sometimes called Diagonalization argument.

Lemma 4.12. (Barbalat theorem) Let $t \longmapsto F(t)$ be a differentiable function with a finite limit as $t \to \infty$. If F is uniformly continue, then $F(t) \to 0$ as $t \to \infty$.

Theorem 4.5. Assume that $\min\left(\mathcal{R}_T, \Omega\right) > 1$ and $c = c_1^*$. Then system 4.21 has a positive non trivial solution satisfying 4.22.

Proof. Assume $\mathcal{R}_T > 1$ and $\Omega > 1$. We consider a decreasing sequence $c_n \in (c_1^*, c_1^* + 1)$ such that $\lim_{n\to+\infty} c_n = c_1^*$. For each c_n following Theorem 4.4, there exists a traveling wave $(\varphi_{1n}, \varphi_{2n})$ satisfying system (4.21) and the asymptotic condition at $z \longrightarrow -\infty$ in (4.22). By Lemma 4.10 $\varphi'_{2n}(z) > 0$ for $z << -1$. Note that a traveling wave solution is invariant in the sense of phase shift (Pan and Lin [75]), so we can assume that

$$\varphi_{2n}(0) = \delta, \;\; \varphi_{2n}(z) \leq \delta \text{ for } z < 0, \tag{4.76}$$

and δ is small enough such that $0 < \delta < 1 - \frac{1}{\Omega}$.

By the Arzela-Ascoli theorem (Green and Valentine [47]), we can find a subsequence of $(\varphi_{1n}, \varphi_{2n})$ again denoted by $(\varphi_{1n}, \varphi_{2n})$ such that $(\varphi_{1n}, \varphi_{2n})$ and $(\varphi'_{1n}, \varphi'_{2n})$ converge uniformly on every bounded interval (and so point-wise on $\mathbb{R}$) to bounded functions (φ_1, φ_2) and (φ'_1, φ'_2) respectively. Moreover, by the dominated convergence theorem we have:

(a) $\lim_{n\to+\infty} J_{L_G} * \varphi_{1n} = J_{L_G} * \varphi_1$,

(b) $\lim_{n\to+\infty} J_{L_T} * \varphi_{2n} = J_{L_T} * \varphi_2$,

(c) $\lim_{n\to+\infty} \phi_{M_2} * \varphi_{2n} = \phi_{M_2} * \varphi_2$,

on every bounded interval. Therefore it follows that (φ_1, φ_2) satisfies system (4.21) and is bounded. Proposition 4.8 is independant of c, then when $c = c_1^*$, we can use this proposition for the asymptotic condition at $z = +\infty$.

Denote $\lim_{z\to-\infty} \varphi_1(z) = \varphi_1^-$ and $\lim_{z\to-\infty} \varphi_2(z) = \varphi_2^-$. By the Barbalat theorem (Farkas and Wegner [38]), we have $\varphi'_1(-\infty) = 0$ and $\varphi'_2(-\infty) = 0$. Then using the dominated convergence theorem in the first equation of system (4.21), we get:

$$\gamma_{TG}\varphi_2^- = \gamma_G\varphi_1^-. \tag{4.77}$$

Using the dominated convergence theorem in the second equation of system (4.21), we have the following possible conclusion:

$$\varphi_2^- = 0 \;\text{ or }\; \gamma_T(1 + \Omega\varphi_2^-)(1 - \varphi_2^-) = \delta_T + \lambda_{fT} f\omega(G_e - \varphi_1^-)\exp(-p\phi_M * \varphi_2^-). \tag{4.78}$$

If the second equation in 4.78 hold then we get:

$$(\delta_T + \lambda_{fT} f)(\mathcal{R}_T - 1) + \gamma_T\Omega\varphi_2^- \left[\left(1 - \frac{1}{\Omega}\right) - \varphi_2^-\right] < 0, \tag{4.79}$$

and this inequality in (4.79) can not hold due to the fact that $\mathcal{R}_T > 1$ and $\varphi_2^- < 1 - \frac{1}{\Omega}$.

Therefore, $\varphi_2^- = 0$ and by the relation in (4.77), $\varphi_1^- = 0$. This ends the proof. □

4.4.4 Non existence of traveling wave when $c < c_1^*$

In this section we shall prove that if $c < c_1^*$, then system (4.21) does not have a positive solution satisfying the asymptotic boundary condition. We consider the following initial value problem:

$$\begin{cases} \dfrac{\partial g(x,t)}{\partial t} & = & (J * g(x,t) - g(x,t)) + \gamma g(x,t)(1-g(x,t)), \\ g(x,0) & = & \Lambda(x), \ \ x \in \mathbb{R}, \end{cases} \tag{4.80}$$

where:

- J satisfies condition (A_2), and γ is a positive constant.
- $\Lambda(x)$ is uniformly continuous and bounded .

Lemma 4.13. (see also Jin and Zhao [56], Zhao et al. [132], Ducrot et al. [33], Dong et al. [32]) Assume that $0 \leq \Lambda(x) \leq 1$. Then system (4.80) admits a solution for all $x \in \mathbb{R}$ and $t > 0$. If $h(x,0)$ is uniformly continuous and bounded, and $h(x,0)$ satisfies

$$\begin{cases} \dfrac{\partial h(x,t)}{\partial t} & \geq (\leq) & (J * h(x,t) - h(x,t)) + \gamma h(x,t)(1-h(x,t)), \\ h(x,0) & \geq (\leq) & \Lambda(x), \ \ x \in \mathbb{R}. \end{cases} \tag{4.81}$$

then, there holds:

$$h(x,t) \geq (\leq) g(x,t), \ \ x \in \mathbb{R}, \ \ t > 0.$$

Define

$$c^* = \inf_{\lambda > 0} \frac{\displaystyle\int_{\mathbb{R}} J(x) e^{-\lambda x} dx + \gamma - 1}{\lambda}.$$

Lemma 4.14. (see also Jin and Zhao [56], Zhao et al. [132], Ducrot et al. [33] and Dong et al. [32]) Assume that $\Lambda(x) > 0$, then for $0 < c < c^*$, we have

$$\lim_{t \to +\infty} \inf_{|x|<ct} g(x,t) = \lim_{t \to +\infty} \sup_{|x|<ct} g(x,t) = 1.$$

Referring to the comparison principle in Lemma 4.13 and 4.14 we get the condition for the non-existence of travelling wave for system (4.18).

Theorem 4.6. Assume that $\mathcal{R}_T > 1$. For any $0 < c < c_1^*$, there exists no nontrivial positive solution $(\varphi_1(z), \varphi_2(z))$ of system (4.18) satisfying

$$\lim_{z \to -\infty} (\varphi_1, \varphi_2) = (G_e, 0) \ \text{ and } \ \lim_{z \to +\infty} (\varphi_1, \varphi_2) = (0, T_2). \tag{4.82}$$

Proof. By contradiction, we assume that there exists some $\tilde{c}_1 < c_1^*$, such that system (4.18) has a positive solution $(\varphi_1(z), \varphi_2(z))$ satisfying (4.82). Then $T(x,t) = \varphi_2(x + \tilde{c}_1 t)$ satisfies:

$$\begin{cases} \dfrac{\partial T(x,t)}{\partial t} & \geq & D_T\left(J_{L_T} * T(x,t) - T(x,t)\right) + \gamma_T T(x,t)\left(1 - \dfrac{1}{\mathcal{R}_T} - T(x,t)\right), \\ & & \\ T(x,0) & = & \varphi_2(x) > 0. \end{cases} \tag{4.83}$$

Let $x(t) = -\dfrac{\tilde{c}_1 + c_1^*}{2}t$, then $|x(t)| = \dfrac{\tilde{c}_1 + c_1^*}{2}t < c_1^* t$. From Lemmas 4.13 and 4.14, we have :

$$\lim_{t\to+\infty} \inf_{|x|=\frac{\tilde{c}_1+c_1^*}{2}t} T(x,t) \geq 1 - \frac{1}{\mathcal{R}_T} > 0. \tag{4.84}$$

On the other land let $x(t) + \tilde{c}_1 t = \dfrac{\tilde{c}_1 - c_1^*}{2}t$, then

$$z = x + \tilde{c}_1 t \longrightarrow -\infty \ \text{ as } \ t \to +\infty,$$

and

$$\lim_{t\to+\infty} \sup T(x(t),t) = \lim_{z\to-\infty} \varphi_2(z) = 0,$$

which yields a contradiction. □

Remark 4.4. From the two previous subsections 4.4.3 and 4.4.4, it follows that c_1^* is the minimal wave speed connecting the grassland to the forest homogeneous steady states of the system. Then referring to the Lemma 4.1, we have:

$$\begin{aligned} c_1^* &= \min_{\lambda>0} \frac{D_T\left(\displaystyle\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda y}dy - 1\right) + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)}{\lambda}, \\ c_1^* &= \min_{\lambda>0} \frac{D_T\left(\exp\left\{\dfrac{(L_T\lambda)^2}{2}\right\} - 1\right) + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)}{\lambda}. \end{aligned} \tag{4.85}$$

Proposition 4.9. Direct calculations show that :

$$\frac{\partial c_1^*}{\partial L_T} > 0; \ \frac{\partial c_1^*}{\partial D_T} > 0; \ \frac{\partial c_1^*}{\partial \gamma_T} > 0; \ \frac{\partial c_1^*}{\partial f} < 0; \ \frac{\partial c_1^*}{\partial \delta_T} < 0. \tag{4.86}$$

Proof. Recall that:

$$
\begin{aligned}
c_1^* &= \min_{\lambda>0} \frac{D_T\left(\displaystyle\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda y}dy - 1\right) + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)}{\lambda}, \\
&= \frac{D_T\left(\displaystyle\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda_T^* y}dy - 1\right) + \left(\delta_T + \lambda_{fT} f\omega(G_e)\right)\left(\mathcal{R}_T - 1\right)}{\lambda_T^*}, \\
&= \frac{D_T\left(\displaystyle\int_{\mathbb{R}} J_{L_T}(y)e^{-\lambda_T^* y}dy - 1\right) + \gamma_T - \delta_T - \lambda_{fT} f\omega(G_e)}{\lambda_T^*}, \\
&= \frac{D_T\left(\exp\left\{\dfrac{(L_T\lambda_T^*)^2}{2}\right\} - 1\right) + \gamma_T - \delta_T - \lambda_{fT} f\omega(G_e)}{\lambda_T^*}.
\end{aligned}
$$

Then:

$$
\frac{\partial c_1^*}{\partial f} = -\frac{\lambda_{fT}\omega(G_e)}{\lambda_T^*} < 0, \quad \frac{\partial c_1^*}{\partial \delta_T} = -\frac{1}{\lambda_T^*} < 0 \text{ and } \frac{\partial c_1^*}{\partial \gamma_T} = \frac{1}{\lambda_T^*} > 0. \tag{4.87}
$$

Similarly,

$$
\frac{\partial c_1^*}{\partial L_T} = D_T L_T \lambda_T^* \exp\left(\frac{L_T^2(\lambda_T^*)^2}{2}\right) > 0 \text{ and } \frac{\partial c_1^*}{\partial D_T} = \frac{1}{\lambda_T^*}\left(\exp\left\{\frac{(L_T\lambda_T^*)^2}{2}\right\} - 1\right) > 0. \tag{4.88}
$$

□

From Proposition 4.9, we deduce that an increase of the tree seeds dispersal range L_T enhances the forest encroachment. Same conclusion holds for the tree biomass dispersal rate (D_T) and the intrinsic growth rate of the tree biomass (γ_T). Conversely, increasing the human activity-induced tree biomass loss parameter (δ_T) or the fire frequency (f) will slow down the forest expansion.

4.5 Numerical illustration

4.5.1 Illustration of some traveling waves

We consider for numerical illustration that:

- the space interval is $[0; 300]$.
- The unit of space considered is meter (m) and unit of time is year (yr).
- The finite difference method is used to discretize the spatial part and on the other hand, the non standard finite difference method (Anguelov et al. [4]) is used to discretize the temporal part of

the problem given by the system (4.1).

By considering the value of parameters in table 4.2 page 163, we obtain the bifurcation diagram presented in figure 4.2, page 163.

Parameters	D_G	D_T	γ_G	δ_G	γ_T	δ_T	λ_{fG}	λ_{fT}	p	g_0	Ω	L_G	L_T
Values	0.01	0.1	2.7	0.1	1	0.3	0.7	0.8	3.4	0.14	5	50	50

Table 4.2: Parameter values for simulation, see also [103, 122].

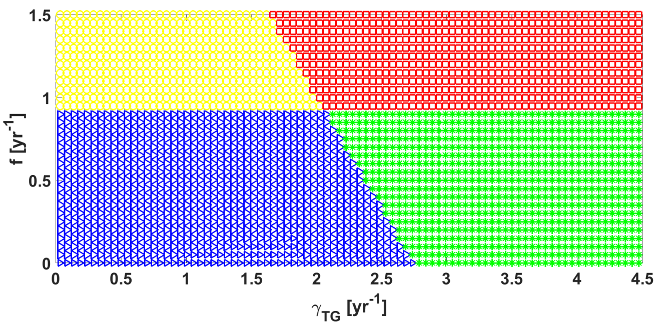
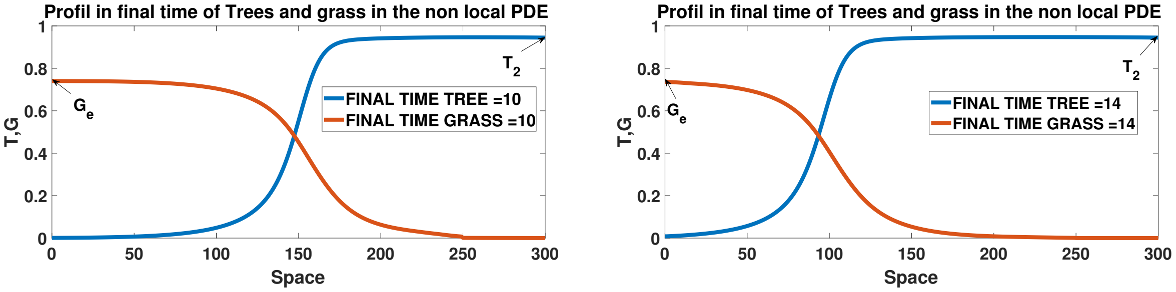


Figure 4.2: A bifurcation diagram for system (4.1) according to the parameter space $(\gamma_{TG} - f)$. The blue zone corresponds to the savanna monostability, the green zone stands for the forest monostability, the red zone denotes the forest-grassland bistability and the yellow zone depicts the grassland-savanna bistability area.

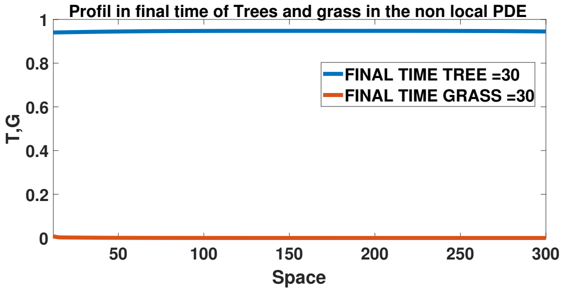
For the rest of numerical illustration, we consider $\gamma_{TG} = 3.5$ and we fix $M = 15m$. We are interested by the wave speed variation, in relation to fire frequency (f), tree seeds dispersal range (L_T) and tree biomass dispersal rate (D_T) as we noted in Proposition 4.9, page 161.

4.5.1.1 Illustration of wave speed variation with increasing value of fire frequency

We first consider $f = 0.85$. In this case, the forest homogeneous steady state $E_{T_2} = (0; 0.9477)$ is locally asymptotically stable and the grassland homogeneous steady state $E_{G_e} = (0.7426; 0)$ is unstable (see for instance figure 4.2, page 163). Then we have the following configuration after 10 years (figure 4.3-(a)), 14 years (figure 4.3-(b)) and 30 years (figure 4.3-(c)):



(a) Tree-grass distribution at $t = 10$ years. (b) Tree-grass distribution at $t = 14$ years.

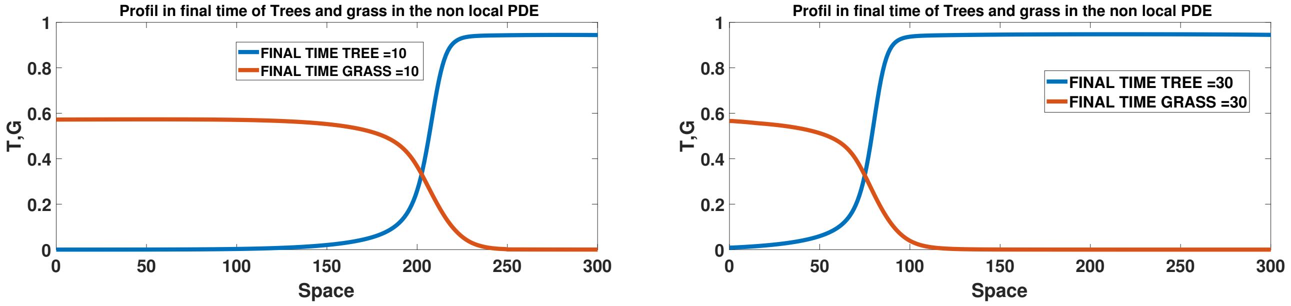


(c) Tree-grass distribution at $t = 30$ years.

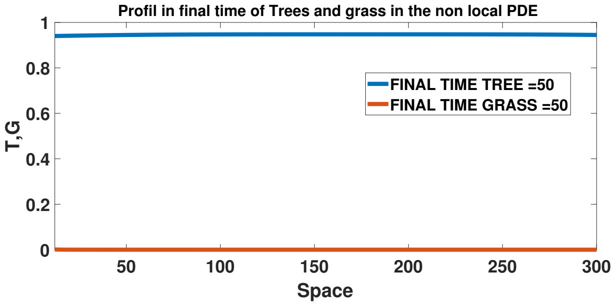
Figure 4.3: Grass and Tree distributions illustrating the progressive elimination of grassland by forest.

Figure 4.3 illustrates the fact that after thirty (30) years, the forest has completely replaced the grassland. In fact in this case, the traveling wave is monostable because (E_{T_2}) is the only stable homogeneous steady state of system (4.1), and it is clear that the forest would take over the grassland and the reverse can't be observed. Figure 4.3 will serve as a basic configuration to illustrate the variations of the wave speed.

Now we increase the fire frequency and we consider $f = 1.5$. Then for the value of parameters in table 4.2, $E_{G_e} = (0.5741; 0)$ and $E_{T_2} = (0; 0.9477)$ are both LAS (see the bifurcation diagram in figure 4.2, page 163). We have the following configuration after 10 years (figure 4.4-(a)), 30 years (figure 4.4-(b)) and 50 years (figure 4.4-(c)):



(a) Tree-grass distribution at $t = 10$ years. (b) Tree-grass distribution at $t = 30$ years.



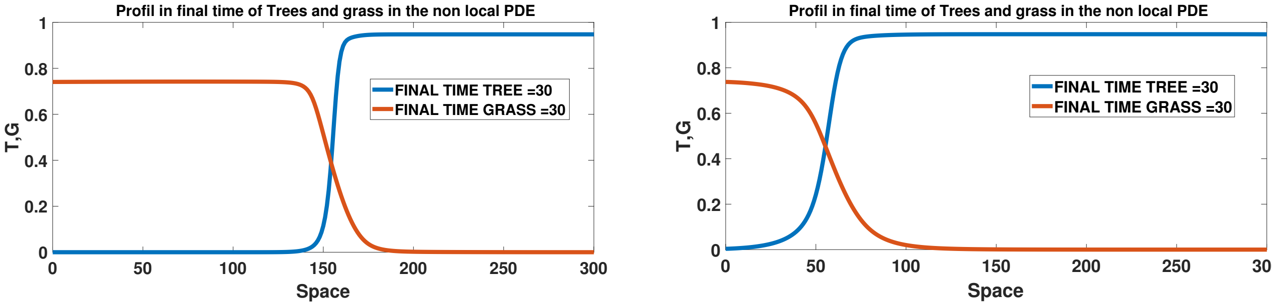
(c) Tree-grass distribution at $t = 50$ years.

Figure 4.4: Grass and Tree distributions illustrating the progressive elimination of grassland by forest with fire frequency $f = 1.5$.

From figure 4.4, we observe that the progression of forest into grassland is slowed down when the fire frequency increases. However, in nearly fifty years, the forest has completely replaced the grassland but it took more longer compared to the previous case in figure 4.3 (see for instance panel (b) of figure 4.4 and panel (c) of figure 4.3). Therefore, figures 4.3 and 4.4 illustrate the fact that the speed c_1^* of the wave could be a decreasing function of the fire frequency.

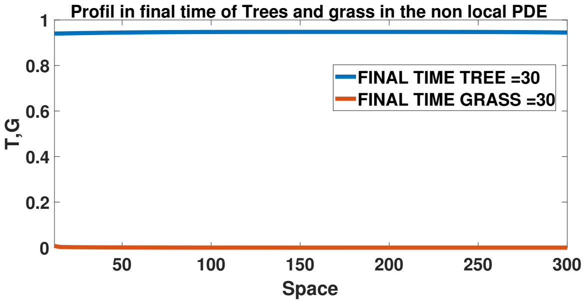
4.5.1.2 Illustration of wave speed variation with increasing values of L_T

In this subsection, we illustrate the fact that, when the range of tree seed dispersal L_T increases, it accelerates the forest encroachment. We fix the final time of illustration at $t = 30$ years and we consider different values of L_T ($L_T = 10m, \ 25m, \ 50m$). For $f = 0.85$ as we saw in the subsection 4.5.1.1, the forest homogeneous steady state $E_{T_2} = (0; 0.9477)$ is locally asymptotically stable and the grassland homogeneous steady state $E_{G_e} = (0.7426; 0)$ is unstable. Therefore we have the following configuration:



(a) Tree-grass distribution at $t = 30$ yr and $L_T = 10m$.

(b) Tree-grass distribution at $t = 30$ yr and $L_T = 25m$.



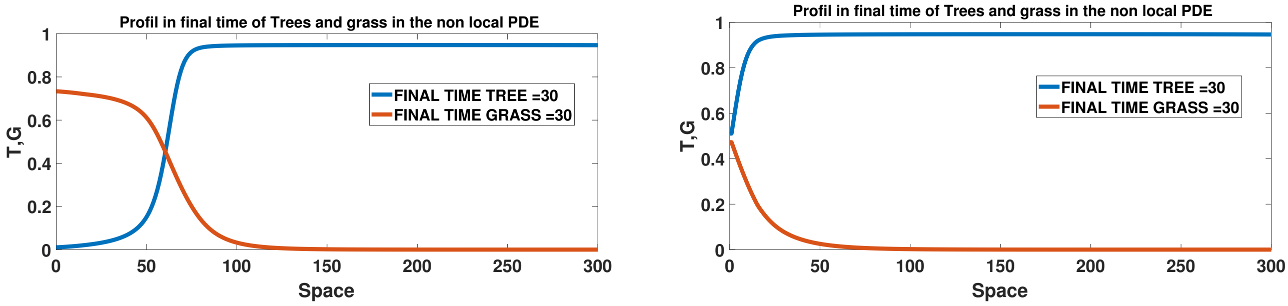
(c) Tree-grass distribution at $t = 30$ yr and $L_T = 50m$.

Figure 4.5: Illustration of Grass and Tree distributions with different values of L_T.

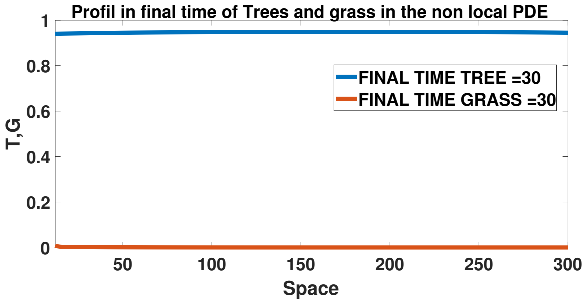
Then, from the panel (a) to the panel (c) of figure 4.5, we observe that the length of tree seed dispersal L_T plays a crucial role on the speed c_1^* of the wave. More precisely, higher values of L_T favor the encroachment of the forest on the grassland patch. In fact, L_T integrates ranges of different mechanisms, since the convolution kernel embodies not only seed dispersal stricto sensu but also facilitative influences of extant tree biomass on germination and seedling survival (probably the most influential mechanism).

4.5.1.3 Illustration of wave speed variation with increasing values of D_T

As in the previous subsection, we consider $f = 0.85$ with the final time $t = 30$ years. Our aim is to illustrate the ability of D_T to provide a quick increase of the wave speed. Therefore for different values of D_T, we have the following configuration:



(a) Tree-grass distribution at $t = 30$ yr and $D_T = 0.02$. (b) Tree-grass distribution at $t = 30$ yr and $D_T = 0.04$.



(c) Tree-grass distribution at $t = 30$ yr and $D_T = 0.1$.

Figure 4.6: Illustration of Grass and Tree distributions with different values of D_T and $L_T = 50m$.

Then, from the panel (a) to the panel (c) of figure 4.6, we observe that, the wave speed increases with increasing D_T and favour the encroachment of the forest into the grassland patch. In fact D_T integrates tree seed germination production along with establishment ability and these could facilitate the extension for forest.

4.6 Discussion

We developed and analysed a reaction-dispersion fire-meditated tree-grass interactions model in order to capture the forest-grassland distribution or transition observable in the climatic context of humid savannas. We proposed an integro-differential model based on Tega II et al. [103] by taking into account, nonlocal dispersal terms in both dynamics of tree and grass biomasses, which is the novelty in this paper. We focus on the potential transition between grassland and forest by studying the existence of a traveling wave that involves the grassland (E_{G_e}) and the forest (E_{T_2}) homogeneous steady states of our system. The key technical point is the construction of a nice pair of upper and lower solutions of the system, and the application of the Schauder fixed point theorem (Pan [73, 74], Zhao et al. [132]). Due to the consideration of nonlocal seed dispersal terms, it is important to asses the role of the ranges of nonlocal dispersal (L_G and L_T), in the existence of traveling wave and/or on the speed of the wave. Therefore we investigated through numerical simulation , the impact of tree seed dispersal range L_T, tree biomass dispersal rate D_T and the fire frequency f in the variation of the wave speed. One may remark that in the context of humid vegetation, current climatic conditions favor the forest expansion (Sagang et al. [83]). Then, to perpetuate the existing vegetation mosaics and preserve land cover diversity, it is necessary to control the speed of the forest invasion wave connecting the forest and the grassland/savanna states.

Based on the mathematical analysis, we proved the existence of a traveling wave with respect to our reaction-dispersion tree-grass system (4.1), connecting E_{G_e} and E_{T_2}. We indeed showed that there exists a real number c_1^* which is the minimal wave speed of the traveling wave connecting the forest to grassland homogeneous steady states of system (4.1). In Lemma 4.1, we proved that the minimal speed c_1^* depends on the tree biomass dispersal rate D_T, the tree seed dispersal range L_T, the fire frequency f, the tree biomass mortality due to human activities δ_T and the tree biomass intrinsic growth rate γ_T (see Remark 4.4, page 161). In fact according to Proposition 4.9, page 161, we have:

$$\frac{\partial c_1^*}{\partial f} < 0; \quad \frac{\partial c_1^*}{\partial \delta_T} < 0; \quad \frac{\partial c_1^*}{\partial D_T} > 0; \quad \frac{\partial c_1^*}{\partial \gamma_T} > 0; \quad \frac{\partial c_1^*}{\partial L_T} > 0. \tag{4.89}$$

We further illustrated the fact that, the increase of L_T or D_T promotes the invasion of grassland by forest (see figures 4.5 and 4.6, pages 166 and 167). Conversely, the increase of fire frequency is able to slow down the progression of forest (see figures 4.3 and 4.4, pages 164 and 165). For a management plan, it is now acknowledged that fire is one of the key disturbances that shapes the physiognomy of vegetation in humid tropical savannas where rainfall is sufficient to promote very high grass biomass production which in turn constitutes the principal fuel for fires during the dry seasons (Tega II et al. [103] and references therein).

In this paper, for mathematical simplicity and as a first attempt, we choose to work with local intraspecific competition between trees. We were able with this choice, to give from a mathematical point of view, a tractable characterization for the existence of the traveling wave solutions involving E_{G_e} and E_{T_2}, and the corresponding minimal wave speed. In fact, if we consider the nonlocal intraspecific competition in the growth term of tree biomass dynamics, it would become difficult (to our knowledge) to show that, the operator $\mathbf{P}$ defined in (4.46), page 148, verify the property $\mathbf{P}(\Gamma) \subset \Gamma$ for the application of Schauder's fixed point theorem that allow proving the existence of the traveling wave. Therefore, another process has to be constructed/advocated to address that case. On the other hand, non-local tree-tree competition is not expected to be a pervasive process in the humid savanna context where rainfall are a priori sufficient to sustain a close canopy forest.

4.7 Conclusion

Understanding the behaviour of forest-grassland dynamics in the humid part of the savanna biome, is fundamental to manage and preserve the current vegetation mosaics against rapid unexpected transitions. In this work, we developed and studied a reaction-dispersion model allowing to render the transitions of vegetation in the humid part of the savanna biome, where forest encroachment has been reported in spite of recurrent fires (Sagang et al. [83] and references therein). Then, we proved the existence of a monostable traveling wave connecting the forest homogeneous steady state to the grassland homogeneous steady state of our system, and, we also characterized the minimal speed of that traveling wave solution. The existence of traveling wave solutions is proved by constructing some suitable upper and lower solutions with the help of the Schauder's fixed-point theorem (see e.g., Pan [74], Zhao et al. [132], Ducrot et al. [33]). On the other hand, the conditions for the nonexistence of traveling wave solutions are obtained by applying the theory of asymptotic spreading for scalar equations. We therefore end up with a characterization of the minimal speed of possible forest-grassland traveling wave. Thereby, we proved that the forest encroachment is shaped by some influential parameters including the fire frequency, the tree biomass mortality, the range of tree seed dispersal and the tree biomass dispersal rate. A line of improvement of this work resides on the consideration of anisotropic seed dispersal in our system (4.1). In fact dispersal of seed and plant propagules are often directional (van Putten et al. [111]) and these latter can be highlighted from a mathematical point of view by the presence of asymmetric kernels functions (Dong et al. [31]) in the dispersal terms of system (4.1).

Conclusion and perspectives

THis thesis focused on the mathematical modelling of the spatio-temporal dynamics of vegetation in humid savannas, with a special attention concerning the effects of fires forcing on spatial self-organisation of vegetation. In fact, many researches dedicated to the description of different archetypes of vegetation landscapes has thrived the past three decades, principally in water limited regions (arid and semi-arid savannas). However, there have been a little use of spatial explicit and tractable mathematical models dedicated to vegetation physiognomies and transitions in humid environmental context. But to understand the self-organization of vegetation along the rainfall gradient, theoretical approaches are required and mathematical modelling is a useful tool to describe dynamics of complex systems. The main objectives in this dissertation were to construct and analyse mathematical tractable models able to give some informations about: (i) the ecological mechanisms that induced periodic vegetation physiognomies and (ii) phenomena of vegetations transitions, specifically on forest encroachment observed in several humid environmental ecosystems.

The first step of this dissertation was to carried out in Chapter 1, a deep literature review about ecological processes that shape the tree-grass coexistence in tropical ecosystems, notably the role of fire, herbivory, competition in term of intra and interspecific relations between trees and grasses, and finally facilitation mechanisms due to recurrent fires. This state of art further support a discussion on existing mathematical models of savannas vegetations and allows us to observe that spatial interactions mechanisms have been scarcely considered in the study of tree-grass dynamics in humid savannas where vegetation mosaics have been regularly reported. Consequently, our modelling choice in chapters 3 and 4, was made in order to be in agreement with the empirical evidence reported and to provide realistic predictions of vegetations physiognomies and transitions in humid environmental contexts. In the second step of this dissertation in chapter 2, we focused on a mathematical method for the existence and uniqueness of solution for reaction-diffusions equations. This chapter gives some important theorems that would help the readability of the rest of the manuscript.

The first contribution presented in chapter 3, is an integro-differential reaction diffusion model able to describe the spatial structuring of vegetation in wet savannas. We considered in this chapter, two states variables (tree and grass) and we assumed that these variables denote the density of tree and grass biomasses. The model developed in chapter 3 is an extension of a previous temporal model studied in Yatat Djeumen et al. [124], by integrating a space variable and kernel functions on both tree and grass dynamic. The aims of nonlocal terms was to describe firstly the area of tree and grass competitive mechanisms, secondly the tree canopy induced shadow effect, and finally to describe the consequence of fire-effect on tree mortality. The main objective of the model was to illustrate by the existence of inhomogeneous solution, the spatial structuring of vegetation in wet savannas area where regular spotted pattern (tree groves) have been casually reported with the presence of high grass production and frequent fires. We showed that the interplay between nonlocal competition and nonlocal facilitation is one of the key mechanisms for the emergence of inhomogeneous space solutions for the model. In fact, thanks to linear stability analysis, we characterized the minimal range of nonlocal interactions for the emergence of patterned space solutions in the model. Moreover in all cases where regular spot pattern were obtained, we characterized the wavelength of the associated periodic solution and we verified that the numerical wavelengths, were in good agreements with theoretical ones.

The second contribution in this dissertation is presented in chapter 4. Precisely, in the previous chapter, we consider as first approximation that both tree and grass biomasses have local propagation through Laplace operator. In fact for mathematical simplicity, plant dispersal is commonly described through diffusion in PDE models. However, diffusion is a local process that neglects nonlocal mechanism involved in the dispersal of seed or propagules. However, comprehensive empirical data on seed dispersal kernels, probability density functions describing the distribution of seed dispersal distances are available. Therefore, we investigate by the mean of a reaction dispersion model the impact of seed dispersal and fire on the process of forest encroachment reported in wet savannas. Mathematically, these empirical observations can be interpreted as a travelling wave that connects a forest state to a grassland state. Therefore, in chapter 4, we focus in a first step, on the existence of travelling wave connecting the forest and the grassland homogeneous steady states of the constructed reaction dispersion model and in second step on the characterisation of the minimal speed of this wave. We proved the existence of travelling wave by : (i) constructing some suitable upper and lower solution, (ii) using the Schauder fixed point theorem. We also provides, the condition of non existence of travelling wave. We therefore end up with the characterisation of the minimal wave speed connecting the two semi-trivial homogeneous steady states (forest and grassland) of the model. Our results imply that the increase of the length of tree seeds dispersal accelerates the wave speed and conversely, the increase of the fire frequency slows down the encroachment of the forest along the domain.

The first line of improvement of this dissertation may consists on the consideration of fire acting as a discrete event. In fact, as pointed out by Tchuinte et al. [101] and Yatat Djeumen et al. [121] fire is not a forcing that continuously removes a small fraction of biomass through time. Instead fire actually suppresses a substantial fraction of biomass at once through punctual outbreaks that shape the ecosystem aspect and immediate post-fire functioning. This principle was implemented with time discrete recurrence equation models, but another framework of impulsive differential equation also proved relevant to gain realism regarding nature and consequences of fire. To our knowledges, there is a lack of theories that deals with pattern formation and travelling wave for impulsive reaction dispersion equation. Hence it is desirable to study a space explicit model with nonlocal diffusion and impulsive fire events in a context on forest-savanna bistability in humid ecosystem. Moreover, in the analyse of the reaction-dispersion model developed in chapter 4, we numerically found that it is possible to obtain a travelling wave connecting a homogeneous solution to an inhomogeneous one, while considering nonlocal terms in the reaction part of the model. Then in our future work, we think to propose and analyse a mathematical model that deals with nucleation process and transition between homogeneous savannas and regular dense cluster of tree groves when nonlocal facilitation and nonlocal competition are simultaneous put in place.

Another challenging outlook of our PhD dissertation is the construction of mathematical models that put an emphasis on the dynamics of boundary between forest and savanna/grassland. In fact one way to circumvent the study of invasion process via travelling wave is to think about the modelling of the dynamics of the boundary between the observed mosaics. From a mathematical point of view, we think to use PDE with moving boundary. Finally, it is also important to improve our numerical schemes done in this dissertation in a one spatial domain (1D). One may obtain multi-scale vegetations pattern on 2D spatial domain.

Appendix

.1 Space of continuous functions

Let $C(\Omega)$ denote the vector space of all continuous functions defined on Ω, where (Ω, d) is a metric space. In order to turn continuous functions into normed space we need to restrict to bounded functions. For this purpose, let:

$$C_b(\Omega) = \{f : f \in C(\Omega), \ \ |f(x)| \leq M, \ \ \text{for some M}\}\,.$$

Proposition .10. $C_b(\Omega)$ is a metric space under the metric induced by the sup-norm. In other words,

$$d_\infty(f, g) = ||f - g||_\infty, \ \ \forall f, g \in C_b(\Omega).$$

Now we gives some basic property of $Cb(\Omega)$.

Proposition .11. $C_b(\Omega)$ is a complete metric space.

Proof. Let $\{f_n\}$ be a Cauchy sequence on $C_b(\Omega)$. For $\epsilon > 0$, there exist n_0 such that $||f_n - f_m|| < \epsilon/4$ for all $n, m \geq n_0$. In particular, it means that $\{f_n(x)\}$ is a Cauchy sequence on $\mathbb{R}$. By the completeness of $\mathbb{R}$,the $\lim_{n\to+\infty} f_n(x)$ exists and we define $f(x) \equiv \lim_{n\to+\infty} f_n(x)$. Assuming $f \in C_b(\Omega)$, by taking $m \to +\infty$ in the inequality above, we immediately obtain $||f_n - f|| < \epsilon/4 < \epsilon$, hence $f_n \longrightarrow f \in C_b(\Omega)$. To show $f \in C_b(\Omega)$, we let $m \to +\infty$ in $||f_n - f_m|| < \epsilon/4$ to get $||f_n - f|| < \epsilon/4 < \epsilon$ for all $n \geq n_0$. Taking $n = n_0$, we get $|f(x)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| \leq \epsilon/4 + ||f_{n_0}||_\infty$, hence f is bounded. On the other hand as f_{n_0} is continuous, for each x, we can find δ such that $|f_{n_0}(y) - f(x)| \leq \epsilon/4$, whenever $d(y, x) \leq \delta$. It follows that for all $d(y, x) \leq \delta$,

$$|f(y) - f(x)| \leq |f(y) - f_{n_0}(y)| + |f_{n_0}(y) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)| \leq \epsilon/4 < \epsilon.$$

□

Proposition .12. $C_b(\Omega) = C(\Omega)$ when (Ω, d) is a compact metric space.

Proof. We need to show that every continuous function on a compact set is bounded. Assume on the contrary that for some continuous function f, there are points $\{x_k\}$ such that $|f(x_k)| \longrightarrow +\infty$. By compactness there exists a subsequence $\left\{x_{k_j}\right\}$ such that $\lim_{j\longrightarrow+\infty} x_{k_j} = z$. But by the continuity of f, we would have $\lim_{j\longrightarrow+\infty} |f(x_{k_j})| = |f(z)| < \infty$, contradiction holds. ☐

.2 Hölder functions spaces

Let Ω be a bounded or an unbounded open domain in $\mathbb{R}^n$, and let $\partial\Omega$ be the boundary of Ω. For each $T > 0$, let $D_T = (0,T] \times \Omega$, $S_T = (0,T] \times \partial\Omega$. Denote by:

- $C^m(\Omega)$, the set of all continuous functions, whose partial derivatives up to the mth order are continuous in Ω.
- $C^{l,m}(D_T)$ the set of functions whose l-times derivatives in t and m-times derivatives in x are continuous in D_T.

Similar notations are used for $C^m(\bar{\Omega})$ and $C^{l,m}(\bar{D}_T)$, where $\bar{\Omega}$, $\bar{D}_T$ are the respective closure of Ω and D_T. When $m = 0$ whe denote by $C(\Omega)$, $C(\bar{\Omega})$, $C(D_T)$, $C(\bar{D}_T)$ the set of continuous functions in Ω, $\bar{\Omega}$, D_T and $\bar{D}_T$ respectively. The norms in $C(\Omega)$ and $C(D_T)$ are defined by:

$$|u|_0^{\Omega} = \sup_{x\in\Omega} |u(x)| \text{ and } |u|_0^{D_T} = \sup_{(t,x)\in D_T} |u(t,x)| \tag{90}$$

Similar norms with respect to $\bar{\Omega}$, $\bar{D}_T$ are defined for $C(\bar{\Omega})$ and $C(\bar{D}_T)$.

Definition .3. A function $u \in C(\Omega)$ is said to be Hölder continuous of order $\alpha \in (0,1)$ if

$$H_\alpha \equiv \sup\left\{\frac{|u(x) - u(\xi)|}{|x-\xi|^\alpha};\ \ x,\xi \in \Omega \text{ and} x \neq \xi\right\} < \infty$$

The Hölder norm of u is defined by

$$|u|_\alpha = |u|_0^{\Omega} + H_\alpha \tag{91}$$

Let D_x^m be nay partial derivative of order m with respect to the variables $x_1, ..., x_m$ and define:

- $|u|_m \equiv |u|_0 + \sum |D_x u|_0 + ... + \sum |D_x^m u|_0$,
- $|u|_{1+\alpha} \equiv |u|_0 + \sum |D_x u|_\alpha$,
- $|u|_{2+\alpha} \equiv |u|_0 + \sum |D_x u|_\alpha + \sum |D_x^2 u|_\alpha$,

where the sums are taken over all partial derivatives of the indicated order.

Proposition .13. The sets of all functions u for which $|u|_m < \infty,\ \ |u|_{1+\alpha} < \infty,\ $ and $|u|_{2+\alpha} < \infty$, are denoted respectively, by $C^m(\Omega),\ \ C^{1+\alpha}(\Omega),\ $ and $\ C^{2+\alpha}(\Omega)$.

The set of Hölder-continuous functions in Ω with finite norm is denoted by $C^\alpha(\Omega)$.

Proposition .14. When the domain Ω is replaced by D_T we defined the Hölder constant by

$$H_\alpha \equiv \sup\left\{\frac{|u(t,x)-u(\tau,\xi)|}{(|t-\tau|^\alpha+|x-\xi|^\alpha)};\ \ x,\xi\in\Omega\ \ \text{and} x\neq\xi\right\}<\infty$$

and the set of all Hölder-continuous functions in D_T with finite Hölder norm is denoted by $C^\alpha(D_T)$.

.3 Contraction mapping theorem

The following theorem is called contraction mapping theorem or Banach fixed point theorem. Let (X,d) be a metric space.

Theorem .7. If $T : X \longrightarrow X$ is a contraction mapping on a complete metric space (X,d), then there is exactly one solution $x\in X$ of :

$$T(x)=x. \tag{92}$$

Moreover, if $y\in X$ is arbitrarily chosen, then the iterates $(x_n)_{n\in\mathbb{N}}$, given by:

$$x_0=y \ \text{ and } \ x_{n+1}=T(x_n),\ \ n\geq 1, \tag{93}$$

have the property that: $\lim_{n\to+\infty}x_n=x$.

.4 A mathematical process for pattern formation: Diffusion driven instability

Spatial or spatio-temporal heterogeneities are one of the characteristic of many ecological systems and have profound effects on the dynamics of invasion, population growth and persistence. These heterogeneities are determining factors of the self-organisation of the studied system. The identification of factors and parameters making it possible, to pass from the stable homogeneous state to the emergence of spatial or spatio temporal patterns is a mater of great importance in the study of biological process or in understanding the evolution of ecosystems. In this paragraph, we are going to give for a model of two species, conditions allowing the formation of non-homogeneous structures. In fact, in the study of non linear chemical system, Turing found that the diffusion of two species may lead to the pattern formation when, they have different diffusion. In the absence of diffusion, both species reach a stable and spatially uniform steady state, while diffusion may be able to destabilize this state (diffusion driven instability) leading to the formation of spatial pattern.

We consider a system of two reaction diffusion equation describe in a one spatial domain the spatial dynamics on two species U and V:

$$\begin{cases} \dfrac{\partial U}{\partial t} &= D_U \dfrac{\partial^2 U}{\partial x^2} + F(U,V), \\ \\ \dfrac{\partial V}{\partial t} &= D_V \dfrac{\partial^2 V}{\partial x^2} + G(U,V), \end{cases} \tag{94}$$

where F and G described the intra and interspecific dynamic of each species U and V; D_U and D_V are the diffusion coefficients. We are interesting, on the study of pattern formation induced by diffusion. For this, we suppose that (U^*, V^*) is a stable homogeneous steady state for the system (94). We set:

$$U(x,t) = U^* + u(x,t) \quad \text{and} \quad V(x,t) = V^* + v(x,t), \tag{95}$$

where

$$|U(x,t)| << U^*, \quad |V(x,t)| << V^*.$$

Linearising system (94) around (U^*, V^*) by using Taylor's expansion, we get the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} &= D_U \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial F}{\partial U}(U^*, V^*)u + \dfrac{\partial F}{\partial V}(U^*, V^*)v, \\ \\ \dfrac{\partial v}{\partial t} &= D_V \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial G}{\partial U}(U^*, V^*)u + \dfrac{\partial G}{\partial V}(U^*, V^*)v. \end{cases} \tag{96}$$

By the Fourier analysis, we can compute:

$$u(x,t) = u_0 e^{\sigma t + ikx} \quad \text{and} \quad v(x,t) = v_0 e^{\sigma t + ikx}, \tag{97}$$

where u_0, v_0 are constant and the function e^{ikx} is periodic and bounded, and k is a wave number, that would indicated the wavelength of emergence of spatial structure:

- if $Re(\sigma(k)) < 0$ for all k, then the solution of system (94) converges to the homogeneous steady state (U^*, V^*).
- if there exits k_0 such that for a range of values of $k > k_0$, $Re(\sigma(k)) > 0$, then (U^*, V^*) is unstable.

Substituting (97) in (94), we get :

$$\begin{cases} \sigma u_0 &= -D_U k^2 u_0 + \dfrac{\partial F}{\partial U}(U^*, V^*)u_0 + \dfrac{\partial F}{\partial V}(U^*, V^*)v_0, \\ \\ \sigma v_0 &= -D_V k^2 v_0 + \dfrac{\partial G}{\partial U}(U^*, V^*)u_0 + \dfrac{\partial G}{\partial V}(U^*, V^*)v_0, \end{cases} \tag{98}$$

We put the previous system in (98) in a matrix form:

$$\begin{pmatrix} \sigma + D_U k^2 - \dfrac{\partial F}{\partial U}(U^*,V^*) & -\dfrac{\partial F}{\partial V}(U^*,V^*) \\ -\dfrac{\partial G}{\partial U}(U^*,V^*) & \sigma + D_V k^2 - \dfrac{\partial G}{\partial V}(U^*,V^*). \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{99}$$

By setting

$$\begin{aligned} M(k) &= D_U D_V k^4 - \left(D_U \frac{\partial G}{\partial V}(U^*,V^*) + D_V \frac{\partial F}{\partial U}(U^*,V^*) \right) k^2 + \frac{\partial F}{\partial U}(U^*,V^*) \frac{\partial G}{\partial V}(U^*,V^*) \\ &\quad - \frac{\partial F}{\partial V}(U^*,V^*) \frac{\partial G}{\partial U}(U^*,V^*), \end{aligned}$$

we get the relation between eigenvalue σ and the wave number k, knows as dispersion relation:

$$\sigma^2 + \left[(D_U + D_V) k^2 - \left(\frac{\partial F}{\partial U}(U^*,V^*) + \frac{\partial G}{\partial V}(U^*,V^*) \right) \right] \sigma + M(k) = 0. \tag{100}$$

The process of "diffusion driven instability" required that the homogeneous steady state (U^*, V^*) be locally asymptotic stable (LAS) for $k = 0$ (i.e LAS in the space implicit system associated to the system (94)), and become unstable (due to the diffusion of two species) for a certain k. Then, we are aimed to establish conditions for the system without diffusion and with diffusion.

- For $k = 0$, (U^*, V^*) is stable if:

$$\begin{cases} \dfrac{\partial G}{\partial V}(U^*,V^*) + \dfrac{\partial F}{\partial U}(U^*,V^*) < 0, \\ \dfrac{\partial F}{\partial U}(U^*,V^*) \dfrac{\partial G}{\partial V}(U^*,V^*) - \dfrac{\partial F}{\partial V}(U^*,V^*) \dfrac{\partial G}{\partial U}(U^*,V^*) > 0 \end{cases} \tag{101}$$

- For $k > 0$. To have $Re(\sigma(k)) > 0$, for a certain k it is necessary that $M(k) < 0$. But, examining $M(k)$ we have:

$$\begin{cases} D_U D_V k^4 > 0 \ \text{ for all } \ k \in \mathbb{R}. \\ \dfrac{\partial F}{\partial U}(U^*,V^*) \dfrac{\partial G}{\partial V}(U^*,V^*) - \dfrac{\partial F}{\partial V}(U^*,V^*) \dfrac{\partial G}{\partial U}(U^*,V^*) > 0, \end{cases} \tag{102}$$

  and then, $M(k) < 0$, if only if

$$D_U \frac{\partial G}{\partial V}(U^*,V^*) + D_V \frac{\partial F}{\partial U}(U^*,V^*) > 0 \tag{103}$$

and

$$\left(D_U\frac{\partial G}{\partial V}(U^*,V^*)+D_V\frac{\partial F}{\partial U}(U^*,V^*)\right)k^2 \;>\; D_UD_Vk^4+\frac{\partial F}{\partial U}(U^*,V^*)\frac{\partial G}{\partial V}(U^*,V^*)- \frac{\partial F}{\partial V}(U^*,V^*)\frac{\partial G}{\partial U}(U^*,V^*). \tag{104}$$

Therefore, we can summarize the necessary conditions for instability in the following system:

$$\begin{cases} \dfrac{\partial G}{\partial V}(U^*,V^*)+\dfrac{\partial F}{\partial U}(U^*,V^*)<0, \\ \dfrac{\partial F}{\partial U}(U^*,V^*)\dfrac{\partial G}{\partial V}(U^*,V^*)-\dfrac{\partial F}{\partial V}(U^*,V^*)\dfrac{\partial G}{\partial U}(U^*,V^*)>0, \\ D_U\dfrac{\partial G}{\partial V}(U^*,V^*)+D_V\dfrac{\partial F}{\partial U}(U^*,V^*)>0, \\ \left(D_U\dfrac{\partial G}{\partial V}(U^*,V^*)+D_V\dfrac{\partial F}{\partial U}(U^*,V^*)\right)k^2>D_UD_Vk^4+\dfrac{\partial F}{\partial U}(U^*,V^*)\dfrac{\partial G}{\partial V}(U^*,V^*)- \\ \dfrac{\partial F}{\partial V}(U^*,V^*)\dfrac{\partial G}{\partial U}(U^*,V^*). \end{cases} \tag{105}$$

Remark .5. It is easy to see that:

$a)$ If $D_U = D_V$, then the first and the third inequality in system (105) contradicts. Then for pattern formation it is necessary that $D_U \neq D_V$.

$b)$ $\frac{\partial F}{\partial U}(U^*,V^*)$ and $\frac{\partial G}{\partial V}(U^*,V^*)$ has different sign. It follows the same for $\frac{\partial F}{\partial V}(U^*,V^*)$ and $\frac{\partial G}{\partial U}(U^*,V^*)$.

$c)$ The conclusion in $b)$ induced that cooperative and competitive model can't lead to pattern formation via diffusion. In these particular cases it is necessary to think about another mechanisms to break up the stability of the homogeneous steady state two give rise to pattern (see the model develops in chapter 3).

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